Connected (n_4) configurations exist for almost all n -- second update

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An $(\mathbf{n_4})$ **configuration** is a family of n points and n (straight) lines in the Euclidean plane such that each point is on precisely four of the lines, and each line contains precisely four of the points. A configuration is **connected** if it is possible to reach every point starting from an arbitrary point and stepping to other points only if they are on one of the lines of the configuration.

In a series of papers leading to [4] it was shown that connected (n_4) configurations exist for all $n \geq 21$, with the possible exception of certain ten values, namely n = 22, 23, 26, 29, 31, 32, 34, 37, 38, 43. Also open was the existence of any configurations with $n \leq 20$; it was conjectured in [5] and [4], and widely believed, that no such configurations are possible.

The last few months have seen considerable advances in the information available on (n_4) configurations. This resulted in the following:

Theorem. Connected (n_4) configurations exist if and only if $n \geq 18$, except possibly if n has one of the eight values 18, 19, 22, 23, 26, 34, 37, 43.

We shall assume that the reader has access to [4], and therefore we restrict attention to $n \leq 20$ and n = 29, 31, 32 and 38. It is convenient to first recall some recent results on configurations of "pseudolines".

A family of simple curves in the (projective or Euclidean) plane is a family of *pseudolines* provided each curve differs from a straight line by at most one segment of the line, and any two curves have at most one point in common, at which they cross each other. (This is equivalent to the definition in [2].) Configurations of pseudolines are defined in complete analogy to configurations of lines. Configurations of pseudolines can be interpreted as topological analogues of configurations of lines, and clearly every configuration of lines can be interpreted as a configuration of pseudolines. On the other hand, it is well known that there exist *unstretchable* configurations of pseudolines, that is, such configurations that are not combinatorially equivalent to configurations of lines.

A result of [2] is the following analogue of our theorem: *There exists an* (n_4) *configuration of pseudolines if and only if* $n \geq 17$. Moreover, in still unpublished work, J. Bokowski and L. Schewe have shown that there is a unique configuration (17_4) of pseudolines, and that it is unstretchable.

Turning now to the proof of our theorem, we see that we need to consider only $n \geq 18$. The cases $n = 18$ and $n = 19$ are still undecided (hence listed in the theorem). However, for $n = 20$ we have the following new construction, illustrated in Figure 1, which can be extended to all $n \geq 20$ divisible by 4.

The construction starts with an astral configuration (10_3); recall from [3] that a configuration (10_3) is *astral* provided there are two orbits of points and two orbits of lines under Euclidean symmetries of the configuration. It is known that there is a unique astral configuration (10_3), which is shown in Figure 1 of [3], it also appears as the heavily drawn part of the illustration of Figure 1 below. The construction of our (20_4) configuration is completed by placing a second copy of the (10_3) configuration, suitably reduced, in such a position that each of its line passes through one of the points of the first configuration, and each of its points lies on one of the original lines. Easy continuity arguments show that such choice is possible in the unique way shown in Figure 1.

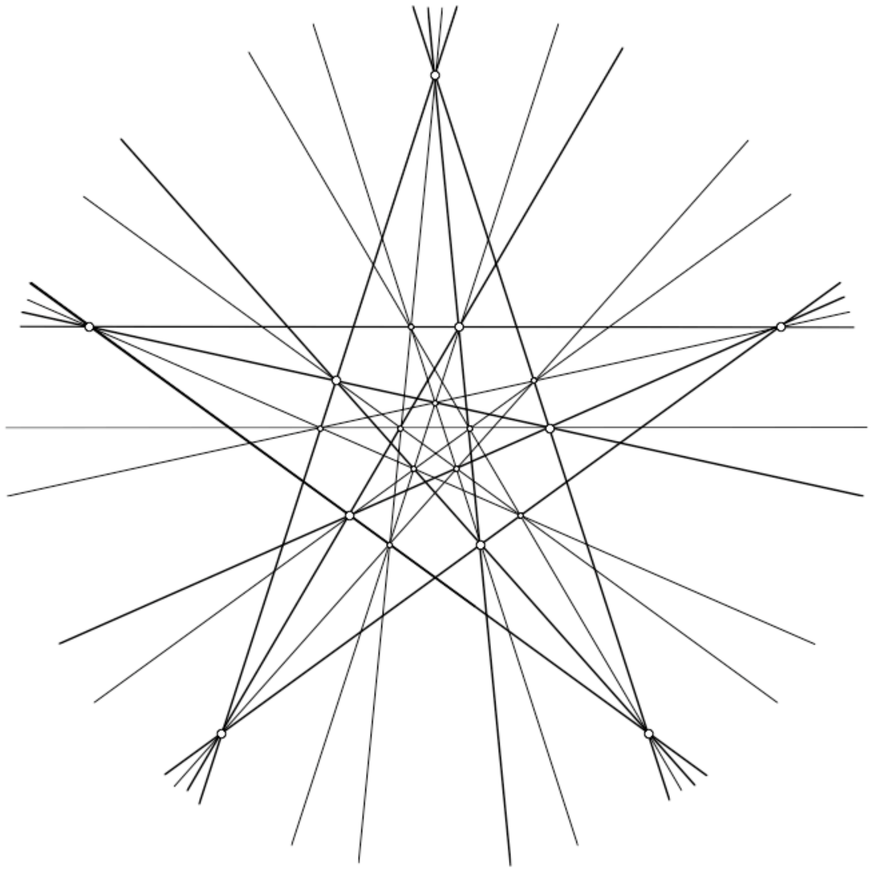

Figure 1. The heavy lines and larger dots indicate an astral (10_3) configuration. Taking a suitably shrunk and rotated second copy yields the desired (20_4) configuration.

As mentioned above, exactly the same construction works for other astral (n_3) configurations, leading to configurations $((2n)_4)$. The one case of interest in connection with our theorem is the case $n = 16$. This leads to the configuration (32_4) – which is one of the now-resolved cases that were open in [4]. This configuration is shown in Figure 2.

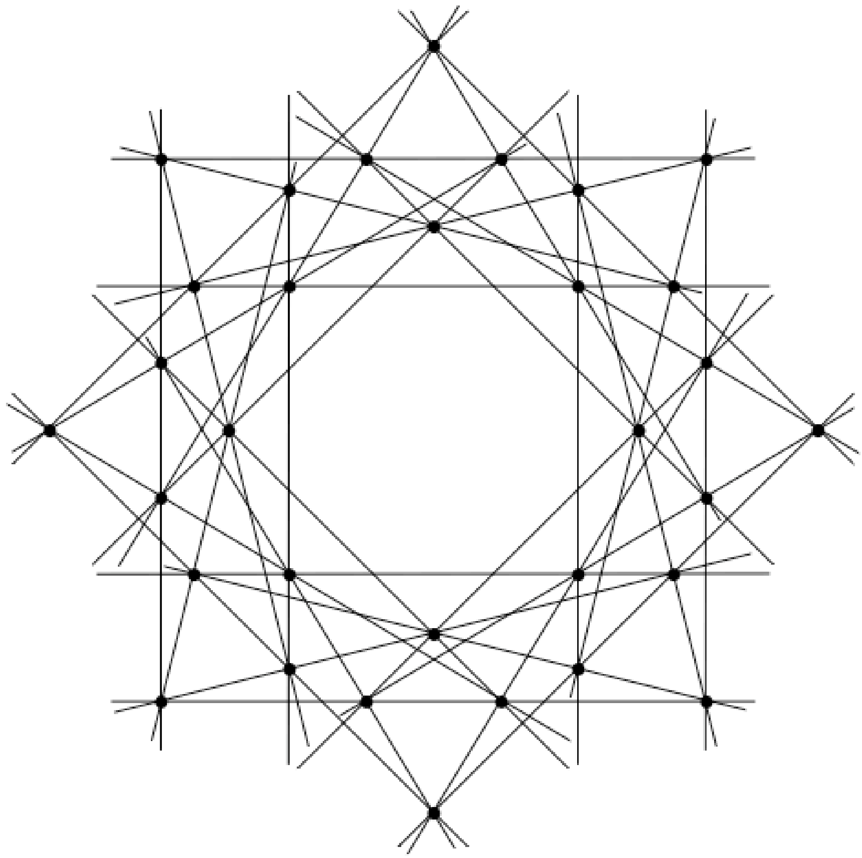

Figure 2. A (32_4) configuration obtained from two copies of an astral (16_3) configuration by the method explained in the text.

Concerning the other two new values of n for which the existence of (n_4) configurations has been established we cannot reproduce here the diagrams. A configuration (38_4) can be constructed by starting with two copies of the configuration (20_4), deleting an appropriate line and an appropriate point from each, and combining suitably distorted images to restore the incidence of each point with four lines. We note that this is a variant of the construction of (53_4) used in [4], see Figure 3 of that paper. The author will gladly e-mail a .pdf image of the (38_4) configuration to interested readers.

Finally, the construction of configurations (29_4) and (31_4), using completely different ideas from the methods used here, has been achieved by Juergen Bokowski and Lars Schewe. It will appear in a separate publication.

This completes the proof of our Theorem.

Remarks.

1. The construction of a (28_4) configuration was described in [5]; however, it has never been drawn in any intelligible manner. The method we used above for the construction of the (20_4) and (32_4) configurations yields various (28_4) configurations, starting from the several (14_3) configurations. One of these (28_4) configurations is shown in Figure 3.
2. The construction of the configurations in Figures 1, 2, and 3 shows a remarkable feature: although the (n_3) configurations used have cyclic symmetry but no mirror symmetry, the resulting configurations have dihedral symmetry.
3. It is an often observed phenomenon that after a first example is found, additional ones come at a fast pace — frequently simpler than the original. In case of (25_4) configurations which have been constructed only recently, by now eight different configurations are known. They are mainly due to still unpublished work of J. Bokowski and T. Pisanski. Other examples of this nature are shown in the following figures.

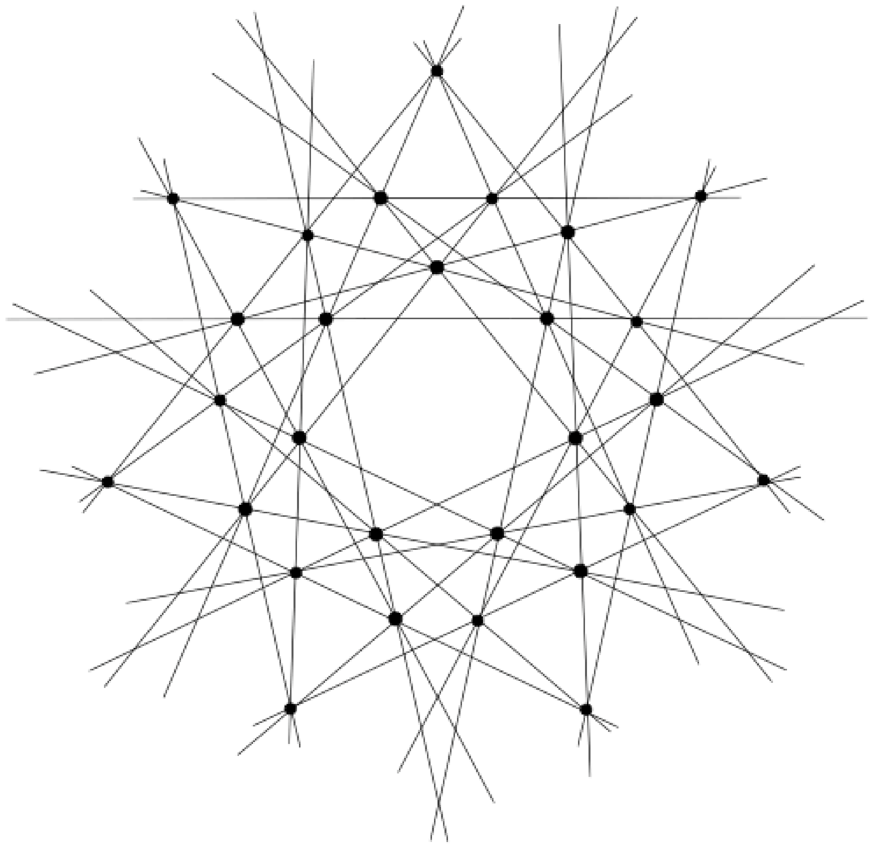

Figure 3. A configuration (28_4) obtained from one of the astral (14_3) configurations.

In Figure 4 is shown a (32_4) configuration distinct from the one in Figure 2. It has two orbits of points and three orbits of lines.

A method of constructing a (35_4) configuration was described in [5]. However, this configuration does not lend itself to intelligible presentation by diagrams. In Figure 5 are shown four simple examples of (35_4) configurations. All are derived from polycyclic (specifically 3-cyclic) configurations (42_4) by omitting one half of

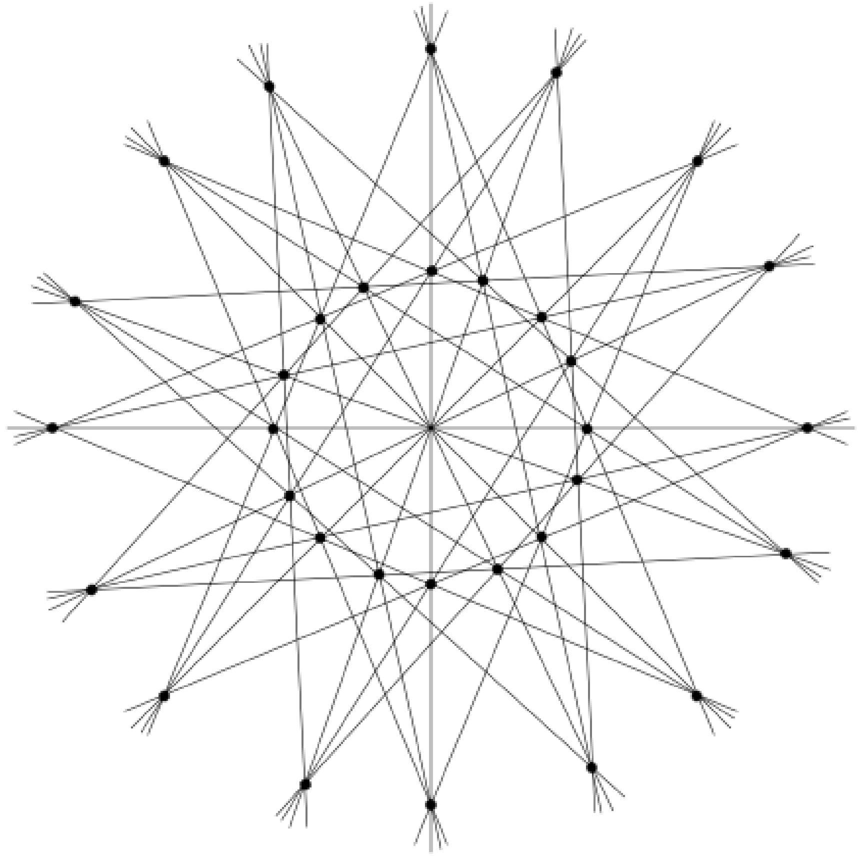

Figure 4. A (32_4) configuration with only two orbits of points.

points of one transitivity class and the lines incident with them, and adding appropriate lines to restore the correct number of incidences. (Concerning *polycyclic* configurations see [1].)

References.

[1] M. Boben and T. Pisanski, Polycyclic configurations. Europ. J. Combin. 24(2003), 431 – 457.

[2] J. Bokowski, B. Grünbaum and L. Schewe, Topological configurations (n_4) exist for all $n \geq 17$. (Submitted for publication)

[3] B. Grünbaum, Astral (n_k) configurations. Geombinatorics 3(1993), 32 – 37.

[4] B. Grünbaum, Connected (n_4) configurations exist for almost all n – an update. Geombinatorics 12(2002), 15 – 23.

[5] B. Grünbaum and J. F. Rigby, The real configuration (21_4). J. London Math. Soc. (2) 41 (1990), 336 – 346.

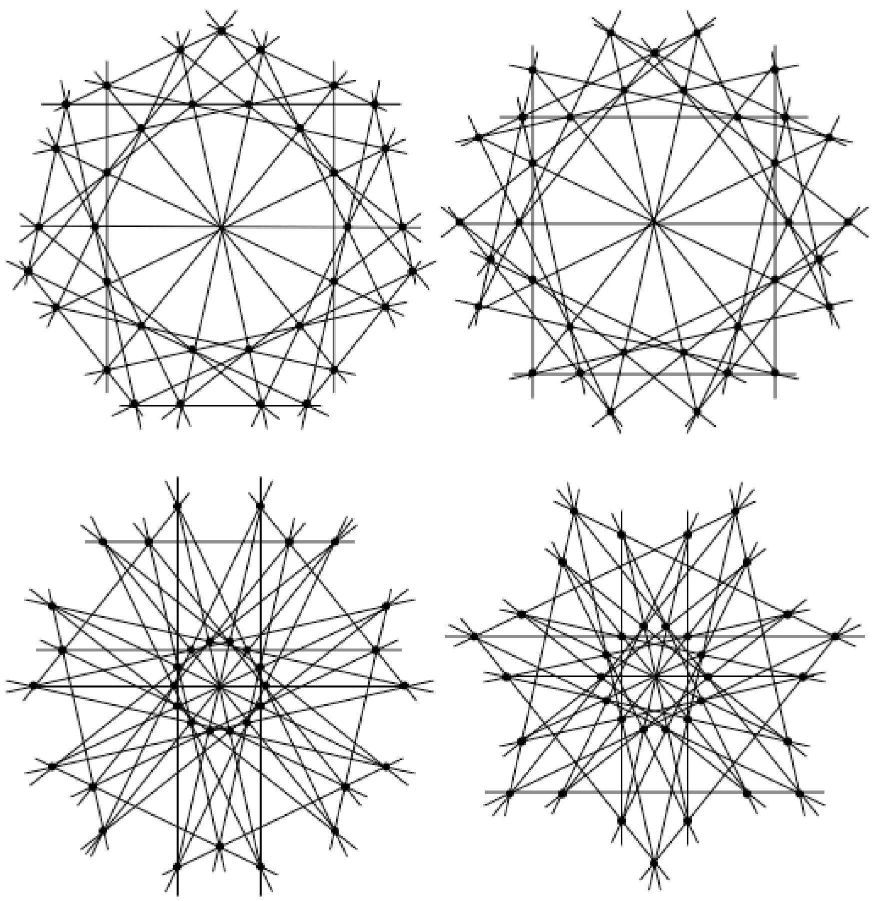

Figure 5. Four examples of configurations (35_4).