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Corresponding Author: Prof. Branko Grunbaum,

Corresponding Author's Institution: University of Washington

First Author: Branko Grunbaum

Order of Authors: Branko Grunbaum

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1. Introduction.

1.1. Early history.

Polyhedra appeared early in the history (as pyramids, dice, and other signs of civilization), and in geometry (Plato, Euclid, Archimedes). However, all these were individual, particular polyhedra. Euclid's enumeration of the regular polyhedra (Platonic solids) was faulted for not defining the class of polyhedra among which he singles out the regular ones. Euclid's defenders have claimed that the term "polyhedron" was understood by the "man in the street" as denoting convex polyhedra; if so, they ancient Greeks were well ahead of our contemporaries. (The lack of precision is one of the unfortunate "traditions" in the theory of polyhedra; at the present time it continues unabated, on the Web and elsewhere.) No general definition of polyhedra appears till much after Euclid, and even then in strange forms. For example, at the end of the XVII century, Ozanam's famous "Dictionnaire Mathematique" declares [54, p.119]:

Le POLYEDRE est un corps terminé par plusieurs Plans rectilignes, & inscriptible dans une Sphere, c'est à dire qu'une Sphere peut être décrite à l'entour. en telle sorte que sa surface touche tous les angles solides du Polyedre ... [The polyhedron is a solid bounded by several straight planes, and inscribable in a sphere, that is, a sphere can be described around in such a way that its surface touches all the solid angles [[vertices]] of the polyhedron.]

Even in the middle of the eighteenth century, Euler discussed his famous theorem $V - E + F = 2$ without specifying what are the polyhedra to which it applies. Apparently he had convex polyhedra in mind, and certainly many of the successors did as well – although they mostly did not define what are the polyhedra about which they are claim to be proving theorems. The discovery of polyhedra to which Euler's theorem does not apply led to the pithy comment by Hessel [37]:

> Andere ausgezeichnete Mathematiker (Legendre, Cauchy, Gergonne, Rothe und Steiner) haben Beweise für die allgemeine Gültigkeit des Satzes geliefert. Indessen leidet derselbe Ausnahmen. [Other excellent mathematicians (Legendre, Cauchy, Gergonne, Rothe and Steiner) gave proofs for the general validity of the theorem. But in fact, it suffers from exceptions.]

1.2. More recent developments concerning convex polyhedra.

During the 19th century, the center of attention concerning polyhedra switched to (more-or-less explicitly declared) convex ones; we shall later return to the less numerous but still important considerations of other polyhedra. However, a sense of naive trust in the benevolent nature of mathematical objects persisted. Even such otherwise very critical thinkers as Thomas Kirkman seemed to believe that if you draw in the plane a figure that seems to represent a polyhedron, there actually is a convex polyhedron that looks much like this figure (see [40]). The first to publicly question this attitude was Ernst Steinitz, in Section 21 of his 1916 contribution [60] to the great "Encyklopedie der mathematischen Wissenschaften". In the following thirteen sections Steinitz shows how one can formulate the criteria that are necessary and sufficient for the existence of a convex geometric polyhedron that is combinatorially given, and establishes that all such convex realizations are determined up to isomorphism of convex polyhedra. The formulation of the result labeled as "the fundamental theorem of the convex types", as given in [60, p. 77] is:

> Jedes K-polyeder ist als konvexes Polyeder realisierbar. [Every K-polyhedron can be realized by a convex polyhedron.]

The difficult chore of deciphering what this means is probably responsible for the long-lasting ignorance of this basic theorem about convex polyhedra. The sketch of the proof given in [60] was elaborated by Steinitz in notes for lectures given in the early 1920's; these notes were posthumously published (after editing by H. Rademacher), see [61].

The formulation of Steinitz's criteria is quite cumbersome, starting from very general 2-dimensional complexes. In retrospect, after absorbing all his definitions and statements it is rather easy to see that Steinitz had reached, in every respect except terminology, the modern formulation, which we shall discuss in the next section. That formulation is graph-theoretic — but in 1916 there was no graph theory he could use. As far as I can tell, a *graph-theoretic* reformulation of Steinitz's theorem was first published in [30] and, in the formulation which is now standard, in [31]. The first direct graph-theoretic proof was published in [18]. Other proofs and generalizations will be discussed in Sections 2 and 3.

1.3. Nonconvex polyhedra.

On the other hand, certain *nonconvex polyhedra* seem to have been first discussed and described in a geometric context by Pacioli [55]. This work contains a number of illustrations of such polyhedra, generally considered to have been drawn by Leonardo da Vinci. Somewhat later, two *regular polyhedra* with *pentagrammatic* faces have been described by Kepler [39]. Although forgotten for almost two centuries, Kepler's work resurfaced after these regular polyhedra, together with two additional ones, were independently discovered by Poinsot [56]. Cauchy [8] soon thereafter proved that there are no other regular polyhedra besides the Platonic one and the ones found by Poinsot.

Later in the nineteenth century, many authors discussed various special kinds of nonconvex polyhedra. A survey of the theory of polyhedra as it existed at the end of that century is the well-known book [7] by Brückner. It presented photographs of a huge number of nonconvex polyhedra. It has been asserted that it gives an exhaustive overview of those polyhedra that have a high degree of symmetry — such as isogonal or isohedral polyhedra (the symmetries of which act transitively on their faces or vertices,

respectively). Unfortunately, this book is completely noncritical and is, in fact, internally inconsistent and misleading. I believe that the problems caused by vague definitions (which are, moreover, often ignored by their authors), contributed to the lack of interest in nonconvex polyhedra throughout most of the 20th century.

The central obstacle to any coherent theory of polyhedra more general than the convex ones is the difficulty of defining precisely what objects should be awarded that designation. In Sections 4 and 5 we shall detail the difficulties and the attempts to overcome them. The solution I shall present in Section 7 is based on description of the combinatorial structure of very general abstract polyhedra by certain graphs characterized by their properties, and presenting geometric polyhedra as realizations of the abstract polyhedra by points, segments and polygons. This development has as a prerequisite the development of an analogous approach to polygons; this will be explained in Section 6. The view of general polygons (which goes back to Poinsot [56]) that is widely accepted at the present time, is unsatisfactory and inconvenient; its shortcomings bear a considerable responsibility for the poor condition of the theory of general polyhedra. The last section will also provide illustrations of previously anomalous situations and cases, which are easily explained by the graph-theoretic approach.

2. Steinitz's theorem and its proofs.

2.1. Formulation and proofs.

The version of Steinitz's theorem that is most relevant to graph theory, and very useful in other contexts, can be formulated as follows:

Theorem 2.1. *A graph* G *is isomorphic to the 1-skeleton of a 3-dimensional convex polyhedron* P *if and only if* G *is planar and 3-connected.*

A graph is said to be 3-connected if any two vertices can be connected by three paths, any two of which have only the two vertices in common. As mentioned above, this formulation of the theorem differs from the original version, which relies on concepts and results concerning 2-dimensional cell manifolds. However, the ideas of Steinitz's original proofs can be reworked in a graph-theoretic setting, gaining in clarity and simplicity. By now, several different proofs of Steinitz's theorem are known; some follow closely his arguments, while others are largely independent. Most of the proofs are of an inductive nature and proceed along the following scheme:

Step 1. A "reduction" process replaces the given planar and 3-connected graph G by a planar and 3-connected graph G' that is "smaller" (in some sense). Repeated "reductions" ultimately lead to one or several "minimal" graphs.

Step 2. For each minimal graph a polyhedron realizing it is exhibited. Then the inductive step is completed by showing how to construct, from any convex polyhedron P' that realizes a reduced graph G', a convex polyhedron P that realizes G.

Different proofs use different reduction methods in Step 1, and correspondingly different geometric arguments in Step 2. In each approach, both steps involve nontrivial arguments.

For example (following [3]) as "reduction" we can use the "deletion" of an edge. By *deleting* an edge E from a graph G we mean that E is removed and, if either of the endpoints of E is 3-valent, the remaining two edges at such a vertex are amalgamated into a single edge. It is a nontrivial graph-theoretic fact that:

Lemma 2.2. *Every 3-connected planar graph G with more than six edges has an edge E such that if E is deleted from the graph, the resulting smaller graph G' is still 3-connected (and planar).*

For the construction of the convex polyhedron P which realizes G, from any realization of G' by a convex polyhedron P', we need another non-trivial result:

Lemma 2.3. *Given a 3-connected planar graph* G *and any edge* E *of* G, *it is possible to arrange the elements of the set which consists of all vertices and all faces (countries, regions) of* G *into such a list that the two vertices and two faces of* G *incident with* E *are the first four elements, and that each element in the list is incident with at most three elements that precede it in the list.*

The proof of Steinitz's theorem is completed by observing that the edge E can be drawn as a chord on a face of P', and that suitable choices (possible because of Lemma 2.3) for the other vertices and faces can lead to a convex realization P of G in which E is an edge (and not a chord of a face) of P.

Another proof (see [63] or [3]) of Theorem 2.1 uses as "minimal" sets the family of "wheels". An *n-wheel* is the graph consisting of an n-circuit together with an additional vertex joined by edges to all vertices of the circuit. Obviously, the n-wheel can be realized by the graph of the n-sided pyramid. As "reduction" we use the procedure given in the following:

Lemma 2.4. Each 3-connected planar graph G which is not an n-wheel for any n ≥ 3, contains either

(a) an edge E' such that the omission of E' yields a 3-connected planar graph G'; or

(b) an edge E" such that the contraction of E" yields a 3-connected planar graph G".

If Lemma 2.4 is used for the reduction step, then Lemma 2.3 can be used on P' in case (a) directly, and in case (b) as applied to a polar of P' and then replacing the resulting polyhedron by its polar.

Other variants of this approach are possible. Examples can be found in [18, Section 13.1] and [66, Lecture 4]. For a radically different approach, via the Koebe-Andreev-Thurston "circle packing theorem" see [6], [57] or [53, Section 2.8], where also references to the additional literature on the topic can be found. For brief accounts see [66] or [27], and Section 3.1 below.

2.2. Some history.

The first account of Steinitz's theorem after [61] is in the well-known book [45] by Lyusternik. Besides the Russian original (published in 1956) there are two English translations [46], [47]. The first was published in 1963, the second in 1966. The various versions of the book have been quoted frequently for some of the results presented in it. Among these, up to the present time, the theorem of Steinitz is often referred to in the Lyusternik version, which purports to follow the original Steinitz formulation. Unfortunately, as was pointed out in [19, p. 1137] and [3, p.32], the Lyusternik treatment is unsatisfactory and inadequate. In fact, prompted by the recent article [62], a renewed scrutiny of the Lyusternik book revealed that it is much more grossly and unexplainably wrong than stated earlier. In most cases, when Lyusternik's book is given as a reference for Steinitz's theorem (see, for example, the recent books [52, p. 428], [35, p. 334]) there is no warning about the shortcomings of the presentation of Steinitz's theorem by Lyusternik; hence such references are misleading.

Among the most important inadequacies of Lyusternik's treatment are:

The "abstract polyhedra" (which take the place of the complexes considered by Steinitz) are not subject to any connectedness restriction in [45] or [46]; in [47] the graphs are required to be connected — but no mention is made of 3-connectedness. Hence examples like the ones in Figure 1 would qualify, despite their obvious non-representability as graphs of convex polyhedra.

Even if 3-connectedness is assumed, the proof of Theorem 2 of [45, p. 95] or [46, p. 76] (this is Lemma 1 of [47, p. 83]), which is supposed to establish a reduction step, is incorrect: it would lead to graphs that are not 3-connected in general. This can happen even with the example presented by Lyusternik as Figure 102.

Sugihara [62] is another example of misattribution and careless formulation of results and proofs. He attributes the above Theorem 2.1 to Steinitz. But the impossibility of Steinitz having used this formulation is obvious from the fact that in 1916 nobody used terms such as "3-connected graph".

The above Lemma 2.2 (which Sugihara formulates as "every polyhedron whose skeleton is planar and 3-connected is resolvable") is attributed by Sugihara to Lyusternik [45] – but there is no such result in [45] or the translations. As explained above, what is there is the assertion that this statement holds without requiring adequate connectedness.

Sugihara bases his proof of our Lemma 2.2 on his Lemma 1. But this lemma is incorrect, as can be shown by easy examples (if V' is a single vertex, and F' empty, or a single face). His proof fails at his inequality (3), for example if all nodes of H are in V, and no faces. Then $|A'| = 0$, $|W'| = 1$, hence $2|A'| < 4|W'|$.

Concerning Lemma 2.3, it may be of some interest to note that it has a structural similarity to a technique Steinitz used much earlier, in his dissertation [59]. The combinatorial result about "configurations of points and lines" established in [59] is a different formulations of what is generally known as Dénes König's theorem on 1-factors in bipartite graphs [42]. For more details about this aspect see Gropp [17]. Its proof in [3] via the vertex-face incidence bipartite graphs is much simpler that the original one due to Steinitz. The same method has since been applied in similar contexts in [6] and [57]. However, the geometric consequence of this combinatorial result, which Steinitz claims to establish in [59], is not valid in the original formulation. For a discussion of this fact, and a corrected version of the Steinitz claim, see [4].

Lemma 2.4 goes back to Kirkman [41]; a more general formulation is due to Tutte [63].

3. Analogues and some consequences of Steinitz's theorem.

3.1. Symmetric and other special polyhedra.

Modifications of the proofs of Theorem 2.1 have been used to show that if the graph G has certain special automorphisms, then there are convex polyhedra P that realize G and admit isometric symmetries that correspond to these automorphisms of G.

For particular symmetries such results have been established in [18, Section 13.2]. For the full group of automorphisms of G this has been shown by Mani [48]; for a different proof see Schramm [57]. A precise formulation of Schramm's result is given (without proof) in [66, Theorem 4.13].

The inductive construction presented in Section 2 can be used to prove that every 3-connected planar graph can be realized by a convex polyhedron all vertices of which are at rational points in a Cartesian coordinate system of the 3-space [18, Section 13.2], [66, Section 4.4]. A quantitative strengthening of this result is mentioned in [27, p. 296a].

Possibly of greatest interest and applicability is the following strengthening of the Koebe-Andreev-Thurston "circle packing theorem":

Theorem 3.1. *If G is a planar 3-connected graph then G can be realized by a convex polyhedron P with the following properties:*

(i) *All edges of P are tangent to a sphere S, and the centroid of the points of tangency is the center of S;*

(ii) *The graph G* dual to G is realized by a convex polyhedron P* dual to P, with all edges of P* tangent to S.*

(iii) *The edges of P and P* that correspond to each other under the duality of G and G* (and of P and P*) are mutually perpendicular.*

Moreover, if S is given then P and P* are uniquely determined, up to isometry.

As mentioned in [6], [57] and [53], this result follows at once from [6, Theorem 6], quoted as Theorem 2.8.10 on page 62 of [53], by the inverse of a stereographic projection. For an application of Theorem 3.1 see [29].

The proofs of Theorem 3.1 establish only the existence of the polyhedron P, which is sometimes called the *canonical representative* of all convex polyhedra with graph G. An iterative approximation algorithm for the actual construction of P given a convex polyhedron with graph G was given by Hart [34]. It works in many situations (see, for example, Figure 2) but seems not to lead to the desired polyhedron in all cases.

3.2. Applications of graph-theoretic methods.

Subsection 3.1 can be interpreted as mainly dealing with realizations of certain graphs by polyhedra with interesting properties. However, Steinitz's theorem can also be fruitfully applied to obtain results on convex polyhedra using graph-theoretic techniques. The reason is simple: steps in various procedures are much easier to justify or legitimize on graphs than on the more highly structured convex polyhedra. Here are a few examples.

If P is a simple (that is, 3-valent) convex polyhedron, and if $f_k(P)$ is the number of k-gonal faces of P, then an easy consequence of Euler's theorem is the relation

$$\sum_{k \geq 3} (6 - k)\, f_k(P) = 12. \tag{*}$$

This relation does not constrain the value of $f_6(P)$, and the natural question what can be said about it was first considered by the blind geometer Victor Eberhard in 1891. In his book [12] he proves:

Theorem 3.2. For every sequence $f_k,\ 3 \leq k \neq 6,$ of nonnegative integers satisfying

$$\sum_{3 \leq k \neq 3} (6 - k)\, f_k = 12\ , \tag{**}$$

there are infinitely many values f_6 such that there is a convex polyhedron P with $f_k = f_k(P)$ for all $k \geq 3$.

Eberhard's proof is long and messy, since he has to accommodate the operations he performs on the convex polyhedra. A simpler proof, utilizing graphs and Steinitz's theorem, appears in [18, Section 13.3]. This was strengthened by Fisher [15], who among other results proved that in Theorem 3.2 the smallest f_6 satisfies

$$f_6 \leq 3 \sum_{3 \leq k \neq 6} f_k\ .$$

Generalizations of these results to convex polyhedra that need not be simple have been given by various authors. The paper by Jendrol [38] is a nice introduction to this topic and its literature. Another result of Fisher [16] deals with the 5-valent case, and shows that every sequence f_k that satisfies the necessary condition derived from Euler's theorem and has $f_4 \geq 6$, then there is a convex polyhedron P with $f_k = f_k(P)$ for all $k \geq 3$.

In another direction, Barnette [2] proved by graph-theoretic reasoning that every 3-polytope contains in its 1-skeleton a spanning tree of maximal valence 3. However, the following related conjecture from [19, p. 1147] is still open:

Conjecture 3.1. Each convex polyhedron P admits a spanning tree T of maximal valence 3 such that the edges of the dual polyhedron P* that correspond to edges of P not in T also form a tree of maximal valence 3.

The number of examples could be greatly expanded, but constraints of time and length dictate we end these comments and move on to the next topic — polyhedra more general than the convex ones.

4. What are nonconvex polyhedra ?

The fruitful interaction between the theories of graphs and convex polyhedra cannot be extended to "general polyhedra" unless we determine the scope of that concept. In the literature one can find many different approaches. First, there is the dichotomy between polyhedra as solids, and as surfaces. For convex polyhedra it is, in essence, immaterial which of the interpretations we chose; the solid and the surface determine each other unambiguously and in a straightforward manner. Hence for them the approach selected depends on what is more convenient for a particular topic.

The situation is different when considering nonconvex polyhedra. The difficulties in the interpretation of polyhedra as solids is well illustrated by the treatment in Hajos [33]. While giving precise and detailed definitions which he illustrates by numerous

examples, the consideration is very quickly restricted to polyhedra homeomorphic to the solid ball, or even to convex polyhedra.

Figure 3(a) shows the two nonconvex polyhedra discovered by Kepler [39]. If they are interpreted as solids, their boundary is a topological sphere bounded by 60 triangles in each case. Under this interpretation, neither of the two polyhedra is regular. However, they are regular if they are interpreted the way Kepler intended, as being constituted by 12 pentagrams ("star pentagons") each. One pentagram on each polyhedron is indicated by heavy lines in Figure 3(b). But clearly pentagrams are unusual polygons, and it is unclear to what general class of polygons they belong, and also how are polyhedra "constituted" by such "polygons". The purpose of our exposition is to provide a general framework for the interpretation of *polyhedra* as being *constituted* by *polygons*. After a discussion of the historical background we shall present precise and explicit definitions of the concepts italicized in the preceding sentence.

Just as Euclid did not define "convex polyhedra" when enumerating the regular ones, so most mathematicians of the 19th and 20th centuries gave no meaningful definition of nonconvex polyhedra in their works on regular polyhedra or other polyhedra with high symmetry. Consider, for example, the following definition of regular polyhedra as given by Cauchy [8, p. 9]:

> Un polyèdre régulier d'une espèce quelconque est celui qui est formé par des polygons réguliers, également inclinés l'un sur l'autre, et assemblés en même nombre autour de chaque sommet. [A regular polyhedron of an arbitrary kind is that one which is formed by regular polygons, equally inclined to each other, and assembled in the same number about every vertex.]

No definition of the class of "polyhedra" here, or in Poinsot [56]. It is interesting that Poinsot mentions the above example of the stellated dodecahedra and states:

> Comme un même polyèdre peut paraître également construit sous tels ou tels polygones, je prendrai pour les *faces*, les plans qui, en plus petit nombre, achèvent complètement ce même polyèdre. [Since the same polyhedron may

equally appear as constructed by these or other polygons, I will take as *faces* the planes which in the smallest number bound the same polyhedron.]

Clearly, this does not define polyhedra in any meaningful way. Similarly uninformative is the definition of polyhedra given by Wiener [64]; this is an otherwise very important work, in which many highly useful insights are first formulated. More damaging for later investigations is the definition given by Brückner [7, p. 46]:

Ein (einfaches) Vielflach, n-flach oder Polyeder ist die Gesamtheit von n ebenen Vielecken, von denen jedes jede seiner Kanten mit einer Kante eines andern Vielecks gemein hat. [A (simple) multiface, n-face or polyhedron is the totality of n plane polygons, each of which shares each of its sides with a side of one other polygon.]

In his book [7], Brückner gives a perfect example of presenting a definition which is later completely ignored. He makes no requirement of connectivity of any kind, or of non-coplanarity of adjacent faces, or of the limitation of faces incident with a vertex to form a single circuit. However, all these and other unstated restrictions are assumed at least in parts of the book.

Another approach which is useful in special cases but does not contribute to define polyhedra in general, is particularly well exemplified by the work of Coxeter *et al.* [9]. They define:

A *polyhedron* is a finite set of polygons such that every side of each belongs to just one other, with the restriction that no subset has the same property.

In itself, this is a rather inadequate definition, but Coxeter *et al.* avoid many of the problems by restricting attention at once to *uniform polyhedra*, that is, polyhedra that are isogonal and have regular polygons as faces.

5. Desiderata for a theory of polyhedra.

The examples in Section 4 illustrate the difficulties inherent in attempts to find an appropriate definition of polyhedra. On the one hand, the concept defined should be general enough to include the many different kinds of objects that have traditionally been considered as polyhedra; this includes the nonconvex polyhedra homeomorphic to the 2–sphere, polyhedra of higher genus (such as picture-frames), selfintersecting polyhedra such as the Kepler-Poinsot regular polyhedra, and many other kinds. But on the other hand, the definition should not be too permissive, in order to avoid objects which generally would not be considered as being a polyhedron, such as two disjoint tetrahedra or the set of eight triangles and three squares determined by the vertices and edges of a regular octahedron. On the other hand, in each of these and other cases one could make an argument for the investigations of such "polyhedral objects" — but not for their inclusion among polyhedra. For the last-mentioned topic, the beginnings of such a study can be found in [28].

Probably most geometers agree that a polyhedron should be in some way constituted by finitely many polygons (but topologists generally do not accept this limitation). Most would also wish to be able to associate with each polyhedron a dual polyhedron. A more controversial desideratum is that the "type" of polyhedron be preserved under continuous deformations. The problem with this last property arises from the prevailing viewpoint of convex polyhedra. As is well known, sequences of convex polyhedra of the same combinatorial type may converge (in the Hausdorff-Blaschke metric) to a polyhedron of a different type. The resulting lower semicontinuity is discussed in detail in [13]. In any case, the three desiderata for the new theory of polyhedra seem to be:

- Great generality, restrained not by tradition but by convenience and usefulness.
- Continuity of type.
- Existence of duals.

In any serious attempt to develop a theory of polyhedra one is inevitably led to the necessity of an appropriate treatment of polygons. In the past I have tried several times

to find the right level of generality in defining polygons and polyhedra, but at present I consider these attempts to have been unsuccessful. Only in the most recent papers ([25], [26]) did I develop an approach that I believe will withstand the test of time and usefulness.   I shall present this approach in the next two sections. But the reader should be forewarned that in order to achieve the desiderata mentioned above, nontraditional "polygons" and "polyhedra" need to be admitted. On the other hand, once the initial discomfort wears off, it will be seen that the present point of view provides a very satisfactory solution to various situations and questions.

6. Polygons.

Let an integer $n \geq 3$ be given.

Definition 6.1. A combinatorial (or abstract) n-gon is a (simple) circuit C of n (distinct) vertices and n edges. A geometric n-gon (or polygon for short) is an image of an abstract n-gon in a plane, such that vertices of C are mapped onto points and edges of C onto segments with appropriate points as endpoints.

This definition may seem at the same time natural and needlessly convoluted. But except for the jargon, it is, in fact, the definition that appears for the first time in Meister [51], and in essentially every other work that attempts to deal with polygons in general. In particular, it coincides with the one given by Poinsot [56], whose lead was followed by almost all later writers.

But there is a rub. Poinsot thought that this (or his) formulation means that *distinct vertices* of the graph C are mapped onto *distinct points*; in other words, that a polygon cannot have distinct vertices represented by the same point. Obviously, if this point of view (which is not justified by the definition itself) is adopted, then there is no continuity among n-gons: a converging sequence of quadrangles may well converge to a triangle, or a segment, or even a single point ("trivial polygon"). Worse, the limit may

well not be a polygon at all — see Figure 4. But the opposite approach, which allows various vertices to be represented by the same point, was advocated by Meister [51] long before Poinsot's paper, and turns out to be one of the crucial steps in developing a satisfactory theory both of polygons and of polyhedra.

Let us examine more closely what is implied by Definition 6.1. Two or more vertices of a polyhedron may be represented by the same point — but this does not affect the *incidences* of vertices and edges; these are inherited from the underlying circuit C. If adjacent vertices are represented by the same point, the edge they determine is a segment of zero length. The segments representing the edges of a polygon may intersect in various ways, contain a point which represents a vertex not incident with the edge, overlap, or even coincide. Simply put, it is not the subset of points of the plane that determines or describes the polygon, but the combinatorial structure imposed on this subset by the graph C. Some of the possibilities are illustrated in Figure 5.

Symmetries are an important part of the theory of polygons and polyhedra. But with the new interpretation of "polygon" the traditional understanding of symmetry as an isometric mapping of the set onto itself is inadequate. Now a symmetry of a polygon is an i*ncidence-preserving automorphism* of the underlying graph *paired* with a *compatible isometry* of the plane. For example, although the 6-circuit underlying the *hexagon* in Figure 6 has 12 incidence-preserving automorphisms, the geometric hexagon in Figure 6 admits only four symmetries:

permutation (1,6)(2,5)(3,4) paired with reflection in the mirror L;
permutation (1,3)(2)(4,6)(5) paired with reflection in the mirror M;
permutation (1,4)(2,5)(3,6) paired with the halfturn about the center of the figure;
the identity symmetry.

In particular, this hexagon does not admit a 90° rotation as part of a symmetry.

Using the concept of symmetry, it is possible to define isogonal, isotoxal and regular polygons as those having a symmetry group acting transitively on the vertices, edges, or flags. (A *flag* is a pair consisting of a vertex incident with an edge.) For example, in Figure 7 are shown all nontrivial regular polygons with at most ten vertices.

(Here, and throughout the paper, we consider individual polygons and polyhedra as representing the class of all those related to them by a similarity.) In a generally accepted notation a regular n-gon whose edges subtend at the center an angle $2\pi d/n$ is denoted by {n/d}. Poinsot's interpretation (or misinterpretation) of his definition of regular polygons led him to assert that a regular polygon {n/d} exists if and only if n and d are relatively prime. This attitude has been generally accepted in the literature, and its rejection is part of our rebuilding of the theory of polygons and polyhedra — but it is not really new: Meister [51] in 1769 (!) adopted the same attitude as we do, and illustrated it by presenting diagrams of *all* the regular polygons {20/d} for $1 \le d \le 10$. Meister's diagram is reproduced in Figure 8. It is an inexcusable perversion of the truth to claim (as do Günther [32, p. 46] and Brückner [7, p. 13]) that Meister presents the same viewpoint as Poinsot.

The utility of our approach can be seen by considering the shapes of isogonal n-gons for even n. (For odd n any isogonal n-gon is regular.) A typical case is shown in Figure 9, which illustrates the general fact that all isogonal polygons fit into continuous families that start and end with regular polygons; similarly for isotoxal polygons. Hence all isogonal n-gons belong to one (and only one) of $\lceil n/4 \rceil + 1$ families. For more details see [21] and [22]. The resulting simple classifications should be compared with the multiplicity of cases that need to be distinguished in the "traditional" approach of Hess [36] and Brückner [7, Sections 21 – 30].

7. Polyhedra.

As for polygons, for polyhedra it is convenient to distinguish the *combinatorial* structure of a polyhedron from its *geometric* realization. Hence we shall start with combinatorial or abstract polyhedra, and obtain geometric polyhedra as images of abstract ones in a Euclidean space.

Definition 7.1. An *abstract polyhedron* is a finite graph, with a special collection of abstract polygons (also called *faces*) formed by its vertices and edges. The vertices, edges and polygons are required to satisfy the following conditions:

(1)   Every edge is incident with precisely two distinct vertices and two distinct faces.

(2)   If a vertex and a face are incident there are precisely two distinct edges incident with both.

(3)   For each face [vertex] the vertices [faces] and edges incident with it form a simple circuit of length ≥ 3.

(4)   If two edges are incident with the same two vertices [faces], then the four faces [vertices] incident with the two edges are distinct.

(5)   Each pair of faces [vertices] is connected through a finite chain of incident edges and faces [vertices].

Definition 7.2. A *geometric polyhedron* (*polyhedron* for short) is the image of an abstract polyhedron under a mapping in which vertices are mapped to points in the Euclidean 3-space, edges are mapped to segments with appropriate endpoints, and faces are mapped to (geometric) polygons. The geometric polyhedron is said to be a *realization* of the *underlying* abstract polyhedron.

With the possible exception of requirement (4) of Definition 7.1, these definitions sound quite natural. Even so, a few comments and explanations may be useful. First, some parts of Definition 7.1 are redundant. They have been included in order to make evident the existence of dual polyhedra: the requirements are unchanged if "vertices" are replaced by "faces" and "faces" by "vertices".

Next, since each abstract polygon can be interpreted as the boundary of a 2-cell, abstract polyhedra can clearly be understood as a particular family of cell-complex decompositions of closed 2-manifolds. This *associated manifold* of an abstract polyhedron is often useful in understanding the structure of the polyhedron, and can be used to decide whether the abstract polyhedron is orientable or not. However, cell complexes with digons, or with cells that fail to satisfy some other conditions of Definition 7.1, are not associated with polyhedra.

Another way of characterizing abstract polyhedra is the one followed by McMullen and Schulte in their recent book [50] and other publications. Here abstract polyhedra are considered as lattices satisfying appropriate conditions. McMullen and Schulte present this approach for polytopes of all dimensions, and their formulation is not limited by finiteness restrictions. For our purposes of detailed geometric investigations, the graph-theoretic approach appears more convenient.

The concept of polyhedra as defined above clearly includes most of the polyhedra that have been considered in the literature. It fact, it can be adapted with essentially no change to polyhedra with non-planar polygons (as considered in [20], [10], [11], [49], [14], [43], [44]) and to polyhedra in spaces other than the Euclidean 3-space; but these extensions are beyond the scope of the present paper.

As with polygons, geometric polyhedra come in many forms, among them some that are quite non-traditional. Many different kinds of coincidences, selfintersections and overlaps are possible, as are *subdimensional* representations, including *trivial* polyhedra. But just as group theory cannot be imagined without the trivial group, excluding trivial (or subdimensional) polyhedra would not only be arbitrary but actually detrimental to any general theory of polyhedra. On the other hand, if in a particular investigation such polyhedra are not desired, it is easy to restrict the topic to fulldimensional polyhedra.

Symmetries of geometric polyhedra have to be treated in analogy to the polygonal case. Each symmetry of a polyhedron is a pair consisting of an automorphism of the underlying abstract polyhedron and a compatible isometry of the point set representing the polyhedron. This makes possible the consideration of different classes of polyhedra with high symmetry. Specifically, it is convenient to define a polyhedron to be isogonal, isotoxal, isohedral, noble, fully transitive, or regular, provided its symmetry group acts transitively of its vertices, edges, faces, vertices and faces, vertices, edges and faces, or flags, respectively. (Here a *flag* is a triplet consisting of mutually incident face, edge and vertex.) Similar terminology applies to the abstract polyhedra in terms of their groups of automorphisms.

Many examples of more or less interesting polyhedra have been presented in various recent publications of the author (such as [21], [25], [26], [23]). In order to save space, these examples are not reproduced here. Instead, we show several examples which illustrate the unusual possibilities of the new setup.

Figure 10 shows a nonorientable isogonal abstract polyhedron and its dual, with subdimensional realizations of both.

Figure 11 shows two fully transitive (but not regular) realizations by "V-shaped quadrangles" of a toroidal map. The polyhedra belong to infinite families described in [21].

An example of a "new" regular polyhedron is shown in Figure 12. It is a realization of the regular map labeled 36.13 in Wilson's catalog [65] of regular maps. The polyhedron is obtained by a technique described in [26].

Figure 13 shows representatives of a continuous family of realizations of an isogonal abstract polyhedron with symbol (5.10.10), which has 12 pentagonal and 12 decagonal faces. All these realizations are isogonal, and the one marked by two asterisks is a "new" uniform polyhedron. It is "new" because it is not included in the enumeration in [9]; this omission is justified under the assumptions on which [9] is based, but the exclusion of this polyhedron is presented in [9] as something exceptional. A similar situation exists regarding realizations of the abstract polyhedron (3.10.10), and truncations of the great stellated dodecahedron {5/2, 3}.

As a final example we consider in Figure 14 representatives of the continuous family of isohedral realizations of the abstract polyhedron [3.6.6]. Each of the 12 faces is a triangle with one 3-valent and two 6-valent vertices.

8. Concluding remarks.

It is my hope that the above presentation may contribute to a clarification of the existing confusion regarding polyhedra more general than the convex polyhedra, and to cross-fertilization of the geometric and combinatorial aspects of these objects. It is obvious that there are many fruitful avenues of research into related topics. To mention just a few obvious ones:

ç   Which abstract polyhedra can be realized by geometric polyhedra that are homeomorphic to a sphere?

ç   The result of Archdeacon *et al.* [1] announced at the Bled conference shows (among others) that every combinatorial triangulation of the torus can be homeomorphically imbedded as a geometric polyhedron in Euclidean 3-space. This solves a longstanding problem. Still unsolved is the question which more general cell-complex decompositions of the torus can be represented analogously. It is known that realizations with convex polygons face various obstructions; see [24] for a more detailed discussion of this and related questions. The situation is even more challenging in view of the result of Bokowski and Guedes de Oliveira [5] that for every $g \geq 6$ there exist triangulations of the orientable manifold of genus g that do not admit a homeomorphic realization by geometric polyhedra.

ç   There seems to be no known characterization of those abstract polyhedra that admit full-dimensional geometric realizations.

ç   Can the Hart algorithm (from [34]) be improved so that it works for all convex polyhedra? Does Theorem 3.1 generalize to a suitable class of polyhedra more general than convex ones? For example, geometric polyhedra that can be considered as multiple covers of the 2-sphere — such as the Kepler-Poinsot polyhedra.

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Branko Grünbaum
Department of Mathematics
University of Washington
Seattle, WA 98195-4350, USA

E-mail: grunbaum@math.washington.edu

Caption for Figures

Figure 1. These graphs (or complexes) are admitted as graphs of convex polyhedra by the definitions in all versions of Lyusternik's presentation of Steinitz's theorem.

Figure 2. A convex polyhedron, and its canonical representative obtained by the Hart algorithm [34].

Small stellated dodecahedron {5/2,5}	Great stellated dodecahedron {5/2,3}

Figure 3. Kepler's two "starshaped" regular polyhedra, presented in "cardboard model" views. In (b) one of the pentagrammatic faces of each polyhedron is emphasized.

Figure 4. Obvious continuous deformations of the hexagon in (a) and (e) lead to intermediate objects (c) and (g) that could not be considered polygons if the approach of Poinsot is adopted.

Figure 5. Examples of different types of quadrangles, with vertices labeled 1, 2, 3, 4 for clarity. The quadrangles in the first two rows are *full-dimensional*, the ones in the last two rows are *subdimensional*. The last one is the *trivial* quadrangle. It should be noted that the traditional approach admits only three types of quadrangles (convex, simple but nonconvex, and self-crossing). In our interpretation there are eighteen different types! The number increases even more if orientation is taken into account.

Figure 6. Illustrations of the symmetries of the geometric hexagon with vertices 1, 2, 3, 4, 5, 6.

Figure 7. Representatives of all nontrivial types of regular polyhedra with at most 10 vertices.

Figure 8. A representation of the different regular 20-gons, presented by Meister to illustrate the existence of regular polygons {n/d} in cases n and d are not relatively prime.

Figure 9. A representation of a continuous family of isogonal 14-gons, that starts with the regular {14/3} and ends with {14/4}. The *rotation number* and the *winding number*

w.r.t. the center (the "density" of the polygon) are equal before the polygon marked by the first asterisk ✶ and after the one marked by the second asterisk, but differ for polygons between these two. The rotation number is not defined for the polygon marked by the first asterisk, and the winding number is undefined for the polygon marked by the second asterisk.

Figure 10. (a) An isogonal subdimensional realization of a nonorientable abstract polyhedron (shown in (b) by its map) with the least possible number of faces. The three faces of the realization are shown separately in (c). Part (d) shows a map of the dual abstract polyhedron. It also indicates how the polyhedron can be realized: by folding the three outer triangles over the middle one, and identifying the *appropriate* edges. Thus the polyhedron is *isomeghetic* with (that is, *looks* like) a triangle. However, each edge of the triangle is replaced by a pair of edges between the same vertices but incident with four different faces.

Figure 11. (a) Two fully transitive realizations of the abstract polyhedron presented as a map (cell complex) in (b). The structure of these polyhedra is most easily visible from that map. In both realizations the faces are the full-dimensional "V-quadrangles", exemplified by the last quadrangle in the second row of Figure 5. Both are selfpolar realizations of the underlying selfdual abstract polyhedron.

Figure 12. (a) A regular polyhedron with Schläfli symbol {12,3}, which realizes the abstract polyhedron given by the map in (b). The map is listed as #36.13 in Wilson's catalog [65]. The polyhedron results from "vertex tripling" of the cube; this operation and related ones are described in [26].

Figure 13. A representation of a part of the continuous family of realizations of an abstract isogonal icositetrahedron (5.10.10), with 12 pentagonal and 12 decagonal faces. Each of the 60 vertices is incident with one pentagon and two decagons. The

polyhedron is orientable, of genus 4. The realization marked by two asterisks is a "new" uniform polyhedron with symbol (5.10/2.10/2). The realizations marked by asterisks are *isomeghetic* (represented by the same point set) as well known polyhedra: one asterisk and two asterisks indicate polyhedra isomeghetic with regular polyhedra (small stellated dodecahedron {5/2,5} and dodecahedron {5,3}, respectively). The polyhedron marked by three asterisks is isomeghetic with the uniform dodecadodecahedron (5.5/2.5.5/2). The polyhedra between those with one and two asterisks are isomeghetic with truncations of the small stellated dodecahedron.

Figure 14. A representation of the continuous family of realizations of an abstract isohedral triakis tetrahedron [3.6.6], with four trivalent and four sixvalent vertices. Each polyhedron is shown in a "cardboard model" view, as well as in a skeletal view. All polyhedra are presented as inscribed in the same sphere. The first and last are approximations to the limit polyhedron, which is isomeghetic to four equiinclined segments. Polyhedra #4 and #11 have equilateral triangles as faces. (The latter was dismissed from the enumeration of isohedral deltahedra by Shephard [58] with the statement "... the construction leads to a set of twelve equilateral triangles which coincide in four sets of three"; this dismissal completely disregards the combinatorial structure of the polyhedron.) Polyhedron #5 is 2-isomeghetic with the cube, and #6 is the Catalan polyhedron [3.6.6] (polar of the uniform polyhedron (3.6.6)). The faces of #8 coincide in pairs; this polyhedron marks the boundary between the acoptic representatives and those with selfintersections in the "cardboard model". The grey segments mark these selfintersections.

Figure 1.EPS
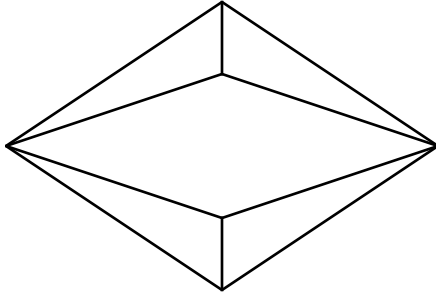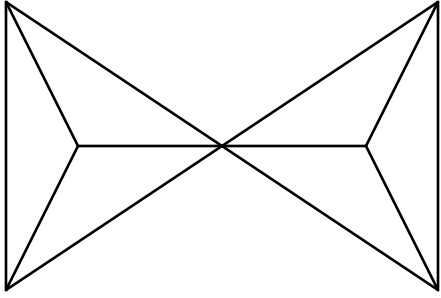
Figure 2.EPS

Figure 3.EPS
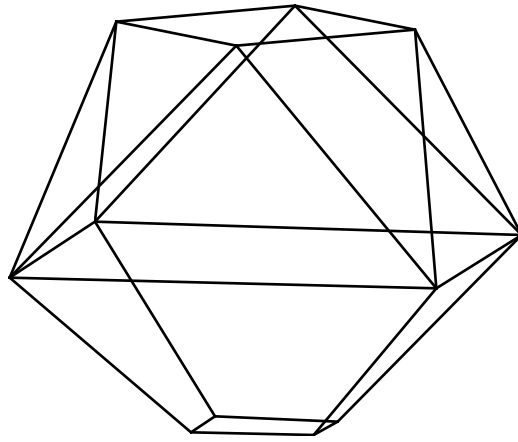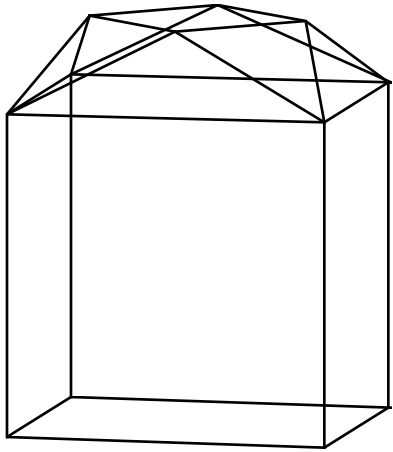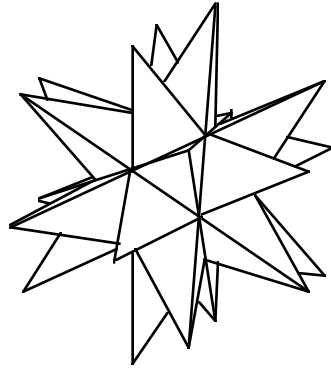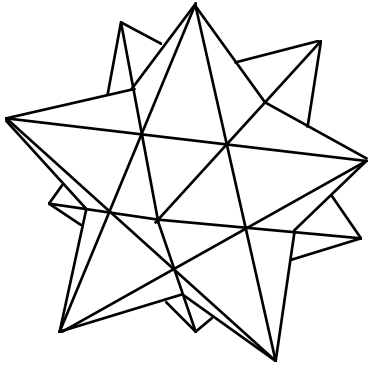
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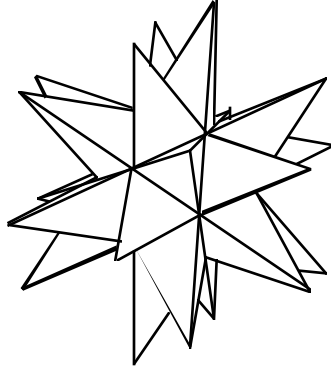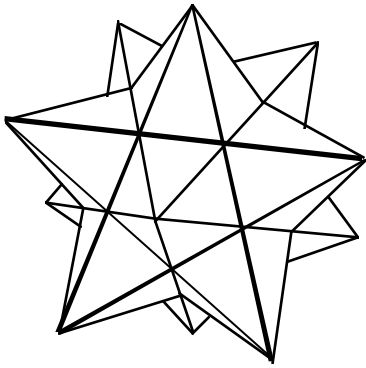
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Figure 4.EPS
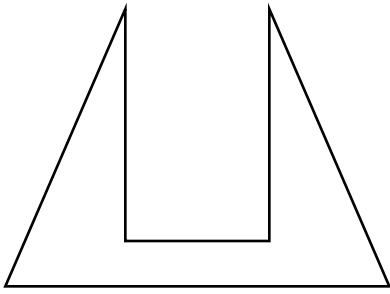
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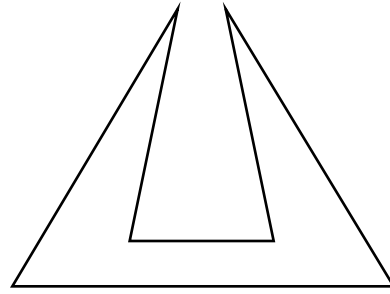
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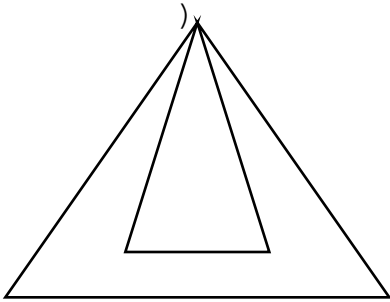
(c)
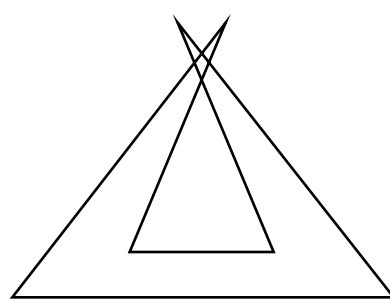
(d)
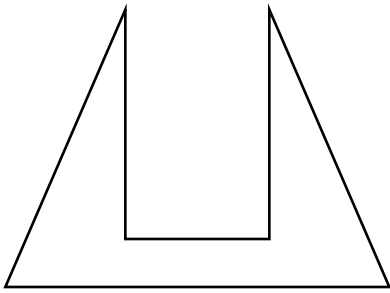
(e)
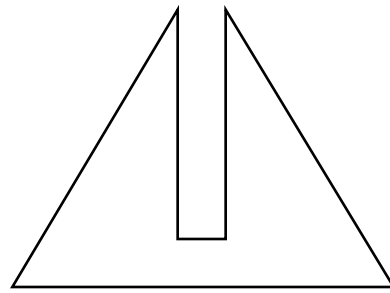
(f)
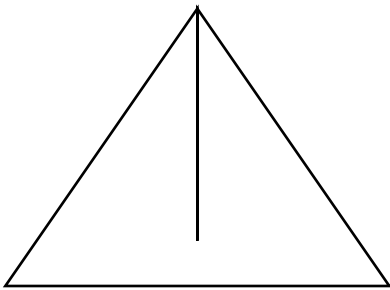
(g)
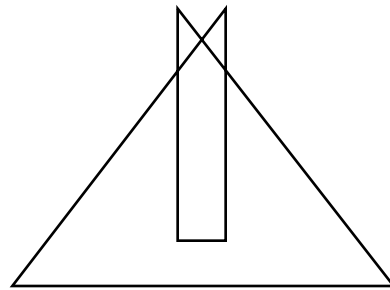
(h)

Figure 5.EPS

Figure 6.EPS

Figure 7.EPS

{3} {4} {4/2} {5} {5/2} {6} {6/2} {6,3}

{7} {7/2} {7/3} {8} {8/2} {8/3} {8/4} {9}

{9/2} {9/3} {9/4} {10} {10/2} {10/3} {10/4} {10/5}
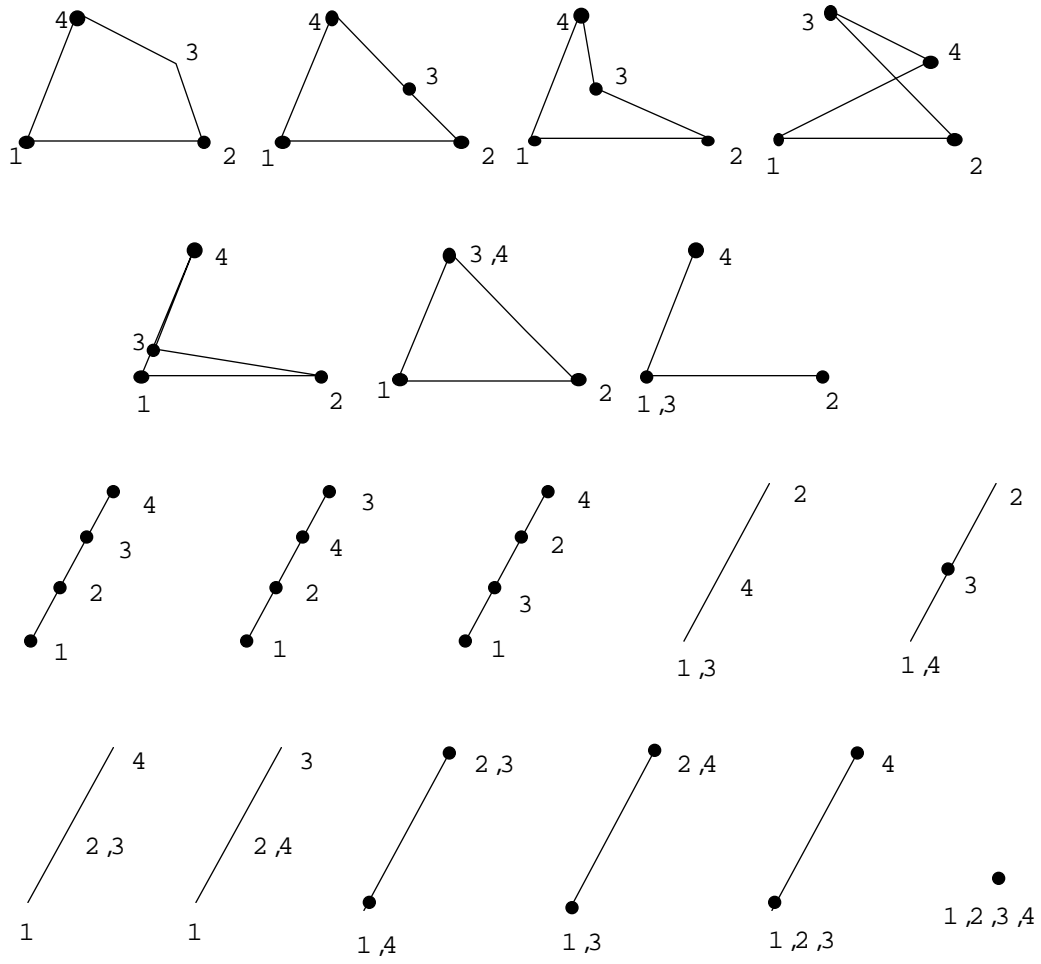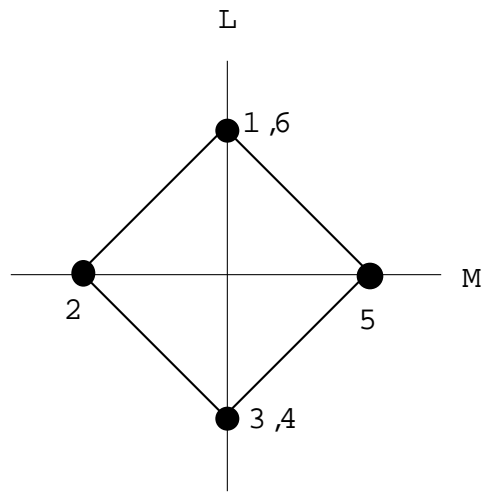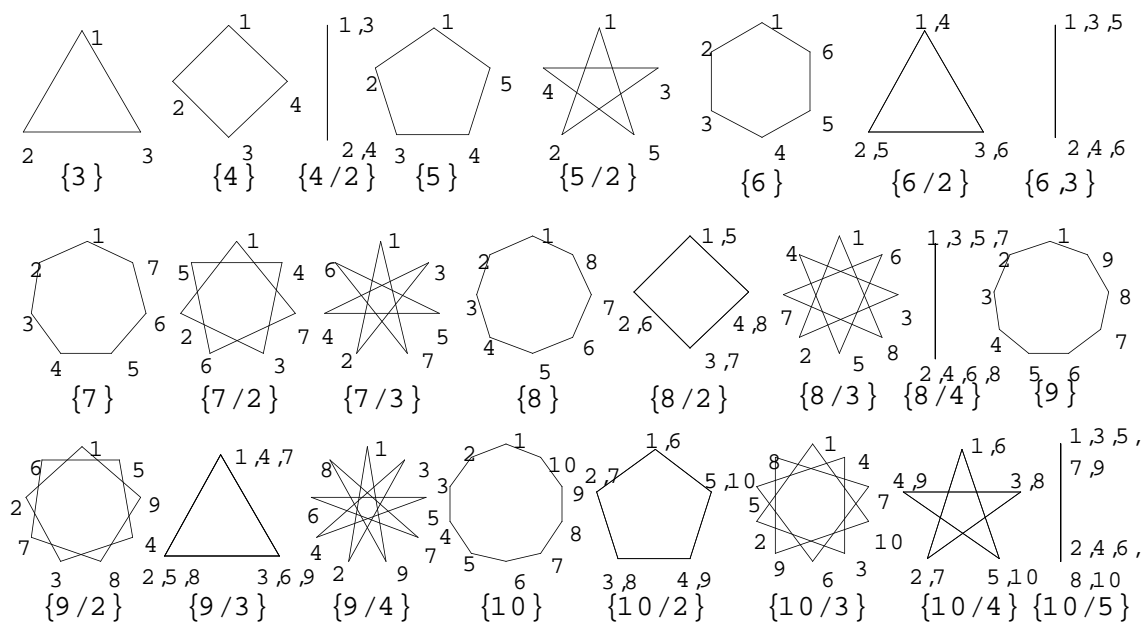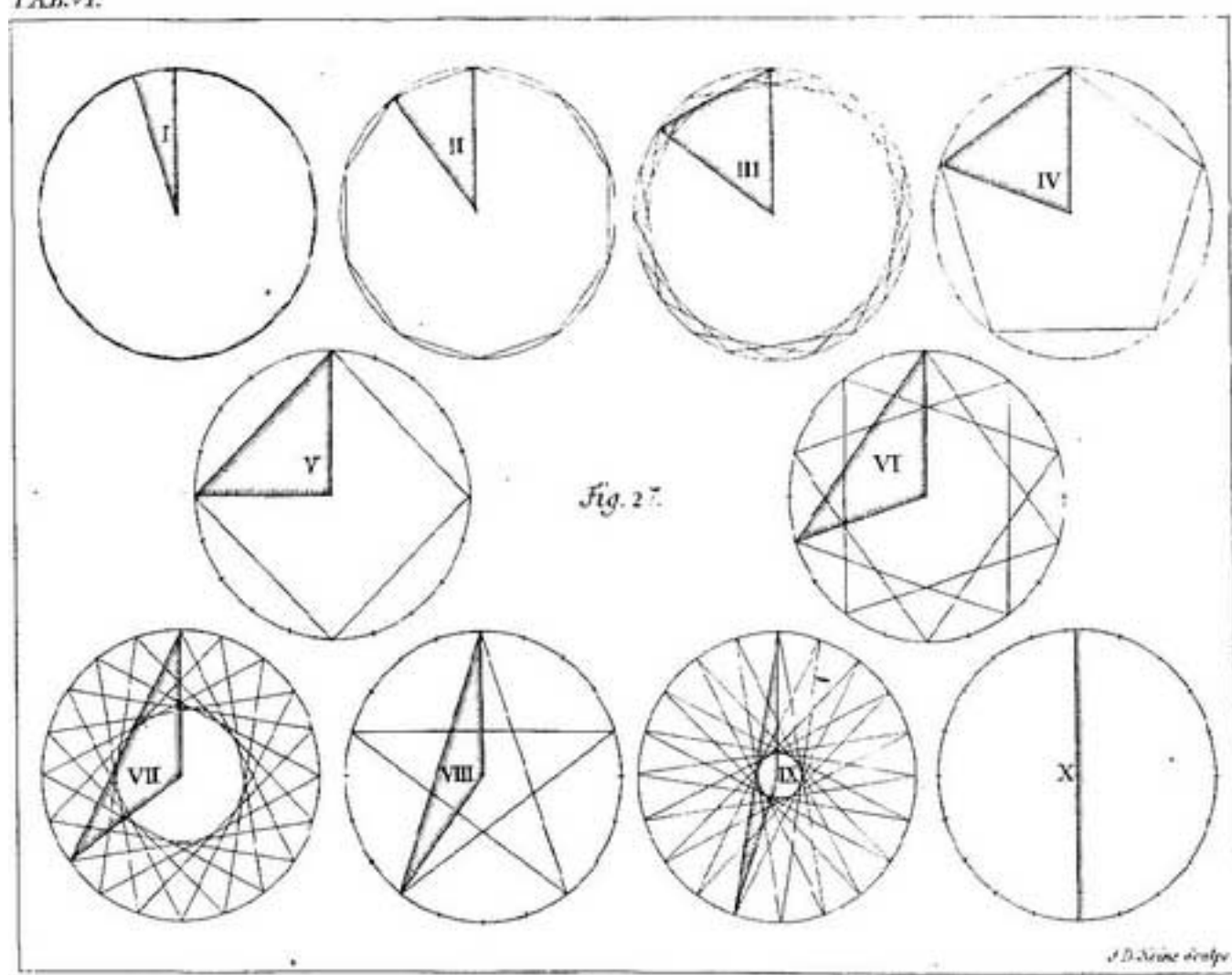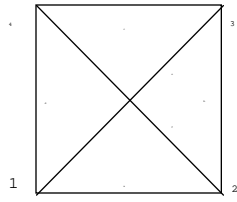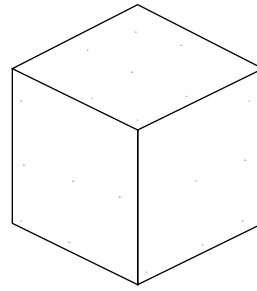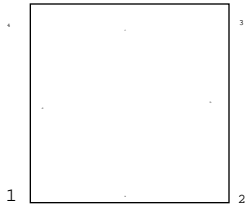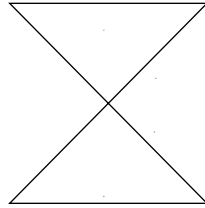
TAB.VI.

Fig. 27.

Figure 9.EPS
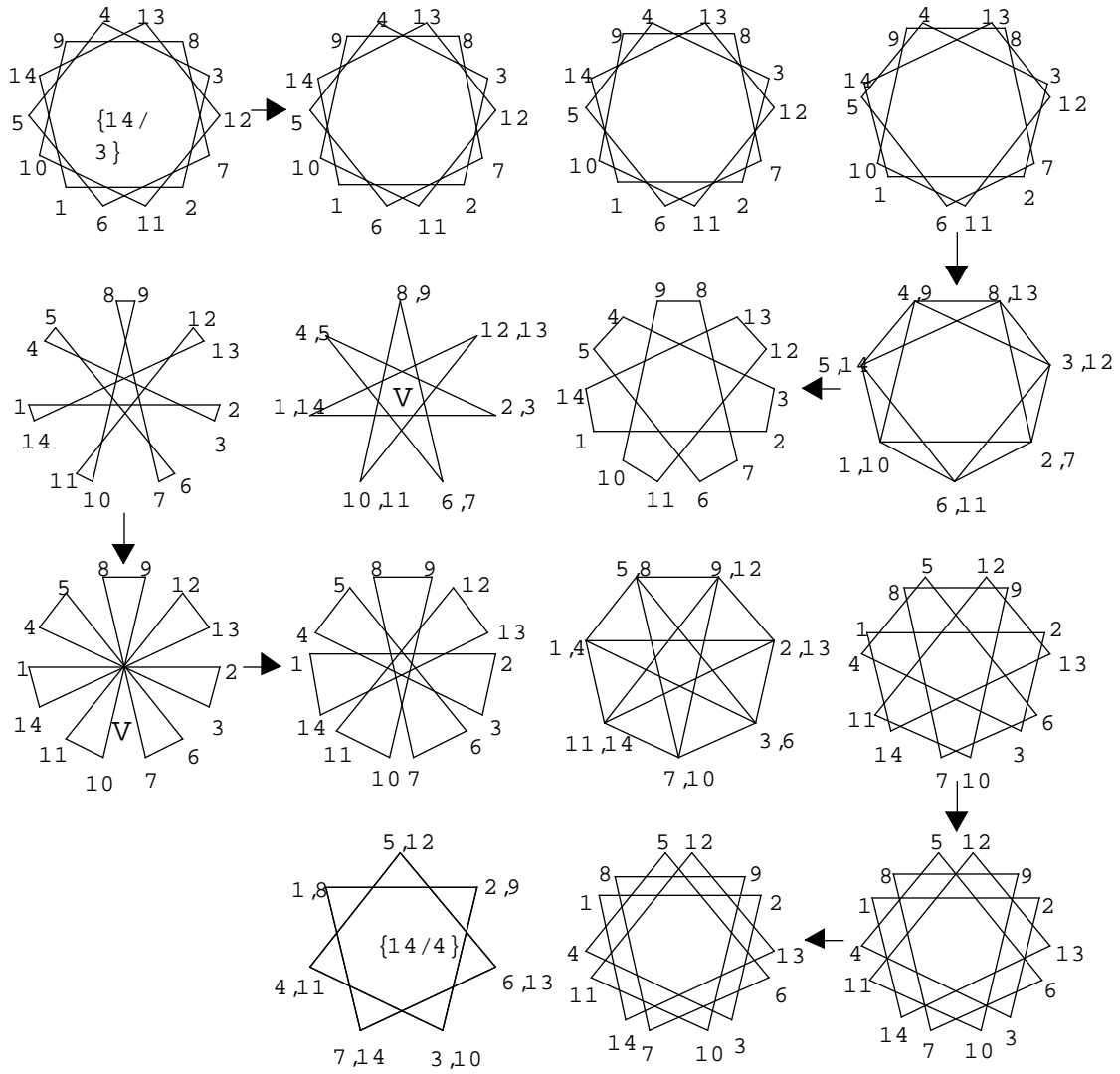
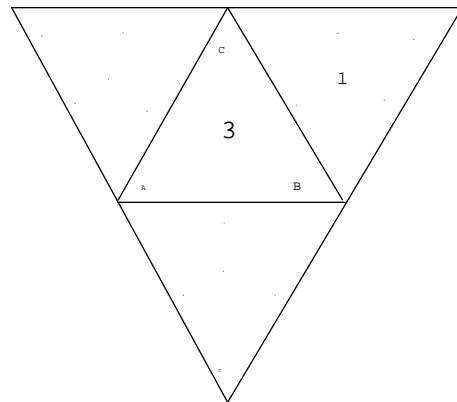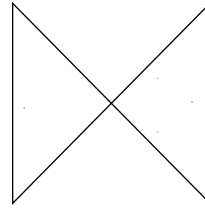**Figure 10.EPS**
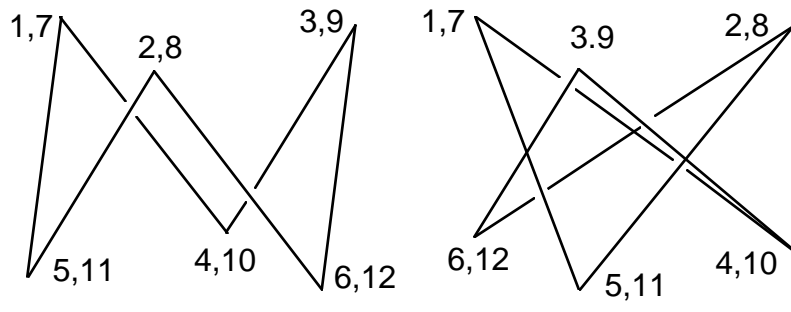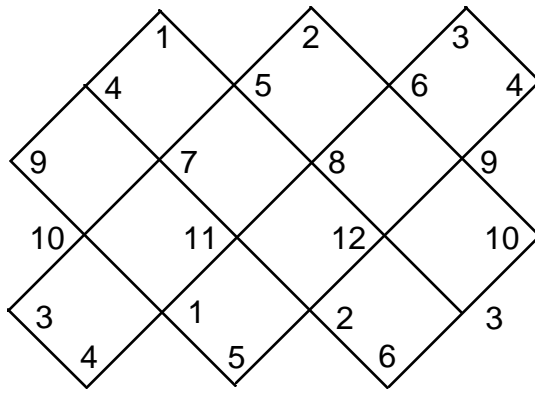
(a)

A =
[1a2b3c4d1]

B =
[1a2f4c3e1]

C =
[1e3b2f4d1]

(b)

A

(c)

(d)

Figure 11.EPS

(a)

(b)

Figure 12.EPS

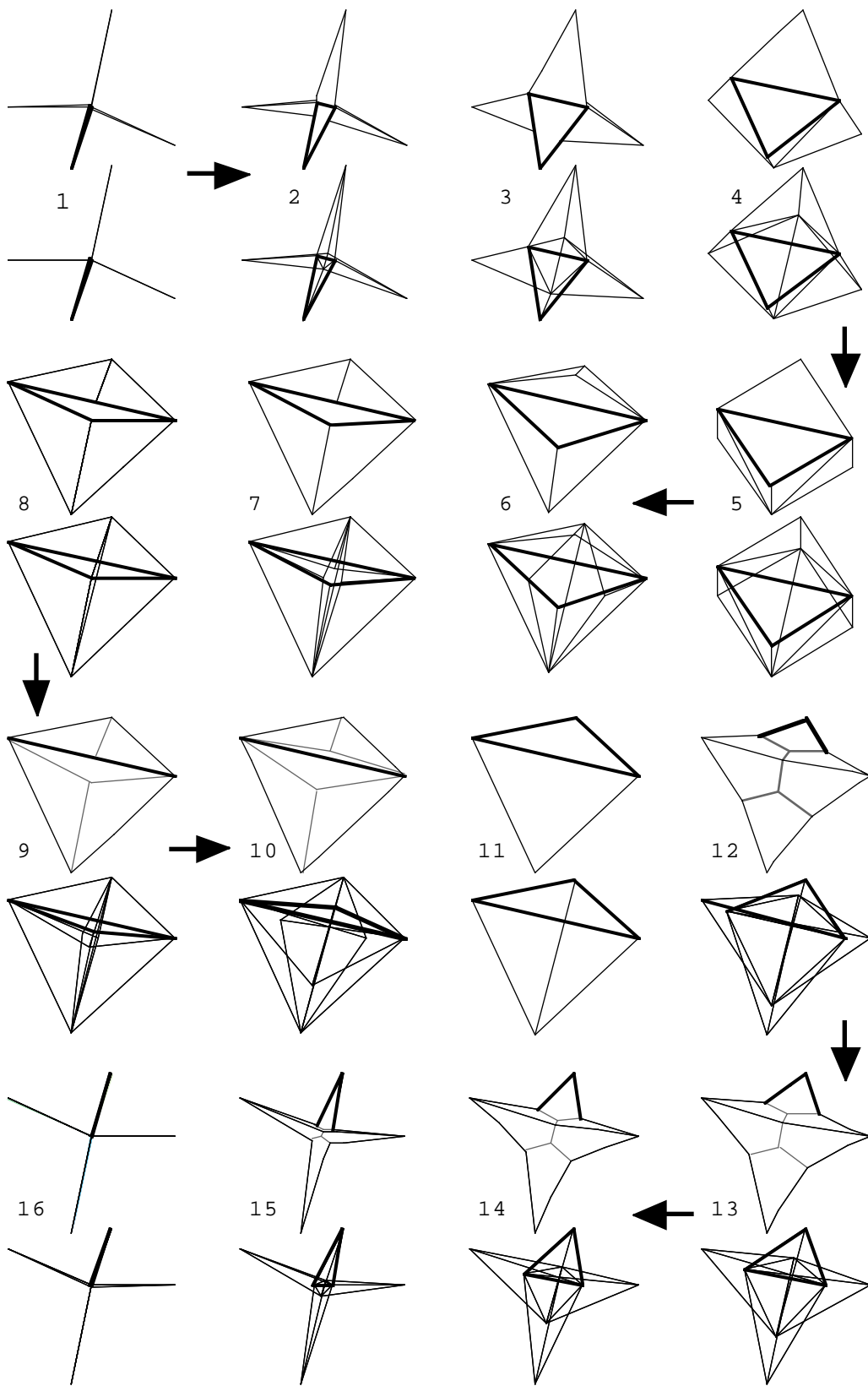(a)   (b)

Figure 13.EPS
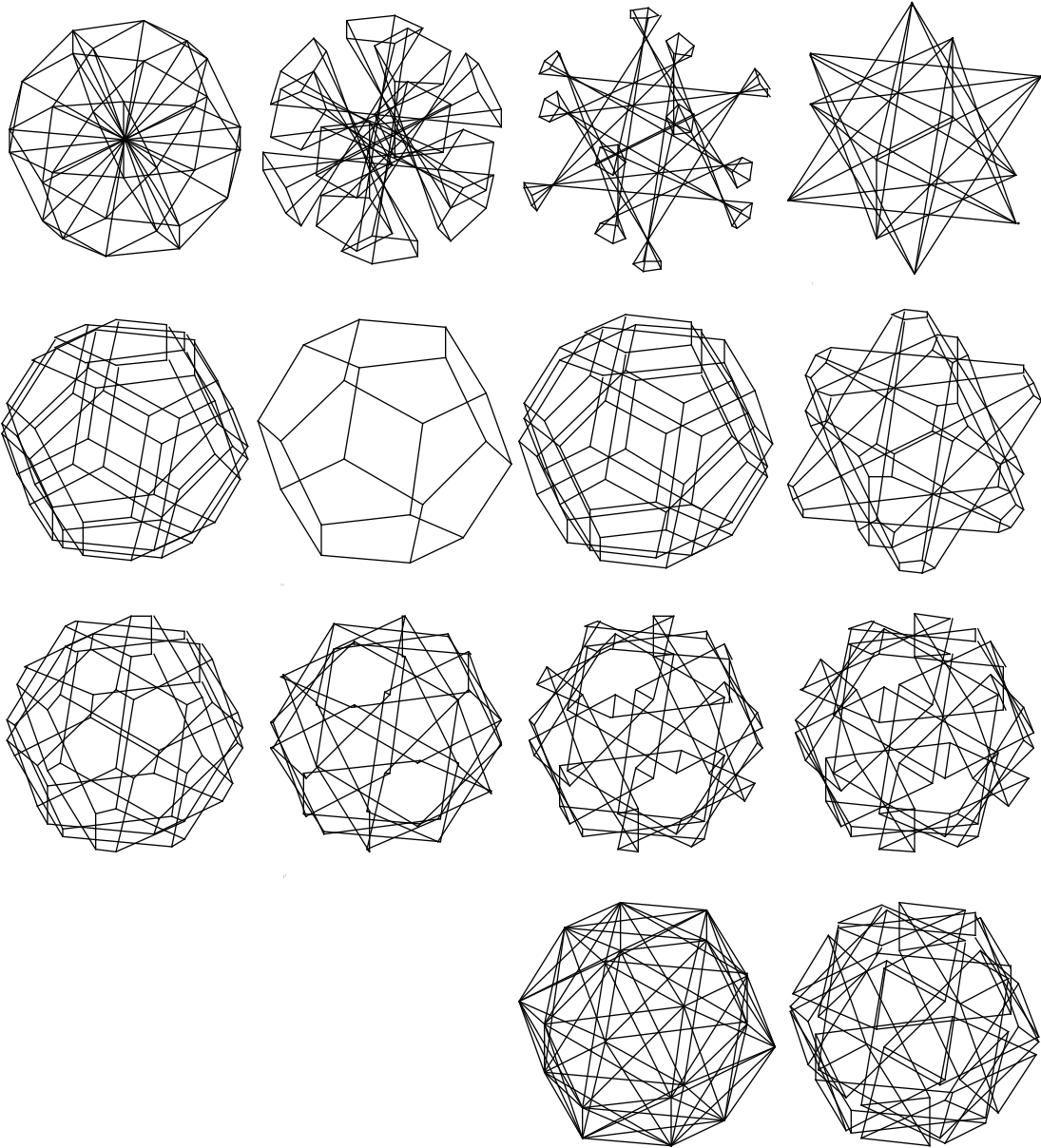
Figure 14.EPS

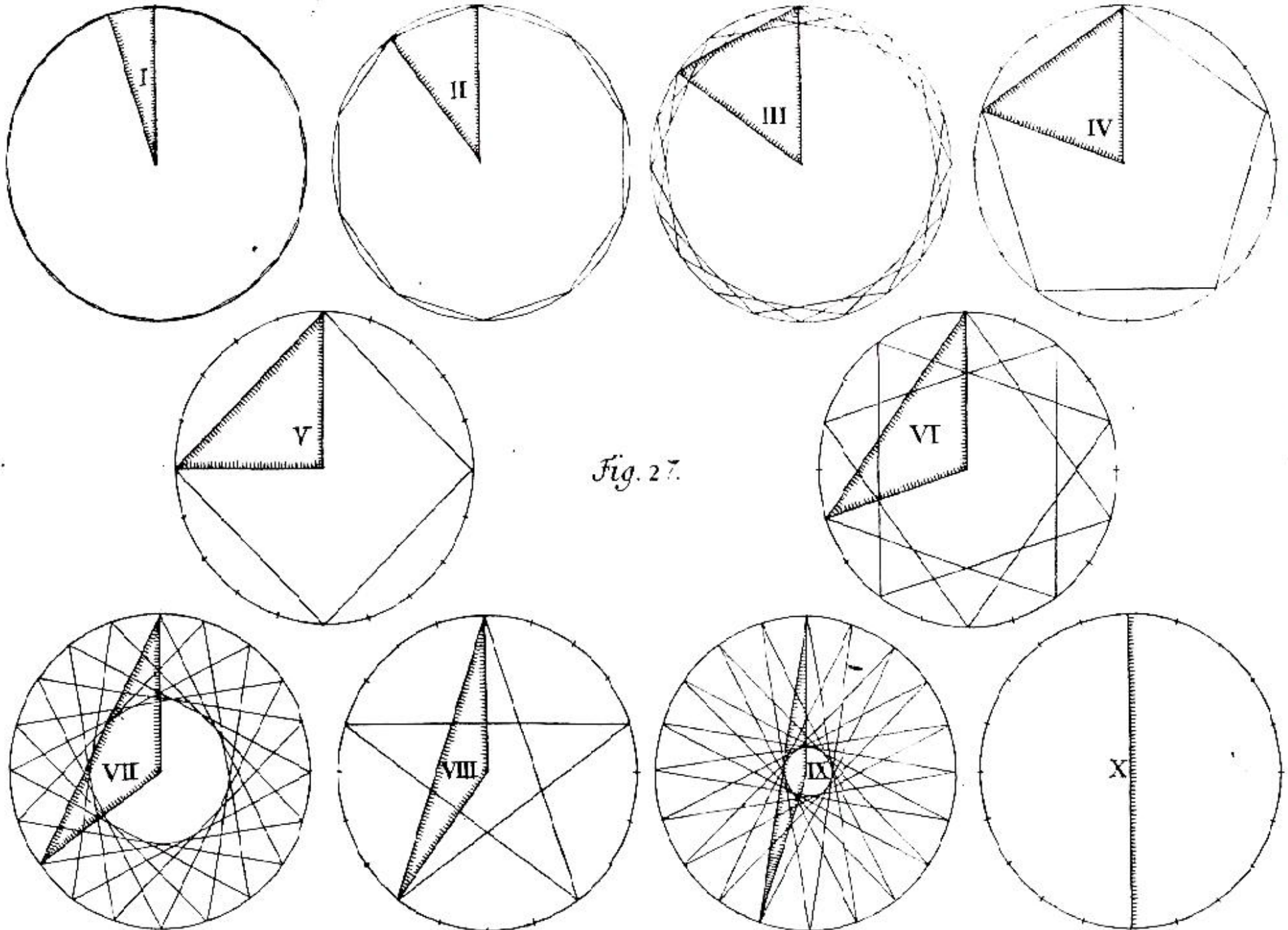This is a paper submitted for the special issue of DISCRETE MATHEMATICS, edited by MOHAR

TAB.VI.

Fig. 27.

J.D.Reine sculps