

# Euler's ratio-sum theorem and generalizations

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Over the centuries, many papers have been written about relations among different parts of a triangle. Well-known mathematicians as well as others have contributed to these discoveries. The main aim of this note is to show how asking the right questions can lead to new facts and to far-reaching generalizations that retain an elementary nature and would have been understandable to mathematicians of ages past.

Our starting point is one of the results of Euler's paper [5], which shows that

$$QB_1/A_1B_1 + QB_2/A_2B_2 + QB_3/A_3B_3 = 1. \quad (*)$$

We use the notation indicated in Figure 1, with  $Q$  an arbitrary point in the plane of the arbitrary triangle  $A_1A_2A_3$ , and  $B_i$  the intersection point of the cevian line  $Q_iQ$  with the side opposite  $A_i$ . Here and throughout, the only restriction is that all the points are well-defined and all the lengths appearing in the denominators are not zero. The lengths are understood as signed lengths; since only ratios of collinear segments are considered, the positive

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<sup>1</sup>Professor Klamkin passed away in the summer of 2004. As a friend and a mathematician he will be sorely missed by many of us. Professor Klamkin was still able to see the referees' comments on our paper, and approve the proposed final version of it. A variety of unfortunate circumstances delayed the sending of that version to the Editor. But this had the silver lining contained in part (vii) of the last section, added September 15, 2005. BG

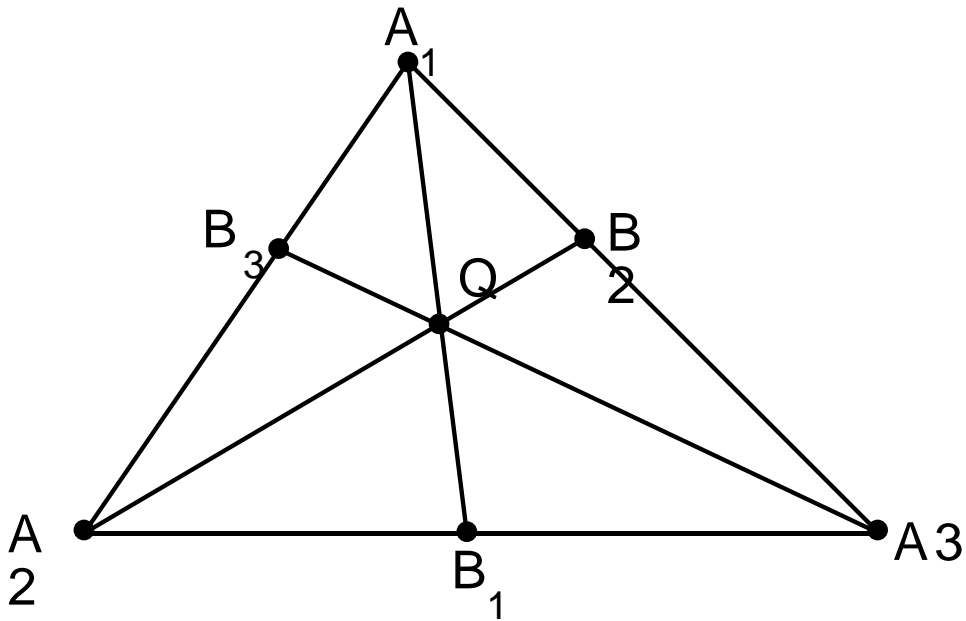


Figure 1: An example illustrating the notation used in Euler's theorem and in Theorem 1

direction on the lines carrying the segments is irrelevant. Euler gives several proofs that use various elementary geometric or trigonometric arguments. We shall provide a simple proof, and show how this result can be generalized in a variety of ways: to analogues of triangles in three and higher dimensions, to polygons with more than three sides and their higher-dimensional analogues, and to other ratio-sums.

We shall first discuss the version of Euler's result that holds for  $d$ -dimensional simplices, that is, the simplest polytopes of dimension  $d$  — the analogues of the triangles in the plane and tetrahedra in 3-space. We shall tie this with a presentation of similar results on the five other ratio-sums that can be defined using cevians; so far, these seem to have received scant attention. In contrast to Euler's result, other constants appear in these ratio-sum; best possible estimates are given in all cases. The generalizations of these results to more

general polygons, polyhedra and polytopes (higher dimensional relatives of polygons and polyhedra) will then be presented, followed by historical and other comments.

## Ratio-sums for simplices

For  $d \geq 2$ , let  $T^d$  denote the  $d$ -dimensional simplex in Euclidean  $d$ -space  $E^d$ , with vertices  $A - i$ ,  $0 \leq i \leq d$ . Thus  $T^d$  can be interpreted as the convex hull of  $d + 1$  points of the Euclidean  $d$ -space  $E^d$  which are not all contained in a hyperplane of smaller dimension. A good model to illustrate the concept and the following arguments is given by the  $d$ -simplex with vertices at the origin and at the unit points of a standard basis of  $E^d$ . For notational convenience we shall occasionally use  $A_{d+1} = A_0$ .

We start by describing the setting of the results. Let  $Q$  be a point of  $E^d$ , and  $F_i$  the facet (that is, the  $(d - 1)$ -dimensional face) of  $T^d$  that is opposite  $A_i$ . Let  $B_i$  be the point of intersection of the line (the “cevia”) through  $A_i$  and  $Q$  with the hyperplane  $H_i$  that contains  $F_i$ . For the definitions, and some of the results, the point  $Q$  need not be in the interior of  $T^d$ ; the only overall restriction on  $Q$  is that all points  $B_i$  must be well-defined, and that the denominators in the various fractions be non-zero. This condition will be assumed throughout, and will not be repeated in the reformulation of our results. We shall be interested in various ratios involving the lengths  $a_i = \|A_i - Q\|$ ,  $b_i = \|Q - B_i\|$ ,  $q_i = \|A_i - B_i\|$  of the segments  $A_iQ$ ,  $QB_i$ ,  $A_iB_i$ . As already mentioned, the lengths in question are to be taken as signed lengths; since we shall always consider ratios of collinear segments, the scale of measurement and the direction chosen as positive on each line are irrelevant. For  $d = 2$  one illustration of the possibilities is indicated in Figure 1, and another in Figure 2.

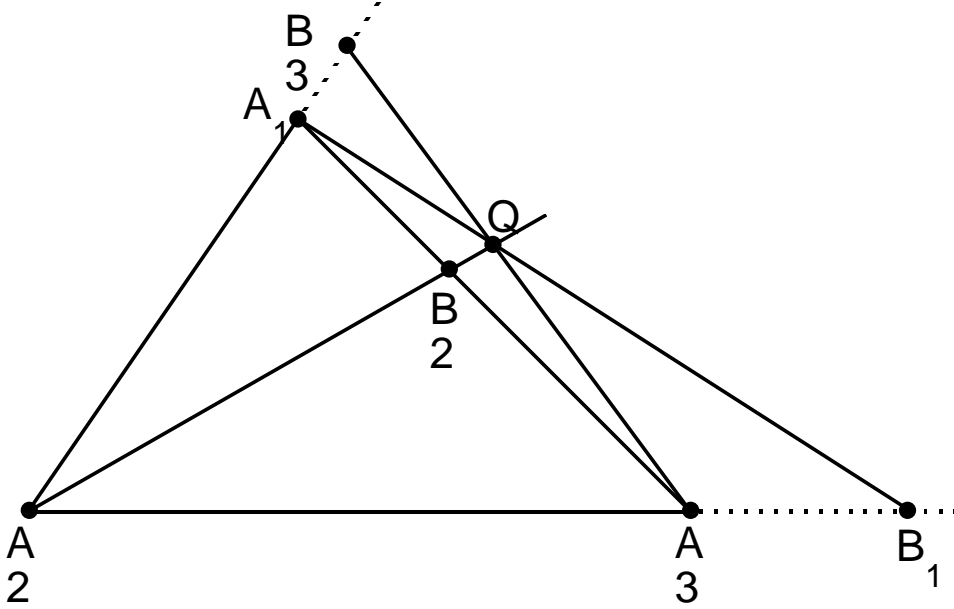


Figure 2: Another illustration of parts (i), (ii), (iv) and (vi) of Theorem 1. Parts (iii) and (v) are not applicable to this example since  $q_2/b_2 < 0$

To begin with, we are interested in the six ratio sums defined as follows:

$$\begin{aligned}
 \rho(b, q) &= \sum_i b_i/q_i; & \rho(a, q) &= \sum_i a_i/q_i; \\
 \rho(q, b) &= \sum_i q_i/b_i; & \rho(q, a) &= \sum_i q_i/a_i; \\
 \rho(a, b) &= \sum_i a_i/b_i; & \rho(b, a) &= \sum_i b_i/a_i;
 \end{aligned}$$

where each sum is over all  $i$ ,  $0 \leq i \leq d$ . We shall prove the following results.

**Theorem 1** *With the above notation the following statements are valid for all  $Q$ :*

- (i)  $\rho(b, q) = 1$ .
- (ii)  $\rho(a, q) = d$ .

- (iii) If  $q_i/b_i > 0$  for all  $i$ , then  $\rho(q, b) \geq (d+1)^2$ ; equality holds if and only if  $Q$  is the centroid of  $T^d$ .
- (iv) If  $q_i/a_i > 0$  for all  $i$ , then  $\rho(q, a) \geq (d+1)^2$ ; equality holds if and only if  $Q$  is the centroid of  $T^d$ .
- (v) If  $q_i/b_i > 0$  for all  $i$ , then  $\rho(a, b) \geq d(d+1)$ ; equality holds if and only if  $Q$  is the centroid of  $T^d$ .
- (vi) If  $q_i/a_i > 0$  for all  $i$ , then  $\rho(b, a) \geq (d+1)/d$ ; equality holds if and only if  $Q$  is the centroid of  $T^d$ .

Before we turn to the proofs, we recall two very useful lemmas.

Let  $T^d$  be a  $d$ -simplex with vertices  $A_i$ ,  $0 \leq i \leq d$ , and let  $S^d$  denote the simplex with vertices  $Q, A_1, A_2, \dots, A_d$ . Then, denoting by  $V(T)$  the signed volume of the simplex  $T$ , we have:

**Lemma 1**  $\|Q - B_0\|/\|A_0 - B_0\| = b_0/q_0 = V(S^d)/V(T^d)$ .

This selfevident fact, which was called the “volume principle” in [11], has been used without a special name by other authors (see, for example, [3, p. 131] for  $d = 3$ ). In case  $d = 2$ , it has been called the “area principle” in [10] and other publications, and it has been used starting at least two centuries ago.

A second well-known tool is the elementary

**Lemma 2**  $x + 1/x \geq 2$  for all  $x > 0$ , with equality if and only if  $x = 1$ .

We shall frequently apply this lemma in the form  $1/x \geq 2 - x$ .

**Proof of Theorem 1.** We shall give here proofs for only the first three parts of Theorem 1, to serve as warm-up for the generalizations presented in Theorem 2.

The result of part (i) follows at once from Lemma 1, on noticing that the signed volumes of the simplices with common apex  $Q$  that are spanned by

the  $d + 1$  facets of  $T^d$  add up precisely to the signed volume of  $T^d$ . For part (ii) it is enough to note that

$$\begin{aligned}\rho(a, q) &= \sum_i a_i/q_i = \sum_i (1 - b_i/q_i) \\ &= (d + 1) - \sum_i b_i/q_i = d + 1 - \rho(b, q) = d + 1 - 1 = d.\end{aligned}$$

For part (iii), using Lemma 2, we have

$$\begin{aligned}\rho(q, b)/(d + 1) &= \sum_i q_i/((d + 1)b_i) \geq 2(d + 1) - \sum_i ((d + 1)b_i)/q_i \\ &= 2(d + 1) - (d + 1)\rho(b, q) = d + 1,\end{aligned}$$

which is equivalent to the inequality of (iii). Equality holds if and only if  $((d + 1)b_i)/q_i = 1$  for every  $i$ , which is a characterization of  $Q$  as the centroid of  $T^d$ . ■

**Further generalizations.** The role that the simplex  $T^d$  plays in the above theorems will become clearer through the considerations of the generalizations of Theorem 1. For their formulation it is convenient to introduce appropriate notation.

Let  $P$  denote a fixed polytope of dimension  $d$  in Euclidean  $d$ -space  $E^d$ . It is simplest to think of  $P$  as a convex polytope, that is, as a generalization of convex polygons in the plane, and convex solids in 3-space — these could be called polytopes of dimension  $d = 2$  or 3. However, the restriction to convex polytopes is in no way necessary. For  $d = 2$  and  $d = 3$  we can admit polygons and polyhedra in the generality described in [8] and [9], that is, selfintersecting polygons, and selfintersecting polyhedra with possibly selfintersecting faces. For  $d \geq 4$  we admit the obvious generalizations of these kinds of polygons and polyhedra. We shall use the term “polytope” for all dimensions  $d \geq 2$ .

We impose the following restrictions on the polytopes considered here.

The polytopes must be orientable, and the  $d$ -polytopes and all their facets must have non-zero content (volume in dimension  $d$  or  $d - 1$ , respectively).

The content of a  $d$ -polytope  $P$  will be denoted by  $V(P)$ . The  $d$ -pyramid determined by a  $(d - 1)$ -polytope  $F$  and point  $X$  will be denoted  $F(X)$ .

Polytopes satisfying these conditions shall be called **star-like**. The traditional Kepler-Poinsot polyhedra — that is, the non-convex analogues of the Platonic regular solids — are star-like both in our sense, and visually. So are many (but not all) of the uniform polyhedra presented in [4], and beautifully illustrated by photos of models in [19]. Many other examples appear in [8] and [9].

Let  $P$  be a star-like  $d$ -polytope. The  $f$  facets of  $P$  are labeled  $F_1, F_2, \dots, F_f$  in an arbitrary order. Let  $A_j$ ,  $1 \leq j \leq f$ , be a collection of points of  $E^d$  such that for suitable points  $B_j$ , with  $B_j$  in the hyperplane determined by  $F_j$ , the line  $L_j = A_j B_j$  is well defined, intersects  $F_j$  only in  $B_j$ , and all lines  $L_j$  pass through a common point  $Q$ .

In analogy to the notation in Section 2, we put:

$$\begin{aligned} \rho(b, q; W) &= \Sigma_j w_j b_j / q_j; & \rho(q, b; W) &= \Sigma_j q_j / (w_j b_j); \\ \rho(a, q; W) &= \Sigma_j w_j a_j / q_j; & \rho(q, a; W) &= \Sigma_j q_j / (w_j a_j); \\ \rho(a, b; W) &= \Sigma_j w_j a_j / (w_j b_j); & \rho(b, a; W) &= \Sigma_j b_j / (w_j a_j); \end{aligned}$$

where  $W = (w_1, w_2, \dots, w_f)$  is an ordered  $f$ -tuple of suitable weights specified below; these weights depend on  $P$  and the points  $A_j$ , but are independent of  $Q$ . All summations are for  $j = 1, 2, \dots, f$ . In all parts of Theorem 2 it is understood that  $P$  is a star-like  $d$ -polytope with  $f$  facets, the points  $Q$  and  $A_j$  satisfy the above condition, and the weights  $W$  are given by  $w_j = V(F_j(A_j)) / V(P)$ . We abbreviate  $w = \Sigma_j w_j$  and  $w^* = \Sigma_j 1/w_j$ . See Figure 3 for an illustration of the notation.

**Theorem 2** *With the above notation the following statements are valid:*

- (i)  $\rho(b, q; W) = 1$ .
- (ii)  $\rho(a, q; W) = w - 1$ .
- (iii) *If  $q_j/b_j > 0$  and  $w_j > 0$  for all  $j$ , then  $\rho(q, b; W) \geq w(2f - 1)$ . Equality holds if and only if  $q_j/(w_j b_j) = w$  for all  $j = 1, 2, \dots, f$ .*

- (iv) If  $q_j/a_j > 0$  and  $w_j > 0$  for all  $j$ , then  $\rho(q, a; W) \geq f^2/(w-1)$ , with equality if and only if  $q_j/(w_j a_j) = f/(w-1)$  for all  $j = 1, 2, \dots, f$ .
- (v) If  $q_j/b_j > 0$  and  $w_j > 0$  for all  $j$ , then  $\rho(a, b; W) \geq w(2f-w) - w^*$ , with equality if and only if  $q_j/(w_j b_j) = w$  for all  $j = 1, 2, \dots, f$ .
- (vi) If  $q_j/a_j > 0$  and  $w_j > 0$  for all  $j$ , then  $\rho(b, a; W) \geq f^2/(w-1) - w^*$ , with equality if and only if  $q_j/(w_j a_j) = f/(w-1)$  for all  $j = 1, 2, \dots, f$ .

**Proof of Theorem 2.** For part (i) we note that an easy generalization of the “volume principle” shows that  $b_j/q_j = V(F_j(Q))/V(F_j(A_j))$ . Hence we have:  $\rho(b, q; W) = \sum_j w_j b_j/q_j = \sum_j w_j V(F_j(Q))/V(F_j(A_j)) = \sum_j V(F_j(Q))/V(P) = 1$ , since the sum of the volumes of the pyramids with apex  $Q$  equals the volume of  $P$ .

For part (ii), in analogy to the above and using Theorem 1(i), we have

$$\begin{aligned} \rho(A, q; W) &= \sum_j w_j a_j/q_j = \sum_j w_j (1 - b_j/q_j) \\ &= \sum_j w_j - \sum_j w_j V(F_j(Q))/V(F_j(A_j)) = \left(\sum_j w_j\right) - 1 = w - 1. \end{aligned}$$

For part (iii), in analogy to the proof of Theorem 1(iii), we have

$$\begin{aligned} \rho(q, b; W)/w &= \sum_j q_j/(w w_j b_j) \geq \sum_j (2 - w w_j b_j/q_j) \\ &= \sum_j 2 - w \sum_j w_j b_j/q_j = 2f - w. \end{aligned}$$

The equality criterion follows from Lemma 2.

For part (iv) we have

$$\begin{aligned} (w-1)\rho(q, a; W)/f &= \sum_j (w-1)q_j Z_j/(f w - j a_j) \geq \sum_j (2 - f w_j a_j/((w-1)q_j)) = \\ &= \sum_j 2 - (f \sum_j w_j a_j/q_j)/(w-1) = 2f - f(w-1)/(w-1) = f, \end{aligned}$$



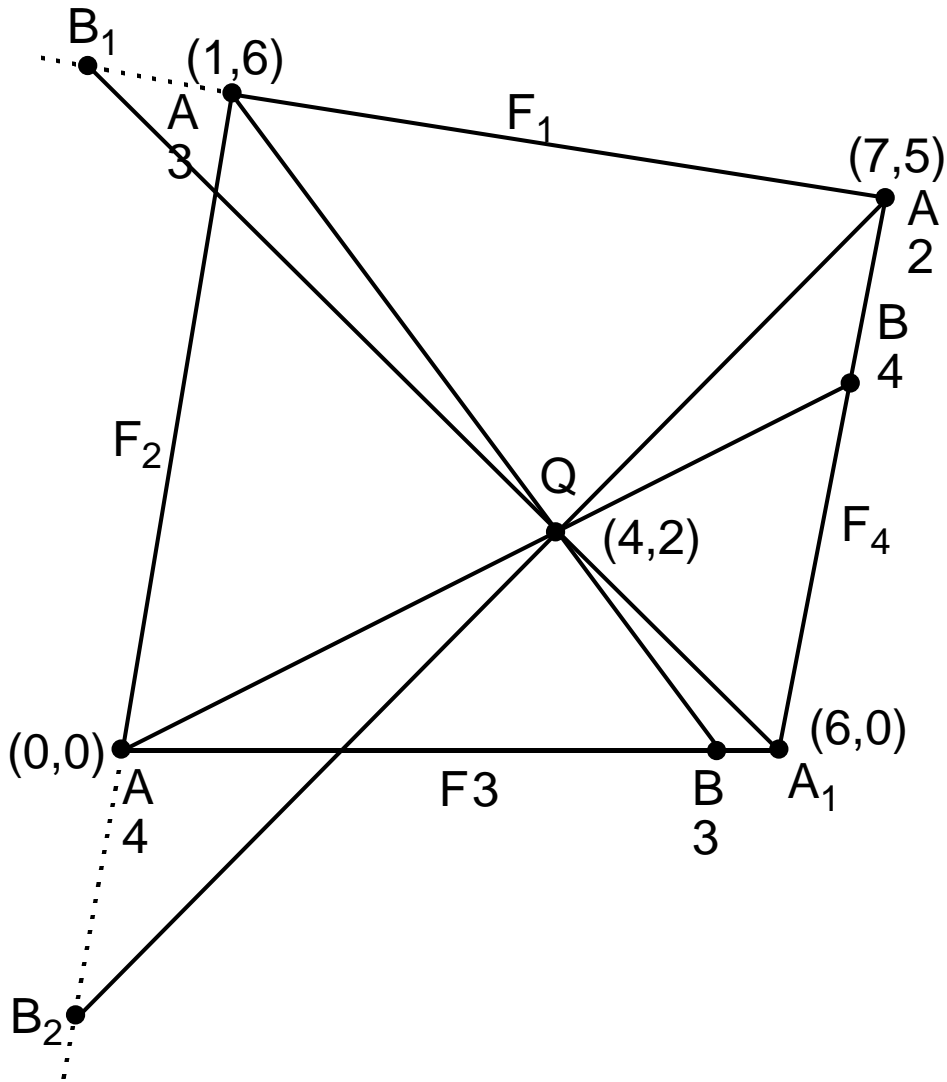


Figure 3: An example of the situation covered by Theorem 2. As is easily verified, in this example the weights are given by  $W = (31/67, 37/67, 36/67, 30/67)$ , hence  $w = 2$

which is equivalent to the claim. The equality condition is again a consequence of Lemma 2.

For part (v), in analogy to the above, and using Lemma 2 and part (i) of the theorem, we have

$$\begin{aligned}
\rho(a, b; W) &= \sum_j a_j/(w_j b_j) = \sum_j q_j/(w_j b_j) - \sum_j 1/w_j \\
&= w \sum_j q_j/(w w_j b_j) - \sum_j 1/w_j \\
&\geq w \sum_j (2 - w w_j b_j/q_j) - w^* = 2w f \geq V w_2 \sum_j w_j b_j/q_j - w^* \\
&= 2w f - w_2 - w^*.
\end{aligned}$$

Equality holds if and only if it holds in part (iii) of the theorem.

For part (vi), in analogy to the above, and using part (iv) of the theorem:

$$\begin{aligned}
\rho(b, a; W) &= \sum_j b_j/(w_j a_j) = \sum_j q_j/(w_j a_j) - \sum_j 1/w_j \\
&= (f/(w-1)) \sum_j (w-1)q_j/(f w_j a_j) - w^* \\
&\geq (f/(w-1)) \left( \sum_j 2 - \sum_j f w_j a_j / ((w-1)q_j) \right) - w^* \\
&= 2f^2/(w-1) - (f^2/(w-1)^2) \sum_j w_j a_j/q_j - w^* \\
&= 2f^2/(w-1) - (f^2/(w-1)^2) (w-1) - w^* = f^2/(w-1) - w^*,
\end{aligned}$$

with equality if and only if equality holds in part (iv).

This completes the proof of all parts of Theorem 2. ■

## Historical and other comments.

- (i) For  $d = 2$  it is clear that Theorem 1(i) contains Euler's result. Euler's theorem has been rediscovered by several authors; first among them is Gergonne [6]. Very few of these mention Euler — even the authoritative work of Zacharias [21] mentions only Gergonne. Surprisingly, the detailed survey of pre-20th century geometry by Simon [18] (which has references to well over 2000 authors!) does not mention the result at all. Without any attribution, Euler's result appears in [1, p. 162]. The extension to higher dimensions is also not new. For  $d = 3$  the earliest mention we are aware of is in [6]. Parts (i) and (ii) of Theorems 1 appear in [2, page 115] and [3, page 131]. For general  $d$ , our Theorem 1(i) appears in [13] and probably in several other places; it was also mentioned in a letter from Prof. H. Gülicher in 1998.
- (ii) We are not aware of any mention of parts (iii) to (vi) of Theorem 1 in the literature. The fact that equality holds in these cases for  $Q$  at the centroid is obvious. In each of them, the characterization of  $Q$  as the centroid in case of equality is due to Klamkin [14].
- (iii) It is easy to verify that the theorems of Section 3 reduce to the results of Section 2 in the special case that  $P$  is the  $d$ -simplex  $T^d$  and the points  $A_i$  are the vertices of  $T^d$ . However, even if  $P$  is  $T^d$  the results of Section 3 are more general since they do not restrict the points  $A_i$  to be vertices of  $T^d$ .
- (iv) Parts (i) and (ii) of Theorems 1 and 2 are somewhat analogous to the classical theorems of Ceva and Menelaus, and the new results on selftransversality (see [11]). These earlier results deal with product of ratios, while here we are concerned with sums of ratios. However, our other results seem not to have any multiplicative analogues.
- (v) The ratios  $a_j/b_j$  for cevians of a triangle appear in Euler's paper [5], in

the following result, formulated in the notation of Section 1:

$$\begin{aligned} & A_1Q/QB_1 + A_2Q/QB_2 + A_3Q/QB_3 + 2 \\ = & (A_1Q/QB_1) \cdot (A_2Q/QB_2) \cdot (A_2Q/QB_2). \end{aligned} \quad (**)$$

This nonlinear relation seems to have been largely forgotten. It has been established in a simple way and its validity extended in the recent paper [17]. An analogue of this result is due to Euler [5]; it deals with ratios of lengths of the segments into which each side of a triangle is partitioned by parallels to the other sides. It was independently found by Gülicher [12].

- (vi) The idea to use weights attached to the ratios originated with Shephard [16]; he kindly sent a preprint of this paper to one of us. For polygons in the plane Shephard establishes in [16] a restricted version of part (i) of our Theorem 2. Associating weights to ratios was also used in [7], for ratio-sums of a slightly different kind.
- (vii) Part (iii) of Theorem 1, as well as some results found in the literature, can be generalized so that the “cevians” are arbitrary segments, that need not have a common point. Let  $T^d$ ,  $A_i$ ,  $Q$ ,  $F_i$  and  $H_i$  have the same meaning as in the discussion leading to Theorem 1. Let  $B_i$  be an arbitrary point of  $H_i$ , and let  $C_i$  denote the point of  $H_i$  such that the segment  $QC_i$  is parallel to the segment  $A_iB_i$ . Figure 4 illustrates a case with  $d = 2$ . Let  $q_i$  and  $c_i$  denote the signed lengths of the segments  $A_iB_i$  and  $QC_i$ , let  $f$  and  $f_i$  denote the  $d$ -dimensional volumes of  $T^d$  and of the simplices with basis  $F_i$ , and let  $u$  and  $v$  be non-negative reals. Using the obvious generalization of the volume principle we have

$$\begin{aligned} \sum_i q_i/(uc_i + vq_i) &= \sum_i 1/(u(c_i/q_i) + v) = \sum_i 1/(u(f_i/f) + v) \\ &\geq (d+1)^2 / \sum_i (u(f_i/f) + v) \\ &= (d+1)^2 / (u + v(d+1)). \end{aligned}$$

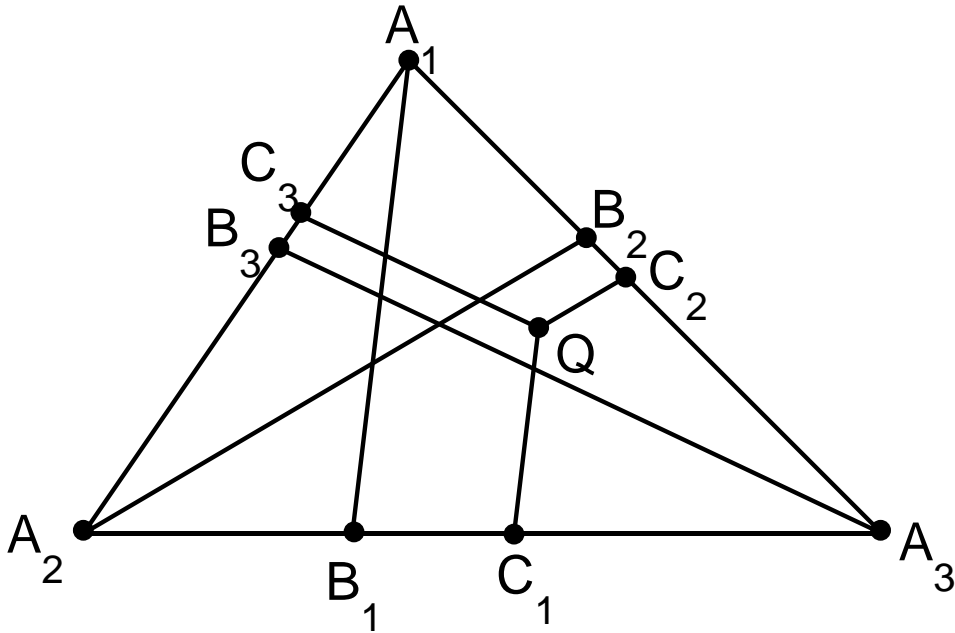


Figure 4: An illustration of the content and notation of comment (vii)

Here we used the fact that the arithmetic mean is greater than or equal to the harmonic mean, and that  $\sum_i f_i/f = 1$ . One could also add the less interesting generalization of part (i) of Theorem 1, namely  $\sum_i (uc_i + vq_i)/q_i = u + v(d + 1)$ . The special case  $d = 3$ , with  $B_i$  the foot of the altitude from  $A_i$ , and  $Q$  the incenter appears in [20] and [15]. In the former,  $u = 3$  and  $v = 1$ , so that the lower bound is  $16/7$ ; Murray Klamkin was one of the solvers of [20]. In [15],  $v = 0$  and  $u = 1$ , hence the lower bound is 16.

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