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ON SOME PROPERTIES OF CONVEX SETS

BY

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1. The following problem has been proposed by H. Steinhaus [7]: *Prove that through each point inside a closed convex surface there is passing such a plane that this point lies on one of the longest chords of the curve of intersection.*

In the present note* we shall disprove this conjecture; on the other hand we shall prove a theorem generalizing some known results [5, 6] of a related nature.

We shall limit ourselves to convex bodies in three-dimensional space, and only briefly indicate the extension to higher dimensional spaces.

2. Let K be a convex body in E^3 and let p be a point of $\operatorname{Int} K$. A set-valued function F which assigns to every plane π containing p a compact convex subset $F(\pi)$ of π, and which is continuous (in the Hausdorff metric [2] for compact sets) in π, shall be called a *proper mapping*. Then we have

THEOREM 1. *If K is a convex body in E^3, if $p \in \operatorname{Int} K$ and if F is a proper mapping, then there exists a plane π (containing p) such that $p \in F(\pi)$.*

Proof. Assuming the assertion of the theorem false, let F be a proper mapping for which $p \notin F(\pi)$ for all planes π containing p. Let $p(\pi)$ be that (unique) point of $F(\pi)$ for which the distance to p is minimal, and let $v(\pi)$ be the vector issuing from p and ending at $p(\pi)$. Obviously $v(\pi)$ depends continuously on π, is parallel to π and different from 0. For each point u of the unit sphere $S_2 = \{u = (x, y, z);\ x^2+y^2+z^2 = 1\}$ let π_u be the plane orthogonal to u and containing p. Then $f(u) = v(\pi_u)$ is a continuous field of non-vanishing vectors, tangential to S_2. But this is a contradic-

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tion to the well-known theorem of Poincaré-Brouwer (see, e. g., [1], p. 484) according to which such a vector field does not exist. This ends the proof of Theorem 1.

We note that Theorem 1 is valid in all dimensions. The above proof transfers immediately to odd-dimensional spaces; for even-dimensional spaces the property $f(u) = f(-u)$ of our mapping is to be used, together with the following theorem: Any continuous field $f(u)$ of tangential vectors on S_{2n-1} satisfying $f(u) = f(-u)$, vanishes for some $u \in S_{2n-1}$. To prove this theorem we observe that otherwise f could obviously be used to define a map $F: S_{2n-1} \to S_{2n-1}$ (where $F(u) = f(u)/|f(u)|$) such that: (1) F is homotopic to the identity map of S_{2n-1} onto itself (see, e. g., [3], p. 130) and thus has degree 1; (2) F may be factored through the projective space P_{2n-1} (because $F(u) = F(-u)$) and thus has even degree, in contradiction to (1).

Theorem 1 obviously implies the results of [5, 6]: Every point p belonging to the interior of a convex body K is the centroid of some plane section of K (resp. of some "cap" of K, resp. of the surface of some "cap" of K). (The proof of Theorem 1 is practically identical with that given in [6]).

Theorem 1 also implies that every point of $\mathrm{Int}K$ is the center of a circle circumscribed about some plane section of K, as well as the "quasi-center" of some section of K (see [3]), etc.

As is easily seen, Theorem 1 may fail if the assumptions on F are significantly weakened. An example to that effect may easily be derived from Theorem 2.

3. We shall now show that Steinhaus' conjecture is false, even if some additional conditions (like smoothness, differentiability, etc.) are imposed on the convex body considered. A point p belonging to the interior is said to have the *diameter-property* if and only if there exists a plane π such that p belongs to some diameter (= segment of maximal length) of the set $K \cap \pi$. A convex body K is said to have the *diameter-property* in case each point of $\mathrm{Int}K$ has the diameter-property. With this terminology we have

THEOREM 2. *The set $\mathcal{K}^* = \{K^*\}$ of convex bodies in E^3 which have the diameter-property is a closed, proper subset of the space $\mathcal{K}$ of all convex bodies in E^3 ($\mathcal{K}$ being considered a metric space in the Hausdorff metric).*

Proof. We first show that $\mathcal{K}^*$ is a closed subset of $\mathcal{K}$. Let $\{K_n\}$ be a sequence of convex bodies having the diameter-property, converging (in the Hausdorff metric) to a convex body K, and let p belong to $\mathrm{Int}K$. Without loss of generality we may assume that $p \in K_n$ for all n. By assumption there exist, for each n, a plane π_n and a diameter D_n of $\pi_n \cap K_n$ such that $p \in D_n$. Taking subsequences if necessary we may assume (by compactness) that the sets $\pi_n \cap K_n$ converge, as well as the diameters D_n. Obviously, the limit of $\pi_n \cap K_n$ is a plane section of K having $D = \lim D_n$ as diameter; but, since $p \in D_n$ for all n, we have $p \in D$. Thus p, and therefore K as well, have the diameter-property.

The proof of Theorem 2 will be completed by exhibiting a convex body which does not have the diameter-property. A very simple such example is the regular octahedron. Indeed, assuming the octahedron K given by

$$K = \{(x, y, z);\ |x|+|y|+|z| \leqslant 1\}$$

in an orthogonal system of coordinates, let $p \in \mathrm{Int}K$ be the point having all three coordinates equal to $1/(3+2\varepsilon)$, for a sufficiently small positive ε (e. g. $0 < \varepsilon < 1/4$). We shall see that p does not have the diameter-property. Each plane section of K is a polygon; since a diameter of a polygon is necessarily a diagonal (or a side) of the polygon, a diameter of a plane section of K has its endpoints on edges of K. An easy computation shows that if p belongs to a segment D whose endpoints belong to edges of K, then the endpoints are either

$$A = \left(\frac{1+\varepsilon}{2+\varepsilon}, \frac{1}{2+\varepsilon}, 0\right) \quad \text{and} \quad B = \left(\frac{-\varepsilon}{1+\varepsilon}, 0, \frac{1}{1+\varepsilon}\right)$$

or one of the five pairs obtained from A, B by permuting the coordinates. Without loss of generality we may assume that $D = \overline{AB}$. Then, assuming that D is a diameter of some plane section $\pi \cap K$ of K, a contradiction results: Any plane π containing D intersects either the edge E_1 of K with endpoints $(1, 0, 0)$ and $(0, 0, -1)$, or the edge E_2 with endpoints $(0, 1, 0)$ and $(0, 0, -1)$ (or both), and this intersection is obviously contained in $\pi \cap K$. But by our choice of p the distance from B to A is strictly less than the distance form B to any point of $E_1 \cup E_2$, and therefore D is not a diameter of $\pi \cap K$.

This ends the proof of Theorem 2.

Added in proof: Another result of Steinhaus [6] is that every convex body K in E^3 has either continuum many points each of which is the centroid of at least two different plane sections of K, or a point which is the centroid of continuum many different plane sections. This theorem generalizes both to higher dimensions and to arbitrary proper mappings. The following related problem seems to be open (except for $n = 2$, in which case an affirmative solution is easily established): If K is a convex body in E^n does there exist a point $x \in \mathrm{Int}\ K$ which is the centroid of at least $n+1$ different $(n-1)$-dimensional sections of K? In particular, is the centroid of K such a point?

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STABILITY OF THE FIXED-POINT PROPERTY

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As is well-known, the fixed-point property is possessed by every compact absolute retract A; if the mapping φ of A into itself is continuous, then some point of A is invariant under φ. We show here that for such an A there is in the following sense a sort of stability about the fixed-point property; if the mapping φ of A into itself is nearly continuous, then some point of A is nearly invariant under φ. An example is given of a plane continuum in which the fixed-point property persists but fails to satisfy the stability condition.

Consider a topological space X and a metric space (M, ϱ). For $\varepsilon > 0$, a mapping φ of X into M will be called *ε-continuous* provided each point x of X admits a neighborhood U_x such that the ϱ-diameter of the set φU_x is at most ε. For $\delta \geqslant 0$, a *δ-invariant* point for a mapping ξ of M into M is a point $p \in M$ such that $\varrho(\xi p, p) \leqslant \delta$; ξ will be called a *δ-mapping* provided each point of M is δ-invariant for ξ.

1. PROPOSITION. *Suppose X and Y are topological spaces, M a metric space, f a continuous mapping of X into Y, φ an ε-continuous mapping of Y into M, and ξ a δ-mapping of M into M. Then $\xi\varphi f$ is an $(\varepsilon+2\delta)$-continuous mapping of X into M.*

Proof. Consider an arbitrary point $x \in X$. Since φ is ε-continuous, there is a neighborhood V of fx such that $\operatorname{diam}\varphi V \leqslant \varepsilon$. And since f is continuous, there is a neighborhood U_x of x such that $fU_x \subset V$. Then $\operatorname{diam}\varphi f U_x \leqslant \varepsilon$. Since ξ is a δ-mapping, for arbitrary $u, u' \in U_x$ we have

$$\varrho(\xi\varphi fu, \xi\varphi fu') \leqslant \varrho(\xi\varphi fu, \varphi fu) + \varrho(\varphi fu, \varphi fu') + \varrho(\varphi fu', \xi\varphi fu') \leqslant \delta + \varepsilon + \delta.$$

Consequently $\operatorname{diam}\xi\varphi f U_x \leqslant \varepsilon + 2\delta$ and the proof is complete.

2. PROPOSITION. *Suppose P is a compact convex polyhedron in a finite-dimensional normed linear space, and φ is an ε-continuous mapping of P*

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