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ARE PRISMS AND ANTIPRISMS REALLY BORING ? (Part 2)

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1. Introduction. In the first part of this paper [5] we considered the unjustifiably low reputation of prisms. In the same spirit we shall in this note attempt to show that antiprisms are not only interesting, but in fact present many challenges and lead to seemingly very hard problems. As in the case of prisms, "antiprisms" mean different things to different people. We shall start by surveying the current usage. Due to the length of the note, it will be presented in two parts. The present installment is mainly devoted to the case of 3-dimensional antiprisms, while the next one will deal with combinatorial antiprisms and their geometric realizations, and with higher dimensions.

2. The traditional approach. Many authors (in print, or on the Web) define prisms and antiprisms only in the context of Archimedean polyhedra (variously called also semiregular or uniform), more or less as follows (see, for example, [3, p. 85]): "A *prism* is formed from two n-sided [regular] polygons separated by a ring of n squares. An *antiprism* also contains two n-sided regular polygons, ... separated by a ring of $2n$ equilateral triangles." This is equivalent to saying — as many other authors do — that an antiprism results from a prism by twisting one of the n-gons by $180/n$ degrees, and adjusting the distance between the two so that the $2n$ triangles that result from the n squares become equilateral. This is illustrated in Figure 1 for $3 \leq n \leq 6$. In slightly mere general meaning, instead of being equilateral, the triangles of the ring are allowed to be isosceles.

Somewhat more general antiprisms are discussed by Aravind in [1]. Here the two n-gons are assumed only to be similar and situated so that the $2n$ mantle triangles are isosceles. An example is shown in Figure 2.

A different extension of the concept appears in [3, p. 13]. "A *prism* is formed from two congruent polygons lying in parallel planes connected by a ring of rectangles. An *antiprism* is similar except that the connecting ring is composed of isosceles triangles." However, it may well be that this definition admits more than was intended: For most writers the notion of antiprism includes the alternation of the bases of the triangles among the sides of the n-gons. As shown in Figure 3, this needs not be the case for Cromwell's concept.

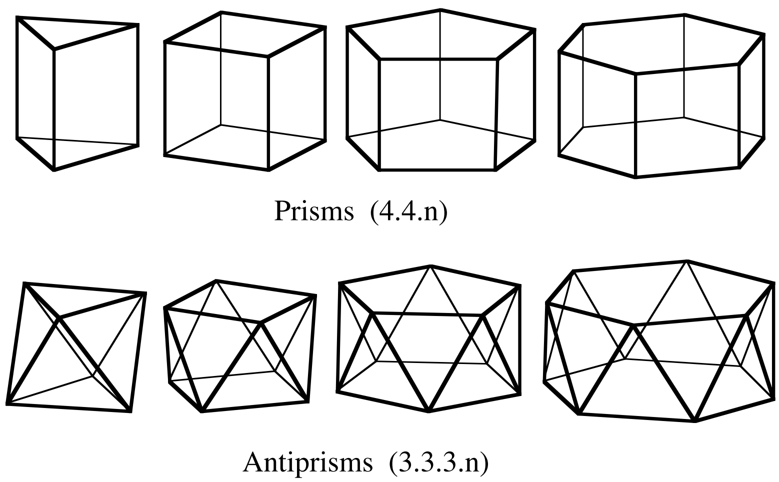


Figure 1. Examples of prisms and antiprisms. The 4-prism is better known as the cube, and the 3-antiprism is the regular octahedron.

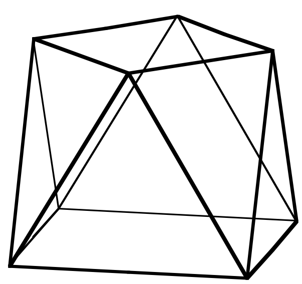


Figure 2. An example of a 4-antiprism in the sense of Aravind [1].

3. New approaches. In [5] we have seen that allowing more generality in the definition of prisms leads to more interesting polyhedra. One may expect that greater generality will be interesting in the case of antiprisms as well. On the other hand, it is quite obvious that any extension of the concept of antiprism to more general polyhedra, or to higher-dimensional polytopes, cannot proceed along the lines discussed above. Fortunately, there is a way to define antiprismatic polyhedra in such a way that fruitful generalizations are possible. Although we could again start by defining abstract *d*-antiprisms, it seems more appropriate to start by restricting attention to convex ones.

For $d \geq 3$, a *convex d-antiprism* P with *bases* P_1 and P_2 is the convex hull of convex (d-1)-polytopes P_1 and P_2 provided:

(i) P_1 and P_2 are situated in distinct parallel hyperplanes, and are dual to each other under a mapping f;

(ii) the only other facets (that is, (d-1)-dimensional faces) of P are the convex hulls of faces F_1 and F_2 of P_1 and P_2, which correspond to each other under f.

For this definition see [4, p. 66]; it was adopted by Broadie [2]. Independently, a similar definition was proposed by Smith [6] in the special case that P_1 and P_2 are 3-dimensional tetrahedra. Since polygons are selfdual, it is obvious that this definition includes the traditionally considered antiprisms, as well as the Aravind concept, but not the one proposed by Cromwell.

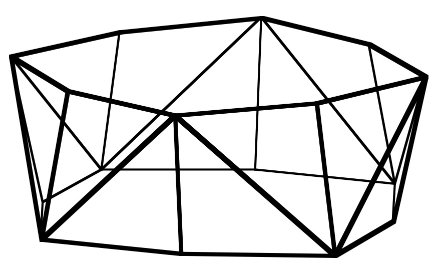


Figure 3. A polyhedron that satisfies the Cromwell restriction, but would not be considered an antiprism in the traditional sense.

It is very easy to show that if P_1 and P_2 are the bases of an antiprism, one can change the distance between them, translate either one arbitrarily, or replace it by a homothetic copy (that is, replace it by a reduced or enlarged copy, without rotating).

Here are a few results that involve our concept.

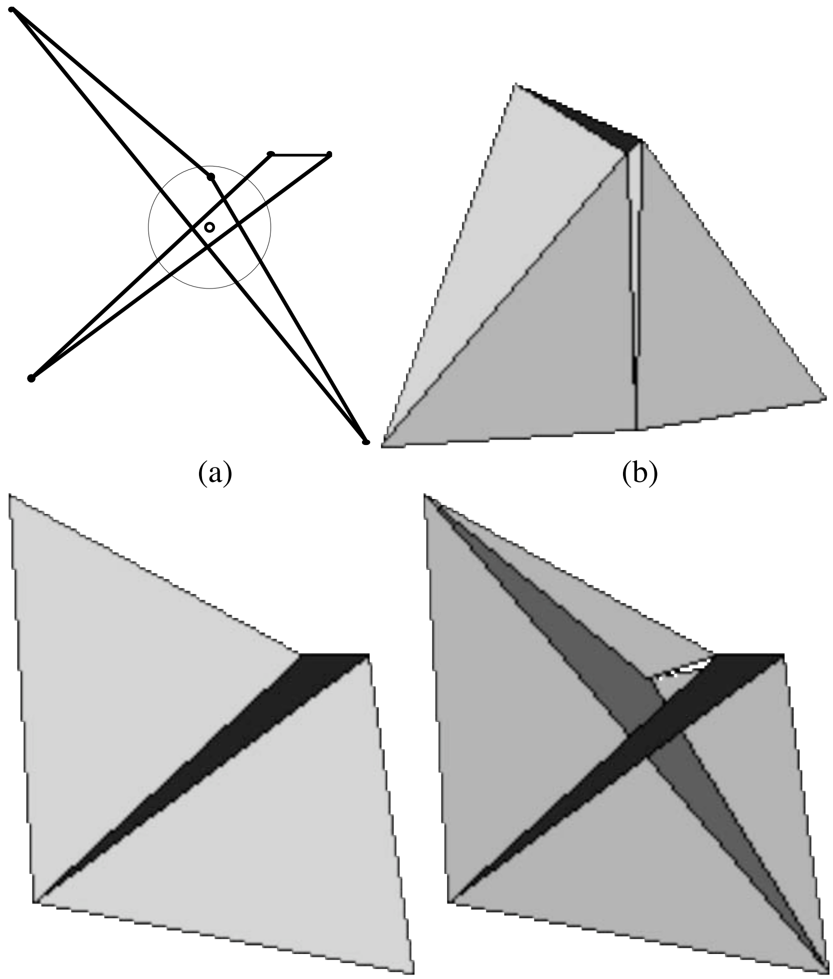
Proposition 1. Every convex polygon P_1 is a basis of a 3-antiprism.

Proof. By construction of a suitable polygon P_2. The only restriction on P_2 is that it has the same number of sides as P_1, and that the triangles determined by the sides of P_1 and P_2 with appropriate vertices of the other be non-overlapping. For a given P_1 the construction of P_2 is easy: At each vertex of P_1 Draw a supporting line that intersects P_1 in that point only; then the intersection of the halfplanes determined by these lines and containing P_1 yields a suitable P_2. Finally, lifting one of them (say) perpendicularly out of the common plane and taking the convex hull yields the required antiprism.

Figure 2 can be interpreted as arising by this construction, with the top square playing the role of P_1.

In the case of regular polygons, and many others, P_2 can be chosen as a *polar* of P_1 with respect to a suitable circle. However, many related questions arise. For example: For each P_1, can P_2 be always chosen as a *polar* of P_1 with respect to a suitable circle? If the answer is affirmative, one could further ask whether *every* polar of P_1 can be chosen as P_2.

Broadie [2] provided an example that shows the answer to the last question is negative. With slight changes made for sake of easier visualization, Broadie's example is shown in Figure 4. The reason for the failure is that the convex hulls of edges of one of the bases with the dual vertices fail to form the "mantle" of the convex hull of the bases. However, it is easy to show that for every triangle, *some* polar is suitable — thus triangles cannot provide a negative answer to the first question. We shall not give here a proof, since



(a) (b) (c) (d)

Figure 4. An example modified from Broadie [2]. (a) shows a triangle and its polar with respect to a circle. (b) is a side-view of the non-convex polyhedron resulting from the antiprism construction. (c) and (d) show views of this polyhedron from straight above; in (d) the side-triangles that have edges in the upper basis have been omitted, for better visibility of the bottom basis and the other sides.

it will follow from Broadie's general result which will be quoted in Part 3. But a small modification of this construction, shown in Figure 5, can be used to establish a negative answer to the first question for quadrangles.

All the above can also be seen as a consequence of the following Proposition 2. For it we need a definition.

Let P be a convex polygon. The *polarity kernel* $\Pi(P)$ of the polygon P consists of all the interior points of P which can serve as centers of circles with respect to which the polar of P and P can serve as bases of an antiprism. With this we have:

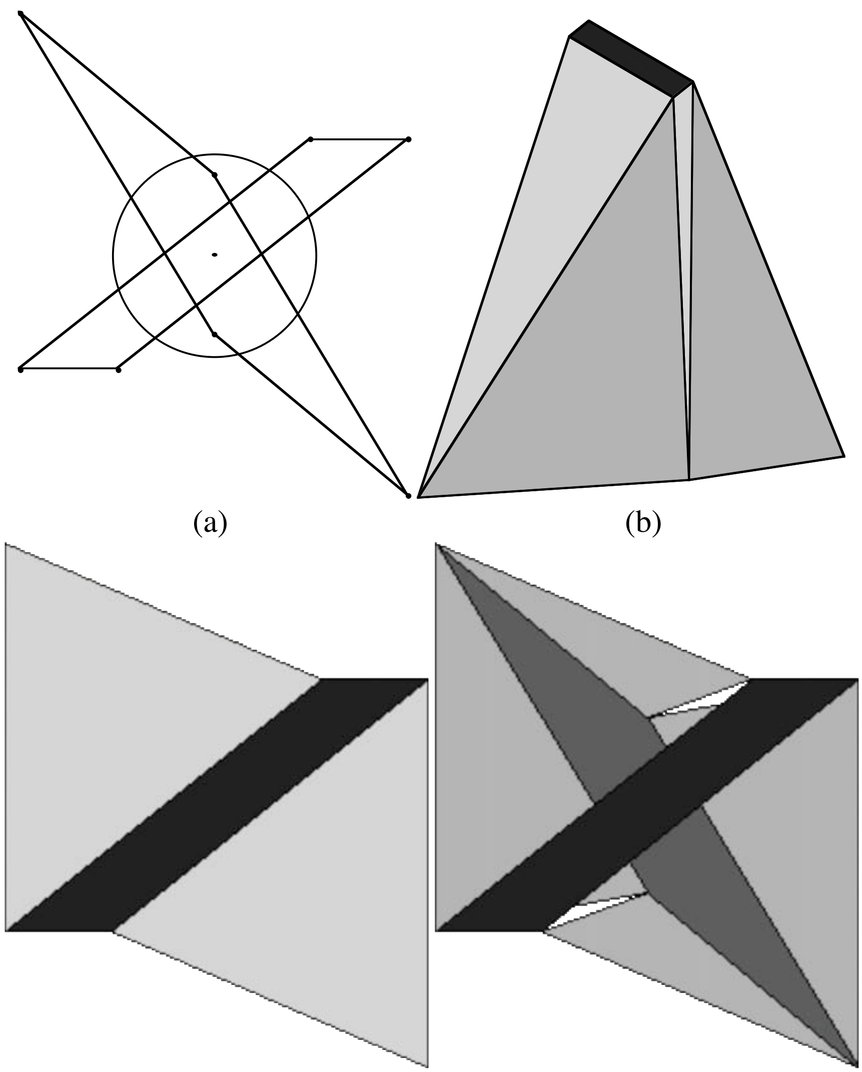
Proposition 2. The polarity kernel $\Pi(P)$ of the convex polygon P consists of all interior points X of P such that the perpendicular from X to the line determined by each edge of P is a point of that edge itself.

To prove the proposition directly it is sufficient to take as the circle any sufficiently large one. Then the polar of each edge of the convex polygon P will be situated "beyond" the edge, and thus no non-convexity can arise.

It is obvious that the polarity kernel of the triangle in Figure 4 does not contain the center of the circle used in the polarity, and that the polarity kernel of the parallelogram in Figure 5 is empty. All the above also suggests that it may be worthwhile to modify the definition of a 3-antiprisms as follows:

Let P_1 and P_2 be two simple (that is, non-selfintersecting) polygons, with vertices $V_1, V_2, \ldots, V_n$ and $W_1, W_2, \ldots, W_n$ respectively. If the triangles $V_1V_2W_1$, $V_2W_1W_2$, $V_2V_2W_2$, ... form a "ring" that is not selfintersecting, then that ring and the bases P_1 and P_2 form an *acoptic* (that is, non-selfintersecting) *3-antiprism*. Clearly, the 3-antiprisms of the original definition are acoptic 3-antiprisms, and so are those in Figures 4 and 5.

Problem 1. Characterize the pairs of simple polygons that can serve as bases of acoptic 3-antiprisms.



(a) (b)

(c) (d)

Figure 5. An example of a polygon (in fact, a parallelogram) which cannot form an antiprism with any polar. The views are as in Figure 4.

Problem 2. Characterize the polarity kernels $\prod(P)$ of simple polygons P.

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