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ARE PRISMS AND ANTIPRISMS REALLY BORING ? (Part 1)

Branko Grünbaum
Department of Mathematics, Box 354350
University of Washington
Seattle, WA 98195-4350
e-mail: grunbaum@math.washington.edu

1. Introduction. Everybody has heard of solids called *prisms*; many – but not all – have also heard of *antiprisms*. The definitions offered by different authors vary somewhat, but these solids are invariably presented as being rather dull and pedestrian. It seems that many writers feel that it is their duty to mention them for sake of completeness, but their heart and interests are really not in it. Some writers fail to mention them when they should: prisms and antiprisms satisfy the definitions of certain types of polyhedra they discus, but their existence is ignored. The present note is meant to show that the presumed triviality of prisms and antiprisms is a bum rap: The *ideas* involved in prisms and antiprisms lead to many unusual polyhedra, and touch on several very modern mathematical questions. Some of these are still open problems. In the present first part we shall concentrate on prisms, to be followed by a discussion of antiprisms in Part 2, to appear in a future issue. In Section 2 we shall present the main results of the note, while Section 3 contains historical remarks, references to the literature, and a proposed solution to a puzzling fact concerning Kepler's enumeration of the Archimedean polyhedra.

2. Prisms. According to various sources ([12], several encyclopedias), prisms appear in some generality for the first time in 1570, in Sir Henry Billingsley's 1570 translation of Euclid's *Elements*. Soon thereafter several slightly different interpretations of the word appear; these continue to be used to this day. On the one hand, a customary statement is that a prism is "a polyhedron having two faces that are polygons in parallel planes while the other faces are parallelograms"; from this follows that the two polygons are translates of each other. On the other hand, many writers restrict the meaning to the case in which the two polygons are regular, and the parallelograms are squares; other call this special case "regular prisms". Even this restricted meaning allows for infinitely many kinds of "regular prisms", since every regular n-gon can be used. The regular prisms satisfy the conditions usually imposed on polyhedra called "Archimedean"; nevertheless, most authors do not include them among the Archimedean polyhedra, stating instead that "there are 13 Archimedean polyhedra". Details about this phenomenon and its history are presented in Section 3. Here it should be sufficient to note that this is one of the needless slurs on the role of prisms.

More interest about prisms can be generated if we ask ourselves how could one define *combinatorially* (as opposed to *geometrically*) a class of polyhedra that would deserve to be called "prisms". It is reasonably clear that the best one could say combinatorially is that a *combinatorial n-sided prism* is a complex that consists of two polygons with the same number n of sides, together with n quadrangles, such that at each vertex two quadrangles and one n-gon meet. In any geometric realization of a combinatorial prism the two n-gons, and each of the n quadrangles must be planar. If we now ask what shapes can the geometric realizations of a combinatorial prism have, we see that there are many possibilities. Two unusual geometric realizations of the combinatorial hexagonal prism are shown in Figure 1.

It should be explicitly noted that we do not restrict attention to convex polyhedra. Just as the well-known regular star-polyhedra

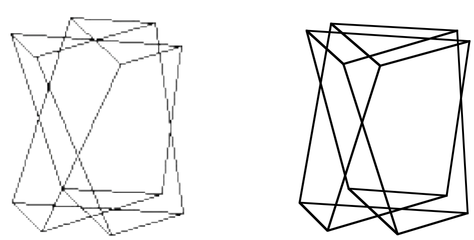

Figure 1.

of Kepler and Poinsot (see, for example, [2], [3], [14]) have selfintersections of various kinds, we consider here "polyhedra" of such kinds. Details of the specific definition would lead us to far to be presented here; they can be found in [5], [6] and, for abstract polytopes of all dimensions in [10].

Every combinatorial prism is *vertex-transitive*; that is, all vertices are mutually equivalent under incidence-preserving automorphisms of the complex. This is a consequence of the fact that the regular prisms is among the geometric realizations of all prisms. Hence it is natural to ask about isogonal geometric realizations, that is, about such geometric prisms on which the group of geometric symmetries acts transitively on the vertices. Somewhat unexpectedly, it turns out that combinatorial n-sided prisms admit, for each odd $n \geq 3$, several distinct isogonal realizations, while for each even $n \geq 4$ they admit a continuum of geometric realizations. Examples for $n = 5$ are shown in Figure 2, while the case $n = 6$ is illustrated in Figure 3. It should be noted that the bases are hexagons in all parts of Figure 3. The apparent triangles in the fourth column are hexagons in which adjacent pairs of vertices are represented by the same point — thus establishing a continuous transition between the prisms in adjacent columns. On the other hand, the apparent triangles in the rightmost column are hexagons of rotation number 2 — each winds twice around the interior. Again pairs of vertices are represented by single points, but in this case these are opposite vertices of the hexagon. They also represent intermediate stages between adjacent prisms.

Figures 2 and 3 are taken from [4], where additional illustrations are also provided. In that paper the concept of prism is generalized to "prismatoid". A *prismatoid* is usually defined as a polyhedron

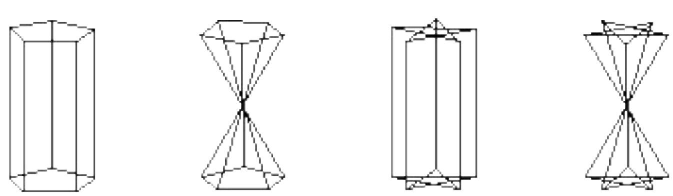

Figure 2.

having all its vertices in two parallel planes; thus they include, besides prisms, also antiprisms, which we shall discuss in Part 2. As shown in [4], even isogonal prismatoids exhibit an astounding variety of possible shapes, of many novel kinds.

In the case of a convex polytope P of any dimension, it is easy to define a *prism* with *basis* P as the convex hull of the union of P with a translate in a direction that is outside of the affine hull of P. This can also be put is a generalized combinatorial form, which includes the earlier definition of 3-dimensional combinatorial prisms.

A *combinatorial 1-prism* is any complex (abstract 1-polytope) consisting of two distinct symbols, interpreted as "vertices", together with the pair of these vertices, interpreted as an "edge". For $d \geq 2$, a *combinatorial d-prism* is a complex (an abstract d-polytope) whose facets are two abstract (d-1)-polytopes P_1 and P_2 isomorphic under a mapping f, together with all the combinatorial (d-1)-prisms determined by facets of P_1 and P_2 that correspond to each other under f.

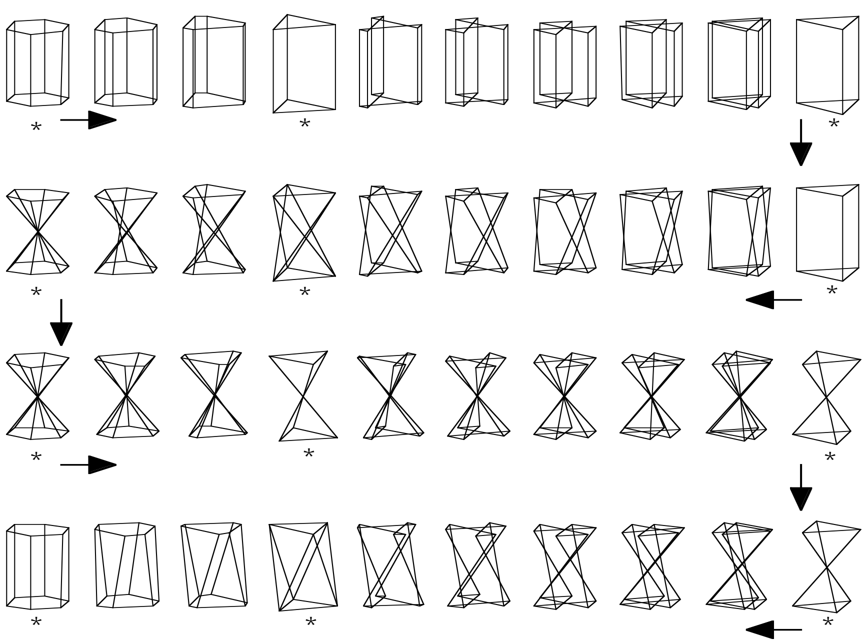

Figure 3.

A *geometric d-prism* (generally shortened to *d-prism*) is the image of a combinatorial d–prism in a Euclidean space, such that each k-dimensional combinatorial face has an image contained in a k-dimensional flat. In *traditional* interpretation, P_1 and P_2 are usually assumed as situated in parallel flats and as translates of each other in a direction perpendicular to their plane, and the d-prism as being full-dimensional (that is, not contained in any flat of smaller dimension).

Thus, a *1-prism* is represented by a segment and its two endpoints. By induction, a *2-prism* is a quadrangle — in fact, *any* quadrangle is a 2-prism. The geometric realizations of 3-prisms have been discussed above. No specific results on d-prisms for $d \geq 4$ seem to be known. We have

Problem 1. Is every d-prism isotopic to a traditional one?

Problem 1 seems to have an affirmative answer for convex d-prisms, but even in this case no detailed proof seems to be available. For general 3-prisms the proof depends on the answer to the following open question:

Problem 2. If two oriented n-gons have the same rotation number and corresponding edges are pairwise parallel in the same direction, are they isotopic through polygons with the same properties ?

We recall that "isotopic" means that one polygon can be changed continuously to the other, while keeping at all intermediate stages the properties mentioned. Concerning rotation numbers see, for example, [7]. For convex polygons Problem 2 has obviously an affirmative answer. An affirmative answer for more general polygons would constitute a strengthening of the polygonal version of the Whitney-Graustein theorem (see, for example, [11]), which asserts that two polygons with the same rotation number are isotopic under some natural restrictions. (It is regrettable that the formulation in [11] uses the term "winding number" which in the context of curves and polygons has a completely different meaning and is irrelevant to the result.)

Problem 3. What are the conditions on isomorphic geometric (d-1)-polytopes P_1 and P_2 which make the construction of a d-prism possible?

3. Remarks and history. To illustrate the lack of serious consideration for prisms and antiprisms, here are some recent instances in otherwise rather nice and well-received texts. In all cases, the only consideration would be of regular prisms, which are polyhedra that satisfy the customary definition of Archimedean solids (also called "semiregular" by some authors) — all faces are regular polygons, not all congruent, and all vertices are "the same".

For example, prisms and antiprisms are not mentioned as belonging among the "semiregular polyhedra" although they should be since they are covered by the definition of given in [13]. In fact, antiprisms are not mentioned anywhere in this text.

In other texts, it is acknowledged that prisms and antiprisms should be listed among the Archimedean solids, but some *ad hoc* reasons are given for their exclusion. In [8, p. 328] the "excuse" is: they do not contain four faces that are contained in the faces of a regular tetrahedron. On the other hand, Cromwell [3, p. 157], quotes and accepts the "reason" Kepler [9] gave for exclusion of prisms and antiprisms (which he described but did not name). Kepler's justification (adapted here from [1, p. 101]) of the exclusion is based on his definition of Archimedean polyhedra, given by his Definition IX (italics and bracketed words are mine):

"Definition IX: A congruence [polyhedron] is perfect, but to a lower degree [than the regular polyhedra], when the plane figures [faces] are regular and all the angles [vertices] lie on the same spherical surface and are similar [congruent] to one another, but the faces are of various kinds, *though the number of each kind must be the same as the number of faces of one of the most perfect figures, that is, not less than four, which is the minimum number of planes to bound a solid figure*."

Obviously, the italicized condition is not natural — is seems to be there just in order to exclude prisms and antiprisms. As mentioned by Cromwell [3, p. 158], Kepler is inconsistent since he includes the snub polyhedra in his list of Archimedean ones although they do not satisfy this requirement (they have 32 and 80 triangles, respectively). It seems that Kepler himself felt that this second part of definition IX is not satisfactory, so he gives somewhat later (see [1, p. 102] another reason for the exclusion of prisms and antiprisms, formulated to apply to the solids shown in one of his figures — heptagonal prism and heptagonal antiprism:

"Note that I have excluded ... suchlike figures ... because only two heptagons are involved, and the figure formed is discus-shaped, like a plane, not globe-shaped, like a sphere."

One cannot escape the feeling that Kepler is working too hard to justify an arbitrary exclusion — or, possibly, to obtain the "right" number 13 of Archimedean polyhedra, which was reported by Pappus (see [3, p. 156]).

It may well be that because of his Definition IX Kepler excluded the pseudorhombicuboctahedron (the 14th "Archimedean" polyhedron) from his list. In an earlier work (see [3, p.152]) Kepler mentions that he found 14 Archimedean solids. In [9], this is reduced to the number 13 reported from antiquity. Indeed, when presenting the rhombicuboctahedron he shows that he is aware of the different roles played by some of the quadrangles. He says: " ... there are 8 triangles and 18 (that is, 12 and 6) quadrangles". For the pseudorhombicuboctahedron he would have to say " ... 18 (that is, 8 + 4 + 4 + 2)...", which is not compatible with Definition IX. By the way, Kepler makes similar comments when discussing the snub cube ("32, that is 20 and 12") and the snub dodecahedron ("80, that is 20 and 60"). Thus, although such considerations "legitimize" the snub cube, for the snub dodecahedron Cromwell's remark still applies.

References.

[1] E. J. Aiton, A. M. Duncan and J. V. Field, *The Harmony of the World, by Johannes Kepler*. Memoirs of the Amer. Philos. Soc., vol. 209, Philadelphia 1997.

[2] H. S. M. Coxeter, *Regular Polytopes*. 3rd ed. Dover, New York 1973.

[3] P. R. Cromwell, *Polyhedra.* Cambridge University Press1997.

[4] B. Grünbaum, Isogonal prismatoids. Discrete and Comput.

Geometry 18(1997), 13 – 52.

[5] B. Grünbaum, "New" uniform polyhedra. In: *Discrete Geometry: In Honor of W. Kuperberg's 60th Birthday*. Monographs and Textbooks in Pure and Applied Mathematics, vol. 253. Marcel Dekker, New York, 2003, pp. 331 – 350.

[6] B. Grünbaum, Are your polyhedra the same as my polyhedra ? In: *Discrete and Computational Geometry: The Goodman-Pollack Festschrift.* B. Aronov, S. Basu, J. Pach and M. Sharir, eds. Springer, New York 2003, pp. 461 – 488.

[7] B. Grünbaum and G. C. Shephard, Rotation and winding numbers for planar polygons and curves. Trans. Amer. Math. Soc. 322(1990), 169 - 187.

[8] J. Kapraff, *Connections*. McGraw-Hill, New York 1990.

[9] I. Keppler [J. Kepler], Harmonice Mundi. Lincii 1619.

[10] P. McMullen and E. Schulte, *Abstract Regular Polytopes*. Encyclopedia of Mathematics and its Applications vol. 92. Cambridge University Press 2002.

[11] K. Mehlhorn and C.-K. Yap, Constructive Whitney-Graustein theorem: Or how to untangle closed planar curves. SIAM J. Comput. 20(1991), 603 – 621.

[12] J. Miller, Earliest Known Uses of Some of the Words of Mathematics. http://members.aol.com/jeff570/mathword.html

[13] S. Stahl, Geometry: From Euclid to Knots. Prentice Hall, Upper Saddle River, NJ, 2003.

[14] M. Wenninger, Polyhedron Models. Cambridge University Press 1971.