(1-2-3)-COMPLEXES

Branko Grünbaum
Department of Mathematics, Box 354350
University of Washington
Seattle, WA 98195-4350
e-mail: grunbaum@math.washington.edu

Introduction. Polyhedra are most often considered as being determined by the planar polygons that are their faces. One of the characteristic properties of the faces of a polyhedron is that every edge is shared by two – and only two – faces. It seems that a question, very natural in this context, has never been asked – much less answered. The question is: Do there exist families of polygons in Euclidean 3-space E^3 such that each edge of every polygon is an edge of precisely three polygons in the family. Such a family will be called a **(1-2-3)-family** or **(1-2-3)-complex**. This name is meant to remind us that, in the language of complexes, for every **1-cell** (that is, edge) the number of **2-cells** (that is, faces) of which it is an edge is **3**.

Are there any such families? In the next section we shall show that the answer is affirmative, even if additional conditions are imposed. In the last section we shall mention various open problems related to this topic.

2. Examples of (1-2-3)-families. The simplest examples can be constructed starting with some of the well known polyhedra — the Platonic (that is, regular) convex polyhedra, and the Archimedean (also known as uniform or semi-regular) polyhedra.

The smallest example can be obtained from the regular octahedron (Figure 1) by noticing that every edge in contained in an "equatorial square". Hence the family consisting of the eight triangles of the octahedron, together with the three squares, is a

(1-2-3)-complex. From the Archimedean polyhedra such complexes can be obtained in the following cases (see Figure 2):

The cuboctahedron (3.4.3.4) together with four regular hexagons.

The rhombicuboctahedron (3.4.4.4) together with six regular octagons.

The icosidodecahedron (3.5.3.5) together with six regular decagons.

The rhombicosidodecahedron (3.4.5.4) together with twelve regular decagons.

All these examples have polygons that intersect each other along segments that are not edges of the polygons involved. Are there any **acoptic** (1-2-3)-complexes, that is, (1-2-3)-complexes without such intersections? (In other words, in acoptic complexes any two polygons that intersect must have either a common vertex, or else a whole edge in common.)

Here is a method to construct acoptic (1-2-3)-complexes. Start with any **3-valent** convex polyhedron P, that is, a polyhedron in which every vertex belongs to precisely three edges, and hence to three faces. (Such polyhedra exist in great profusion. Three of the Platonic polyhedra and seven of the Archimedean ones are 3-valent. For every even number n of vertices there exist 3-valent polyhedra with n vertices; moreover, the number of such

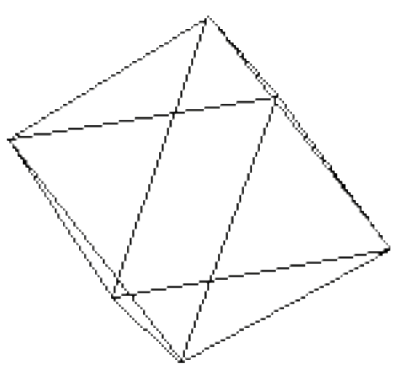
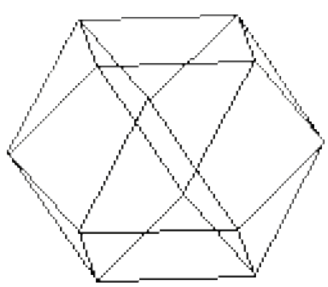


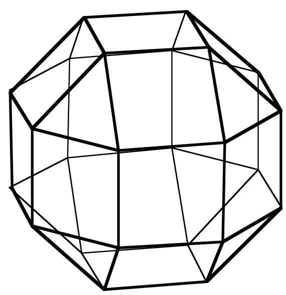
Figure 1. Octahedron {3,4}

polyhedra with n vertices increases very rapidly with n.) Next, from some interior point O of P make a uniformly expanded (stretched) copy $P^\#$ of P. Notice that any edge of P is parallel to the corresponding edge of $P^\#$, and the latter lies in the plane determined by the former and O. Finally, add to the polygons of P and of $P^\#$ the quadrangles (trapezes) determined by an edge of P and the corresponding edge of $P^\#$. The resulting family is a (1-2-3)-complex. Indeed, each edge of P or $P^\#$ is now in three polygons – the third one being one of the added quadrangles – and each other edge of the complex lies on three of the added quadrangles. The construction is illustrated in Figure 3 for the case P is a cube.

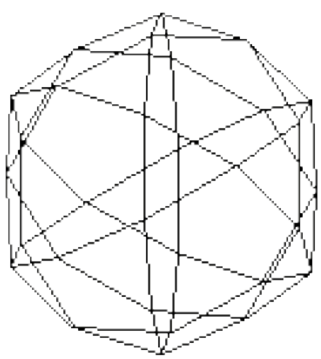
This construction can be considered as a special case of the following. Start with a 4-dimensional convex polytope in which



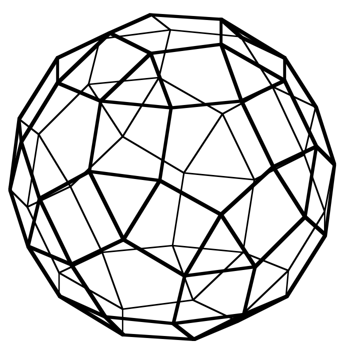
Cuboctahedron (3.4.3.4)



Rhombicuboctahedron (3.4.4.4)



Icosidodecahedron (3.5.3.5)



Rhombicosidodecahedron (3.4.5.4)

Figure 2.

This construction can be considered as a special case of the following. Start with a 4-dimensional convex polytope in which every vertex belongs to four 3-dimensional cells, and hence to four edges and six 2-dimensional faces. Using any of the 3-dimensional faces as a "window", form a "Schlegel diagram" of the polytope. The vertices, edges and polygons of this diagram form a (1-2-3)-complex. The previous construction corresponds to the case in which the 4-dimensional polytope is the prism with P as basis, and Figure 3 can serve as (the well known) Schlegel diagram of the 4-dimensional cube.

A different generalization of the above construction is less obvious, and we formulate the result as:

Theorem. The edges of every convex polyhedron P form a subgraph of the 1-skeleton of an acoptic (1-2-3)-complex.

Proof. We start by constructing $P^{\#}$ as before – but now adding quadrangles would not work if some vertex of P has valence greater than 3. In that case, we construct $P^{\#\#}$ by the following modification of $P^{\#}$: cut off each vertex of valence ≥ 4 by a plane sufficiently close to that vertex so that the various cuts do not affect each other or the 3-valent vertices. Thus every vertex

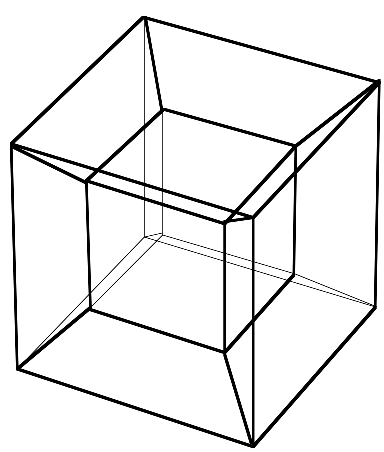


Figure 3.

of $P^{\#\#}$ is 3-valent, and each vertex of P of valence $k \geq 4$ has been replaced by k vertices which determine a k-sided face of $P^{\#\#}$. The k edges of such a face are joined by triangles to the vertex of P which led to them. All edges of P are joined to the corresponding edges of $P^{\#\#}$ (which may have been truncated once or twice). Adding the faces of P and of $P^{\#\#}$ completes the construction of the required (1-2-3)-complex. The construction is illustrated in Figure 4 in the case P is the octahedron..

In view of the well-known theorem of Steinitz (see, for example, [G, Section 13.1]], [2, Section 2.8], [3, Chapter 4]), this result is equivalent to saying that every 3-connected planar graph is isomorphic to a subgraph of the 1-skeleton of an acoptic (1-2-3)-complex. From this formulation, the following question arises naturally:

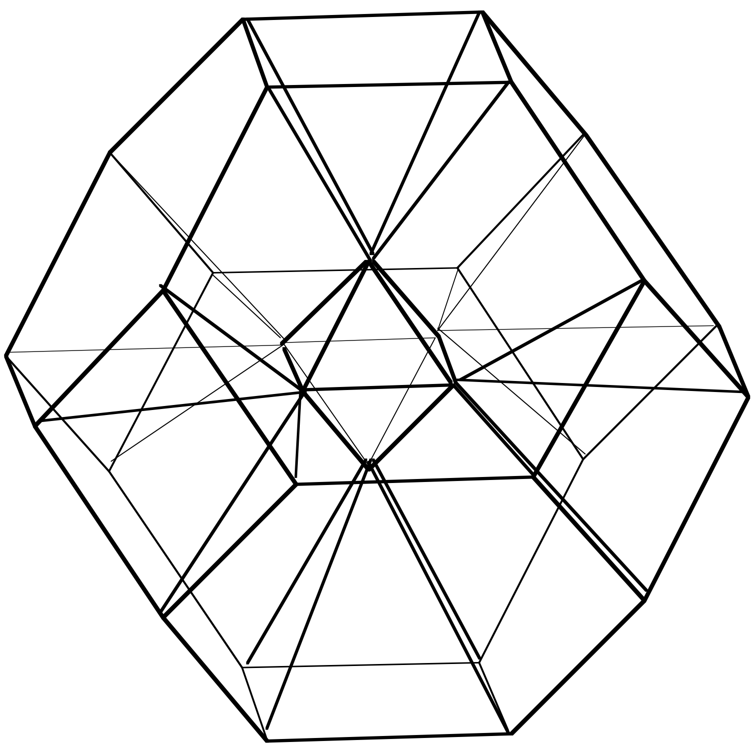


Figure 4.

Problem 1. Is every 3-connected graph is isomorphic to a subgraph of the 1-skeleton of an acoptic (1-2-3)-complex ?

The answer is affirmative if the complex is not required to be acoptic. The proof can be obtained by modifying the construction used in the proof of the above theorem; let $P^{\#}$ be obtained by suitable translation of an imbedding of the graph in E^3, and $P^{\#\#}$ by a suitable modification of the truncation step. If the problem has a negative answer, would it make a difference if the complex were allowed to be in 4-dimensional space ?

It is easy to verify that if the polyhedron P in the theorem has v vertices and e edges, then the (1-2-3)-complex constructed has fewer than $2v + e/2$ vertices.

Problem 2. Is there a better way to construct the (1-2-3)-complex, so that fewer vertices are required ?

Related topics. The definition of the complexes considered so far can be modified in various ways. Possibly the simplest it to declare a **(1-2-4)-complex** as a collection of polygons such that each edge of each polygon is an edge of precisely **four** of the polygons. A simple example is the Platonic icosahedron (Figure 5(a)), if to the twenty triangles one adds the twelve pentagons determined by its edges. Another example is obtained by adding to the twelve pentagrams of the small stellated dodecahedron $\{5/2, 5\}$ the twenty triangles determined by its edges (Figure 5(b)).

It is not hard to see that the edges of every 4-valent convex polyhedron are part of a (1-2-4)-complex in E^3.

Problem 3. Is every 4-valent planar graph isomorphic to a subgraph of the 1-skeleton of an acoptic (1-2-4)-complex in E^3 ? In E^4 ?

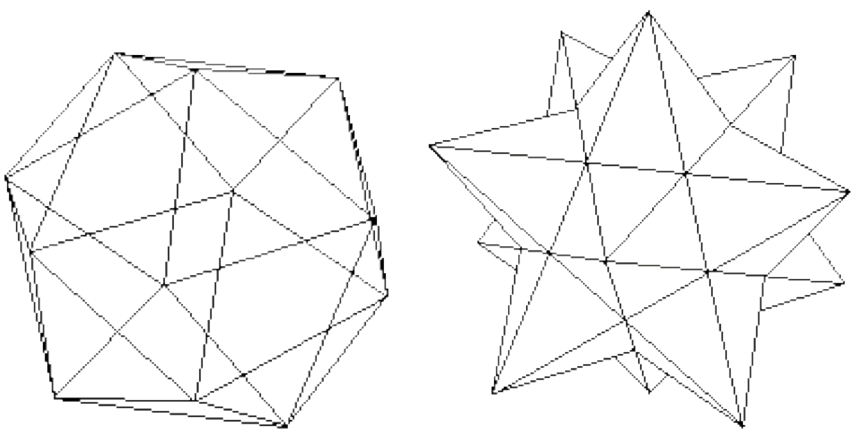
Similar questions can be asked for 4-connected graphs, with or without the planarity requirement.

References.

[1] B. Grünbaum, *Convex Polytopes*. 2nd edition, Springer, New York 2003.

[2] B. Mohar and C. Thomassen, *Graphs on Surfaces*. Johns Hopkins Univ. Press, Baltimore 2001.

[3] G. M. Ziegler, *Lectures on Polytopes*. Graduate Texts in Mathematics #152. Springer-Verlag, New York 1994, ix + 370 pp.



(a) Icosahedron {3,5} (b) Small stellated dodecahedron {5/2, 5}

Figure 5.

Comment (Added August 19, 2003).

Two papers that may seem relevant to the topic of this note came to my attention since its submission.

The first is the Web version of a paper [4] by Guy Inchbald. He considers certain polyhedra-like objects in which some of the edges are contained in three faces. The precise definition of these objects is not clear to me, but it is quite different from the one adopted here. Indeed, Inchbald does not admit the smallest example presented above (the eight triangles and three squares of the complex in Figure 1), but accepts the object obtained from it by deleting two disjoint triangles.

The second is a long paper by Jacek Swiatkowski [5]. It deals with polygonal 2-complexes in which each edge belongs to precisely three 2-faces. The center of attention are very symmetric complexes of this kind. However, despite the geometric connotation of the title, only purely combinatorial complexes and their symmetries are considered.

References

[4] G. Inchbald, Trimethoric (and trisynaptic) polyhedra. *Mathematics and Informatics Quarterly* 2/2001, Volume 11. Reproduced in:
www.queenhill.demon.co.uk/trimethoric/trimethoric.htm

[5] J. Swiatkowski, Trivalent polygonal complexes of nonpositive curvature and Platonic symmetry. *Geom. Dedicata* 70(1998), 87 – 110.