ON THE GEOMETRY OF MINKOWSKI PLANES

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The following propositions of elementary Euclidean geometry are well-known.

If D is the orthocenter of the triangle with vertices A, B, C, then each of the points A, B, C, D is the orthocenter of the triangle having as vertices the three other points. The circum-circles of the four triangles have all the same diameter.

In the present note we shall show that these and other propositions of Euclidean geometry remain, to some extent, valid also in Minkowski planes. Moreover, some of the results yield characterizations of centrally symmetric convex curves, or of ellipses, in terms of properties of triangles.

In the sequel C shall denote a bounded, closed, strictly convex and smooth curve in the plane, which has the origin 0 as center of symmetry. Any curve of the type $x + \lambda C$ (where x is a point and λ a positive real number) derived from C by similarity and translation, shall be called a Minkowski circle, or a circle, for short, with center x and radius λ. The union of $x + \lambda C$ and its interior shall be denoted by $x + \lambda D$ and called a disc.

The following facts are obvious for any Minkowski circle C.

1. Given any three non-collinear points there exists exactly one circle $x + \lambda C$ containing them.

2. If $x_1 \neq x_2$ then $(x_1 + \lambda_1 C) \cap (x_2 + \lambda_2 C)$ contains at most two points.

3. If $x_1 \neq x_2$ and $y_1, y_2 \in (x_1 + C) \cap (x_2 + C)$ with $y_1 \neq y_2$, then $x_1 + x_2 = y_1 + y_2$.

Using these properties we shall establish

Theorem 1. *Let* p_i, $i = 1, 2, 3, 4$, *be points in the plane, no three collinear, and let* $x_i + \lambda_i C$, $i = 1, 2, 3, 4$, *be circles such that* $p_i \in x_j + \lambda_j C$ *for all* $i \neq j$. *If* $\lambda_1 = \lambda_2 = \lambda_3 = 1$, *then* $\lambda_4 = 1$.

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THEOREM 2. *Let* p_1, p_2, p_3 *be distinct points of* C, *and let* $y_j + C$, $j = 1, 2, 3$, *be the three circles different from* C *each of which contains two of the three points* p_i. *Then* $\bigcap_{i=1}^{3} (y_i + C)$ *is not empty, and consists of precisely one point* (the C-orthocenter of the triangle with vertices p_1, p_2, p_3).

Since the two theorems are proved quite similarly (and also easily deducible from each other) we shall prove only the first one.

Proof of Theorem 1. Using the property 3 stated above, it follows from the assumptions of the theorem that $x_i + x_j = p_k + p_4$ whenever $\{i, j, k\} = \{1, 2, 3\}$. Let $x_0 = p_2 + p_3 - x_1$. Then $\{p_2, p_3\} = (x_1 + C) \cap (x_0 + C)$ and, since $x_0 + x_2 = p_1 + p_3$, also $\{p_1, p_3\} = (x_2 + C) \cap (x_0 + C)$. Therefore $\{p_1, p_2, p_3\} \subset x_0 + C$; since $\{p_1, p_2, p_3\} \subset x_4 + \lambda_4 C$, it follows from the above property 1 that $x_0 = x_4$ and $\lambda_4 = 1$. This ends the proof of Theorem 1.

Remark 1. From the above equations it follows that $\frac{1}{3} p_4 + \frac{2}{3} x_4 = \frac{1}{3} (p_1 + p_2 + p_3)$. In other words, the centroid of the triangle with vertices p_1, p_2, p_3 belongs to the segment determined by the center x_4 of the " circumcircle " $x_4 + C$ and by the intersection-point p_4 of the three circles obtained by " mirroring " $x_4 + C$ on the midpoints of the sides of the triangle; moreover, the centroid divides this segment in the ratio 1 : 2.

Remark 2. It is easily seen that each of the points p_1, p_2, p_3, p_4 is the C-orthocenter of the triangle determined by the other three points. If C is a Euclidean circle, the C-orthocenter coincides with the orthocenter, and the equation of Remark 1 expresses in this case the well-known relation between the centroid, the circumcenter and the orthocenter of a triangle; they determine Euler's line, which may, therefore, be generalized to Minkowski planes.

Remark 3. In both the Euclidean and the Minkowski case, the three points on Euler's line (centroid c, orthocenter h, and circumcenter r) of any triangle T may be " completed " by a fourth point $c^* = \frac{1}{3} r + \frac{2}{3} h$, which is the centroid of the associated triangle T*, congruent to T, whose vertices are obtained by mirroring the circumcenter of T in the midpoints of the sides of T. The above becomes particularly clear if the complete symmetry of the relationship between T and T* is noted; thus $(T^*)^* = T$, $r^* = h$; $h^* = r$.

The fact that we used the central symmetry of C in the proof of Theorem 1 is not accidental. Indeed, we have

THEOREM 3. *A strictly convex, smooth, closed curve* K *has a center of symmetry if (and only if) it has the following property. For any three (different) translates* K_1, K_2, K_3 *of* K, *no two of which are mutually tangent and all three passing through a common point* x, *there exists a translate* K_4 *of* K *passing through the three points of intersection* $K_i \cap K_j$, $i \neq j$, $i, j = 1, 2, 3$ *different from* x.

Proof. Given any chord of K there is a unique parallelogram inscribed in K which has the given chord as one of its sides. This parallelogram is degenerate exactly in the case when the (unique) supporting lines at the end-points of the chord are parallel. We shall show that the diagonals of a non-degenerate parallelogram are such " degenerate parallelograms ". Let the origin be in the center of the non-degenerate parallelogram, so that we may denote its vertices by a, b, $-a$ and $-b$. Suppose that the diagonal $[a, -a]$ is a side of another non-degenerate parallelogram inscribed in K, whose other two vertices we may denote by $a + c$ and $-a + c$. Put $K_1 = K$, $K_2 = K + a - b$ and $K_3 = K + 2a$. These three translates all intersect at the point a, and so by the conditions of the theorem there must be a fourth translate K_4 passing through the points $-b$, $2a - b$ and $a + c$, which belong respectively to $K_1 \cap K_2$, $K_2 \cap K_3$ and $K_3 \cap K_1$. Thus, the translate $K_5 = K_4 - a + b$ passes through a, $-a$ and $b + c$, which means that one has either $K_5 = K$ or $K_5 = K - c$. The first case is impossible, since it would imply $b = a$ or $b = -a$. The second case would mean that $b + 2c \in K$. We then repeat the whole argument once more with $K_2' = K - a + b$ and $K_3' = K - 2a$ instead of K_2 and K_3 and find that also $-b + 2c \in K$. This is absurd, hence we have proved that the diagonals of any parallelogram inscribed in K are thamselves degenerate inscribed parallelograms, i.e. that they connect the points of contact between K and two parallel supporting lines. Consider now two such chords in K. Take a chord connecting an endpoint of one of the previous chords with an endpoint of the other and construct its parallelogram. Then, by the above, the diagonals of this parallelogram have the parallel tangent line property, hence they coincide with the two original chords. But now we have proved the theorem, since we have constructed a center of symmetry for K, namely the common center of all its inscribed parallelograms.

Theorem 3 may be thought of as the converse of Theorem 1. In the same way Theorem 2 has a converse, which is easily deducible from the three preceding theorems.

THEOREM 4. *Let* K *be a strictly convex, smooth, closed curve. Suppose that* K *has the property that whenever four of its translates* K_i, $i = 1, 2, 3, 4$, *satisfy the conditions that* $\bigcap_{j=1}^{4} K_j$ *is empty but* $\bigcap_{j \neq i} K_j$ *are non-empty for* $i = 1, 2, 3$, *then* $\bigcap_{j \neq 4} K_j$ *is also non-empty. Then* K *has a center of symmetry.*

Proof. Take three translates K_1, K_2 and K_3 of K that satisfy the conditions for Theorem 3. Suppose moreover, that the chord in K_1 which connects the intersection points of K_1 with K_2 and K_3 respectively which are different from the triple intersection, is not a chord whose endpoint tangents are parallel. Let K_4 be the unique translate of K different from K_1 which also contains this segment as a chord. By the conditions of Theorem 4, K_4 passes through the remaining double intersection point of K_2 and K_3. However, the above mentioned chord in K_1 is never of the "degenerate parallelogram" type, since if it were, we could find the desired translate K_4 passing through the intersection points of K_1, K_2 and K_3 outside of $K_1 \cap K_2 \cap K_3$ by a passage to the limit. Hence Theorem 3 is applicable and we have proved Theorem 4.

Remark 4. In distinction from theorems of a similar nature given in [6, 7, 8], the properties used in Theorems 3 and 4 to characterize centrally symmetric convex curves K make no reference to the point which is to be shown to be the center of K. The characterization in [6], which may equivalently be formulated as " There exists a point x such that each point of K is the vertex of an affine-regular hexagon with center x, all of whose vertices belong to K ", fails if the centers of the hexagons are not assumed to be fixed. Indeed, the curve $K(\varepsilon) = \{(\sin\varphi; \cos\varphi + \varepsilon(1 - \cos 6\varphi); 0 \leq \varphi \leq 2\pi\}$ is easily seen to be convex for sufficiently small positive ε, not to have a center for $\varepsilon > 0$, and to allow an inscribed regular hexagon (of side 1) to rotate in it. (Similar curves were studied in [4].)

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The notion of the Feuerbach (or " nine-points ") circle of a triangle also (partially) generalizes to Minkowski planes. The Feuerbach circle (in a Minkowski plane) of a triangle with vertices x_1, x_2, x_3 and circumcircle C is the circle $\frac{1}{2}(x_1 + x_2 + x_3) + \frac{1}{2}C$.

THEOREM 5. *In any Minkowski plane, the Feuerbach circle of a triangle passes through six " remarkable " points; the midpoints of the sides of the triangle, and the midpoints of the segments determined by the* C*-orthocenter and the vertices.*

Proof. The theorem may be established by an easy computation. Indeed, since $x_3 \in C$, the midpoint $\frac{1}{2}(x_1 + x_2)$ of the opposite side of the triangle satisfies $\frac{1}{2}(x_1 + x_2) = \frac{1}{2}(x_1 + x_2 + x_3) - \frac{1}{2}x_3 \in \frac{1}{2}(x_1 + x_2 + x_3) + \frac{1}{2}C$, and similarly for the two other midpoints. On the other hand, for the midpoint $\frac{1}{2}x_i + \frac{1}{2}(x_1 + x_2 + x_3)$ of the C-orthocenter and a vertex we have, obviously, $\frac{1}{2}(x_1 + x_2 + x_3) + \frac{1}{2}x_i \in \frac{1}{2}(x_1 + x_2 + x_3) + \frac{1}{2}C$; this ends the proof of the theorem.

Remark 5. As in the Euclidean case, it is easily established that the four triangles derived from a given triangle T and its C-orthocenter, have the same Feuerbach circle; it is also the Feuerbach circle of the four triangles derived from the " associated " triangle and its C-orthocenter.

In Euclidean geometry the following property of the Feuerbach circle is easily established:

(*) The Feuerbach circle of any triangle passes through the three intersections of a side of the triangle with the line determined by the opposite vertex and the orthocenter.

THEOREM 6. *The only Minkowski planes with the property* (*) (with C-orthocenter substituted for orthocenter) *are those whose circles are ellipses.*

Proof. It is well-known ([1], p. 143) that ellipses are the only centrally symmetric convex curves with the following property:

(**) The midpoints of any pair of parallel chords are collinear with the center.

We shall show that property (*) implies (**). Let $y_i \in C$, $i = 1, 2, 3, 4$, be four points such that the chord with endpoints y_1 and y_2 is parallel to that with endpoints y_3 and y_4. Then $x_i = y_1 + y_2 + y_3 - 2y_i$, for $i = 1, 2, 3$, are vertices of a triangle with circumcenter $r = y_1 + y_2 + y_3$ and circumcircle $r + 2C$, whose Feuerbach circle is C and whose C-orthocenter is $h = -r = -(y_1 + y_2 + y_3)$. Now if y_4 is (as assumed in (*)) the intersection of the line determined by x_3 and h with that determined by x_1 and x_2 (which also contains y_3 and is parallel to the chord y_1, y_2), the collinearity of $\frac{1}{2}(y_1 + y_2)$ and $\frac{1}{2}(y_3 + y_4)$ with 0 follows from the fact that the lines determined by x_3 and h, by $\frac{1}{2}(y_1 + y_2)$ and 0, by r and y_3 are parallel, and $h = -r$. Thus (*) implies (**) and Theorem 6 is proved.

A great number of theorems in the geometry of circles in the Euclidean plane remain valid in Minkowski geometry if it is assumed that all the circles are of the same size. As an example we cite the following theorem, due to Miquel for Euclidean circles of arbitrary sizes ([3], pp. 86/87):

THEOREM 7. *Let four points* x_i *of* C *be given and let* C_i, $i = 1, 2, 3, 4$, *be the four translates of* C *(different from* C*) determined by pairs of neighboring points. Then there exists a translate of* C *containing the four points* y_i, *where* $y_i \in C_i \cap C_{i+1}$, $(C_5 = C_1)$, *but* $y_i \notin C$.

The proof of Theorem 7 is very similar to that of Theorem 1 and we omit it. The circle containing the points y_i is $x_1 + x_2 + x_3 + x_4 + C$.

Some results transfer verbatim from the Euclidean to the Minkowski case (usually because the equality of size is assumed, explicitly or implicitly, in the Euclidean case). An example of this kind is a " chain of theorems " due to Coolidge [2] (reproduced in [3], p. 94).

* * *

A. Florian [5] mentions, in an account of some unpublished results of J. Molnár, the following proposition: If a circle C in the Euclidean plane is covered by the union of three circular discs D_i, $i = 1, 2, 3$, of diameters not exceeding that of C, then the disc D, bounded by C, is also covered by $D_1 \cup D_2 \cup D_3$.

We shall prove for Minkowski planes

THEOREM 8. *If* $C \subset \bigcup_{i=1}^{3} (x_i + \lambda_i D)$ *and* $\lambda_i \leq 1$ *for* $i = 1, 2, 3$, *then* $D \subset \bigcup_{i=1}^{3} (x_i + \lambda_i D)$.

Proof. Assuming $C \neq x_i + \lambda_i C$ for all i, let p_1, p_2, p_3 be points of C such that $p_i \in (x_j + \lambda_j D) \cap (x_k + \lambda_k D)$ for $\{i, j, k\} = \{1, 2, 3\}$. We define $y_i = p_j + p_k$ for $\{i, j, k\} = \{1, 2, 3\}$. By property 2 (p. 300), it follows that $D \cap (x_i + \lambda_i D) \supset D \cap (y_i + D)$. On the other hand, the points p_1, p_2, p_3 and the circles C, $y_1 + C$, $y_2 + C$, $y_3 + C$ satisfy the conditions of Theorem 2. Therefore, there exists a point $p \in \bigcap_{i=1}^{3} (y_i + C)$. To complete the proof we have only to show that $p \in D$; then, since each point of D belongs to a segment with endpoints p and some $x \in C$, and each such segment is contained in one of the discs $y_i + D$, it follows that D is contained in $\bigcup_{i=1}^{3} (y_i + D)$, and thus also in $\bigcup_{i=1}^{3} (x_i + \lambda_i D)$, as claimed. But if $p \notin D$ is assumed, a contradiction is readily reached: Let L be the line determined by p and 0, and let p^* be the point of $L \cap C$ with the greater distance from p. Since $p^* \in C$, for a suitable i we have $p^* \in y_i + D$. But $p \in y_i + D$ which is impossible since the segment with endpoints p and p^* is longer than the diameter of D parallel to it, and therefore may not be covered by any translate of D.

This ends the proof of Theorem 8.

Obvious examples show that the restriction $\lambda_i \leq 1$ in Theorem 8 may not be omitted.

Remark 6. It is easily seen that Theorem 8 is valid also if the circle C is not assumed to be *strictly* convex and smooth. The argument is completely elementary but somewhat lengthy, and we omit it. On the other hand, Theorems 1 qnd 2 have to be properly reformulated in order to be applicable (and valid) for circles which are not strictly convex and smooth.

Remark 7. It is easily seen that Theorems 1 and 2 do not generalize to higher-dimensional spaces. Theorem 8 is probably valid for spaces of any dimension (with $n + 1$ " solid " spheres covering the surface of another one in the n-dimensional case), although no proof seems to be known even in the case of Euclidean spheres in three-dimensional space.

Note. After the present note was completed, the paper " Zur elementaren Dreicksgeometrie in der komplexen Ebene " (*Enseign. Math.*, *4* (1958), 178-211), by J. E. Hofmann, came to our attention. In this paper the geometry of triangles in the Euclidean plane is developed (in part) in a way closely related to the method used in the present paper.

REFERENCES

1. Bonnesen, T. and W. Fenchel, *Theorie der Konvexen Körper.* Berlin, 1934.
2. Coolidge, J. L., Circles associated with concyclic points. *Ann. of Math.*, *12* (1910), 39-44.
3. —— *A treatise on the circle and the sphere.* Oxford, 1916.
4. Fischer, H. J., Kurven, in denen ein Zug von Sehnen gleicher Länge sich unabhängig vom Ausgangspunkt schliesst. *Deutsche Math.*, *4* (1939), 228-237.
5. Florian, A., Zum Problem der Überdeckung einer Kugel durch Kugeln. *Monatsh. f. Math.*, *63* (1959), 351-355.
6. Kelly, P. J., A property of Minkowskian circles. *Amer. Math. Monthly*, *57* (1950), 677-678.
7. Süss, W., Über Eibereiche mit Mittelpunkt. *Math.-Phys. Semesterber.*, *1* (1950), 273-287.
8. Viet, U., Umkehrung eines Satzes von H. Brunn über Mittelpunktseibereiche. *Math.-Phys. Semesterber.*, *5* (1956), 141-142.

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