Families of point-touching squares

Branko Grünbaum
Department of Mathematics, Box 354350
University of Washington
Seattle, WA 98195-4350
e-mail: grunbaum@math.washington.edu

The literature contains many investigations dealing with questions concerning familes of nonoverlapping sets. Even so, the recent paper [1] by E. Friedman introduces an apparently new type of interesting problems. In order to formulate them we need some definitions. Throughout the paper, without additional mention, we shall assume that each family C considered consist of a finite number of nonoverlapping copies of a convex body C.

A family C is said to be **k-touching** provided each element of C has nonempty intersection with precisely k other elements of C. Figure 1 shows an example of a 3-touching family of circles.
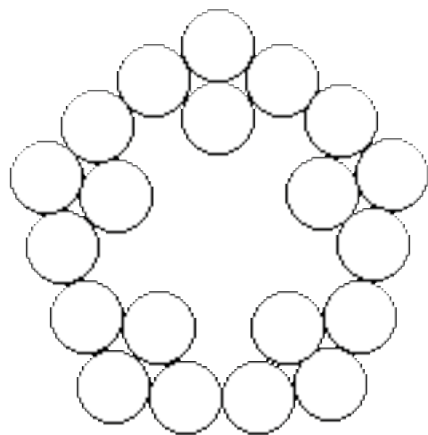
Figure 1. A 3-touching family of 20 circles.

It is easy to demonstrate that there are no 4-touching families of circles.

A k-touching family $\mathcal{C}$ is said to be **point-k-touching** if any two touching sets have only a single common point; $\mathcal{C}$ is **segment-k-touching** if each touching pair has a segment of positive length in common. The family in Figure 1 is point-3-touching. Most of [1] deals (in slightly different terminology) with segment-touching families of squares. Friedman also gives a diagram (shown in Figure 2 below) said to represent (in our terminology) a 4-touching family of squares. It consists of 50 squares, and Friedman conjectures that it is the smallest possible one.

Our first goal is to provide a negative answer to this conjecture, using the 4-touching family of 25 squares shown in
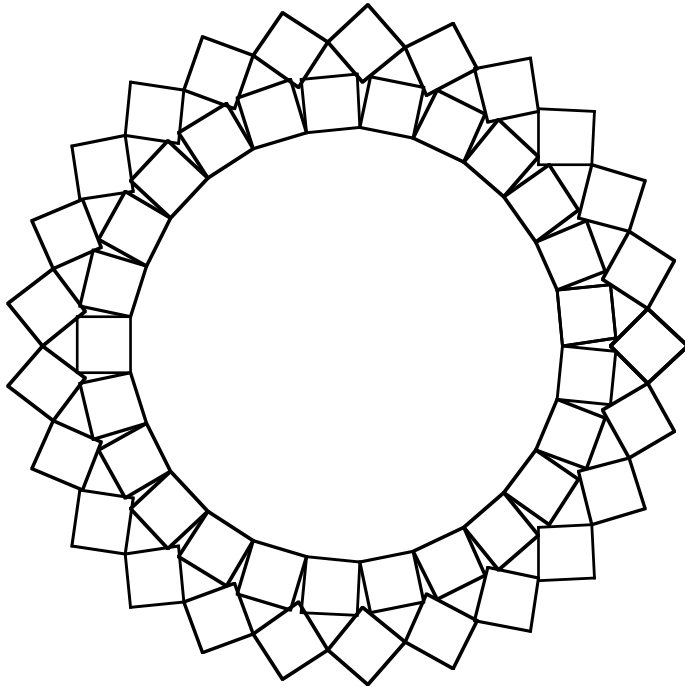
Figure 2. Friedman's family of 50 squares.

Figure 3. It may be conjectured that there is no 4-touching family of squares with fewer than 25 members.

On the other hand, one might think that Friedman's example is the smallest possible point-4-touching family – which may in fact have been Friedman's intention. However, this is not the case, for two reasons. First, this is not a valid example: despite appearances, congruent squares cannot touch in the manner that is implied by Figure 2.

To establish this impossibility, we consider Figure 4, in which the squares have side 2. The notation is self-explanatory, except for the clarification that F denotes a vertex of one of the squares, while G is on the line of symmetry and is a vertex of the right triangle OEG. If we assume that there is a total of $2n$ squares in the family (so that $n = 25$ as in Figure 2) then we can easily verify that $\alpha = 180/n$, $OA = \sqrt{2}/\sin \alpha$, $\beta = 45 - \alpha$, $AD = 1/\tan \beta$, $OE = OA - AD - 2$, $EG = OE \tan \alpha$. If the proposed
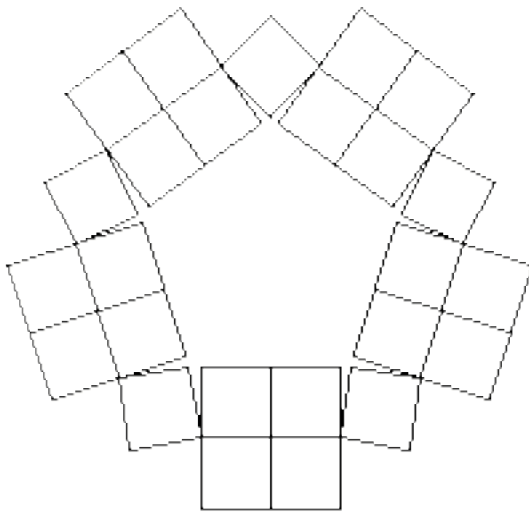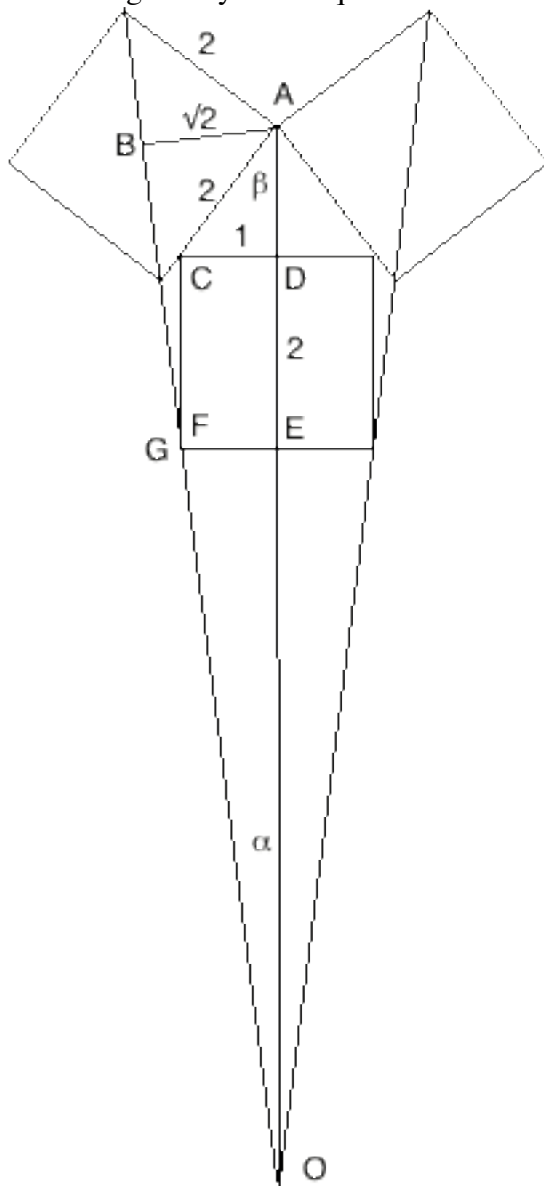
Figure 3. A 4-touching family of 25 squares.

Figure 4. The notation used in proving the impossibility of the
purported arrangement in Figure 1.

arrangement of squares is to work as suggested by the diagram, then we would have $G = F$, so that $EG = 1$.

However, carrying out the calculations shows that (with $n = 25$) $EG = 1.0099321 \neq 1$. This means that the squares of the inner ring do not touch. Naturally, the difference of less than 1% is too small to be noticed, or to interfere with the impression of touching given by Figure 2.

If the calculations are carried out with $n = 24$, the result is that $EG = 0.9915389 < 1$, hence the squares of the inner ring overlap. Again, the difference is less than 1%, hence a diagram as convincing as Figure 2 could be drawn with only 48 squares.

The second reason for the failure of the conjecture is that there do exist 48-member families of point-4-touching squares. An example is shown in Figure 5. Suitable "tilting" of the squares in the inner ring leads to the elimination of the overlap mentioned above. However, this method does not work for $n \leq 23$. It is not clear whether there are any point-4-touching families with fewer than 48 squares.

Clearly, there are many related problems that seem to be of sufficient interest to warrant investigation. In particular, are there any 5-touching families? The affirmative answer follows from Figure 6. What is the largest k for which there exist point-k-touching families of equilateral triangles? Figure 7 shows that $k \geq 4$. Is $k = 5$ possible? Is it possible for any other triangles? It may be conjectured that there is no 4-touching family $\mathcal{C}$ of convex sets C with smooth boundary. I conjecture that the same holds for any family $\mathcal{C}$ in which no three elements have a common point.

What happens if the convexity of C is not required? If the concepts are considered in 3- or higher-dimensional spaces?

Some final remarks seem in order. Computer graphics, as well as other uses of computers, are very useful in discovering facts. Sometimes, a diagram is essentially all that is needed to see the validity of a claim. (Figures 1, 3, 6, and 7 are examples of this situation, since only obvious arguments based on symmetry and continuity are needed to prove that the circles or squares touch as suggested by the diagrams.) However, in many other situations additional arguments are needed to distinguish between apparent and actual relations.
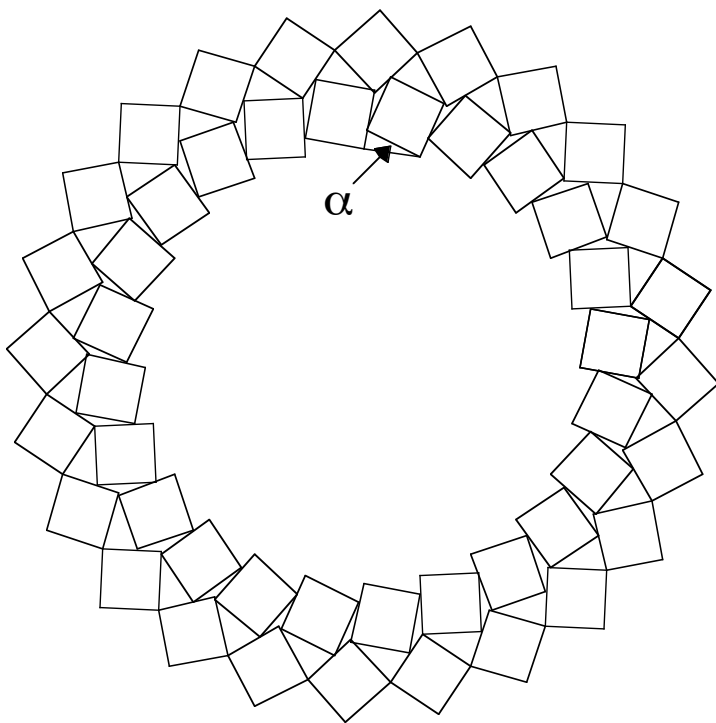

Figure 5. A point-4-touching family of 48 squares.

In the case of Figure 2, there are more incidences that need to happen than can be expected on general grounds: there are no degrees of freedom, and the only variable is the number of squares. While in certain cases such supernumerary and unexpected incidences do occur, the claim that this happens needs full justification; in the present case, the incidences just do not occur.

On the other hand, the diagram in Figure 5 can be justified. Detailed calculations (using Mathematica® software) show that the suggested incidences happen when the tilt angle α equals 18.14510793…°. The many decimals in this value of α may be comforting – but the existence of an appropriate value can be inferred by continuity from the fact that some values of the tilt angle lead to overlaps while other values lead to separation of the squares in the inner ring.
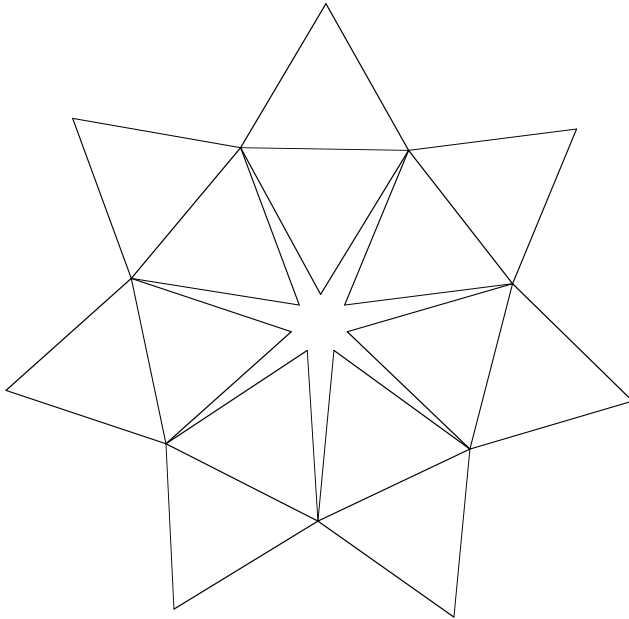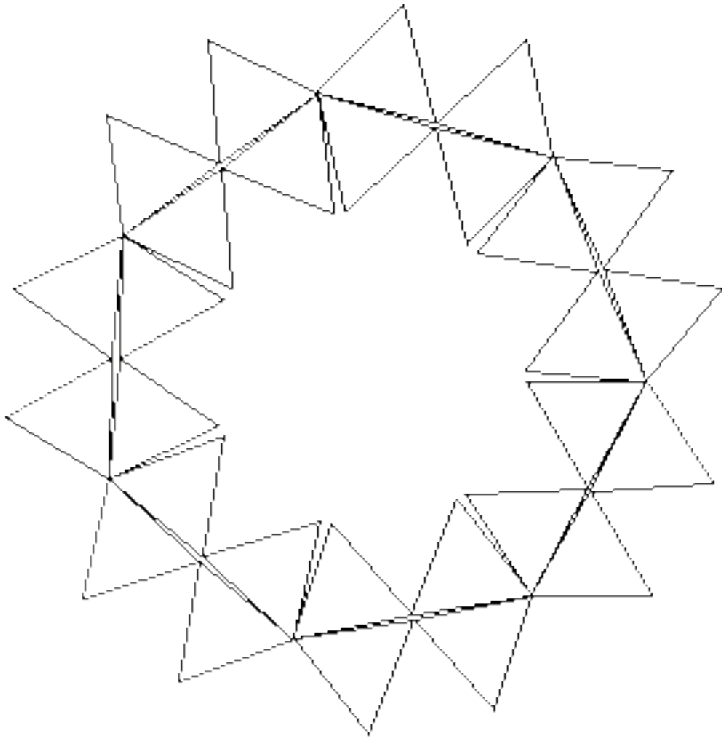

Figure 6. A 5-touching family of equilateral triangles.

Figure 7. A point-4-touching family of 28 equilateral triangles.

Reference.

[1]   E. Friedman, Squares touching a constant number of other squares. *Geombinatorics* 12(2002), 55 – 60.