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NO-NET POLYHEDRA

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Dedicated with admiration and love to H.S.M. Coxeter on his 95th birthday

1. Introduction. It is often said that when the conditions are right, discoveries are made by different people at the same time. The discovery of the non-Euclidean geometry, and the invention of calculus are the most frequently mentioned examples. Here is another example (with an incomparably less significant subject) which I noticed recently. My note [4] described a starshaped polyhedron which admits no "net", as this term is understood by anybody building cardboard models of polyhedra. The note was motivated by the paper of Tarasov [7] published in 1999. Tarasov constructed a no-net starshaped polyhedron with convex faces, and thereby solved a problem posed by N. G. Dolbilin. (Throughout this note, we consider only starshaped polyhedra with convex faces. All nets are assumed to be obtained by cuts along the edges of the polyhedra in question.)

After the publication of [4] I became aware of the paper [1], also published in 1999, in which a different no-net polyhedron is constructed.

While preparing the present note, a message from Craig Kaplan led me to the more detailed account [2] of the results of [1]; this seems not to have been published in print so far, but is available on the Internet. Various no-net polyhedra are presented there, including one with only 24 faces. Another result of [2] is an affirmative resolution of Conjecture 2 of [4], establishing the existence of no-net polyhedra with only triangles as faces.

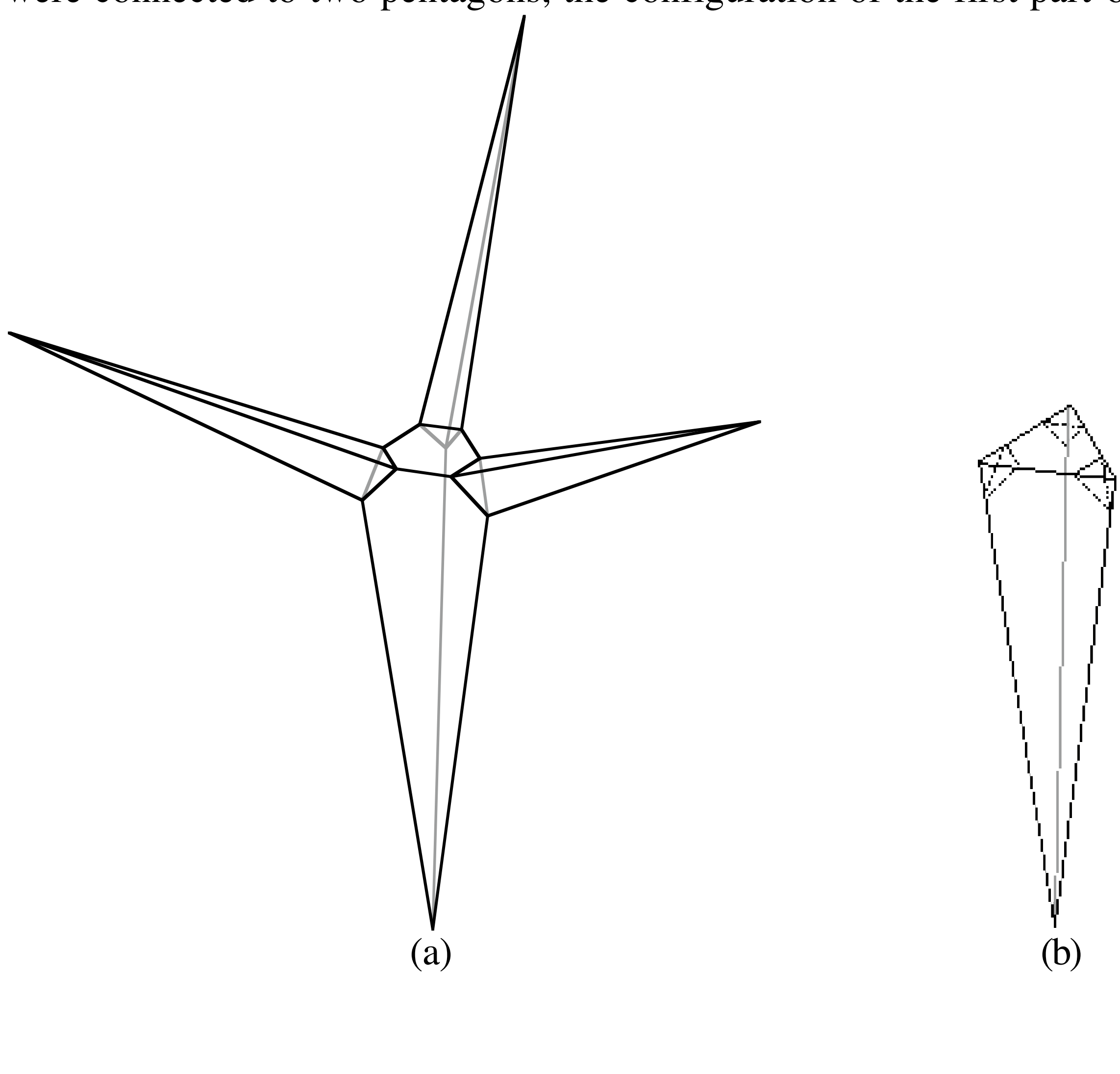
The main purpose of this note is to present a no-net polyhedron with only 13 faces; this is done in Section 2. Section 3 discusses certain related questions, as well as some other remarks.

2. A small no-net polyhedron. We shall show:

Theorem. There exist convex-faced, starshaped no-net polyhedra with 13 faces.

Proof. A polyhedron establishing this result is shown in Figure 1(a). It is constructed as follows. We start from a sufficiently tall pyramid, with an equilateral triangle as basis. This is truncated at each of the basis vertices, to obtain a polyhedron with seven faces — three triangles, three pentagons and one hexagon. This step is shown in Figure 1(b), where the pyramid is pointed downwards for easier visualization. The construction is completed by replacing each of the three triangles by the mantle of a tall pyramid, resulting in the polyhedron of Figure 1(a).

To complete the proof we have to show that this polyhedron does not have a net. Due to the small number of faces, this is very easy. We note first that the hexagonal face could be connected in the net to one or more pentagonal faces either directly, or via the triangles. However, as evident from the first part of Figure 2, the hexagon would overlap any pentagon to which it would be connected via triangles. (Note that, for simplicity, the edges of the pentagons that meet at the apex of the pyramid are drawn as parallel, as are the long sides of the triangles. Since the overlap has interior points, any sufficiently tall pentagons and triangles would would lead to the same kind of overlap.) Similarly unacceptable is connecting two of the pentagonal faces via triangles; this is demonstrated by the second part of Figure 2. It follows that the hexagon and the pentagons mut be connected (in a suitable arrangement) to each other. However, this is impossible as well. Indeed, if the hexagon were connected to two pentagons, the configuration of the first part of



(a) (b)

Figure 1. (a) A no-net polyhedron with 13 faces. (b) The start of the construction of the no-net polyhedron in part (a).

Figure 3 would result, and there would be no way to attach two of the three triangles incident with the three other faces along the heavily drawn edges. But if the hexagon is directly connected to a single pentagon, then the configuration of the second part of Figure 4 is unavoidable, with the same impossibility. Thus, the polyhedron has no net, and the theorem is established.

3. Comments.

(i) The conjecture that every convex polygon has a net (which I made in [3] in 1991) was made earlier (in 1987) by Catherine Schevon and Joseph O'Rourke; see [5] (probably also made in [6], which I had no opportunity to see). It is still open and there is no apparent method of attacking it.

(ii) The proof of the Theorem in Section 2 is similar to the proof presented in [4]. Due to the small number of faces of the polyhedra in question, it is simpler than the proof in [4]. We may make:

Conjecture 1. Every convex-faced starshaped polyhedron with at most 12 faces has a net.

In other words, such polyhedra are not no-net polyhedra. Translated in the terminology used in [1] and [2], where no-net polyhedra are said to be "ununfoldable", one may rephrase Conjecture 1 as

Conjecture 1*. All convex-faced starshaped polyhedra with at most 12 faces are unununfoldable.

(iii) It is well known that every net of any *convex* polyhedron arises by cuts that form a tree in the graph of edges and vertices of the polyhedron. However, it appears to have been first noted in [2]

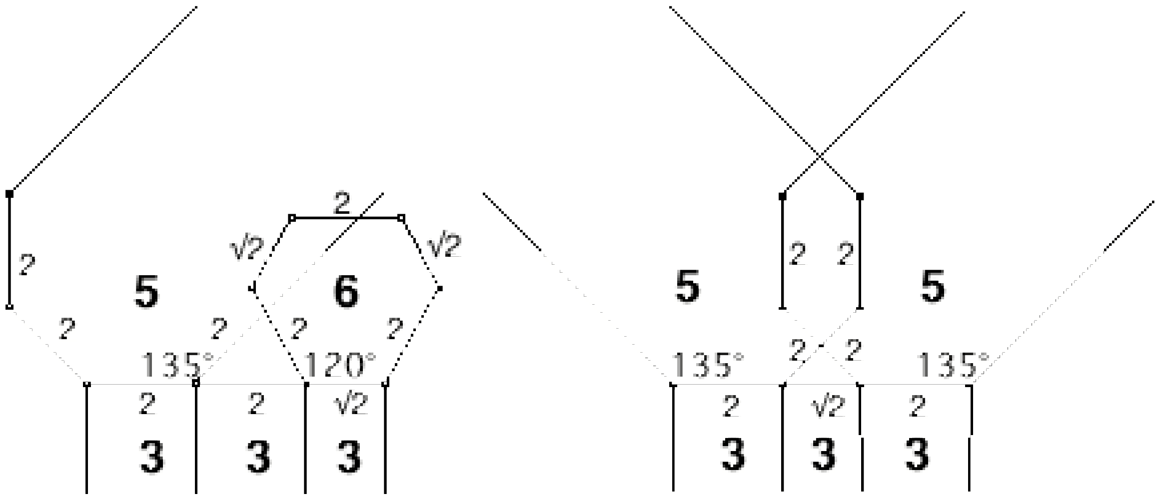


Figure 2. The hexagon and the pentagons cannot be connected via triangles.

that this is not always the case with starshaped polyhedra with convex faces. The cuts may form a forrest with more than one connected component. Clearly, in such a case the net itself is not a simply-connected polygon, but a polygon with one or more polygonal "holes". (The diagrams in Figure 2 of [2] are incomplete, but the omissions can be corrected easily.) The example in [2] has 8 faces, and — taking into account Conjecture 1 — we have:

Conjecture 2. Every convex-faced starshaped polyhedron with at most 7 faces has a net generated by a cut along the edges of a tree.

Conjecture 3. If a convex-faced starshaped polyhedron has a net, it has a net generated by a cut along the edges of a tree.

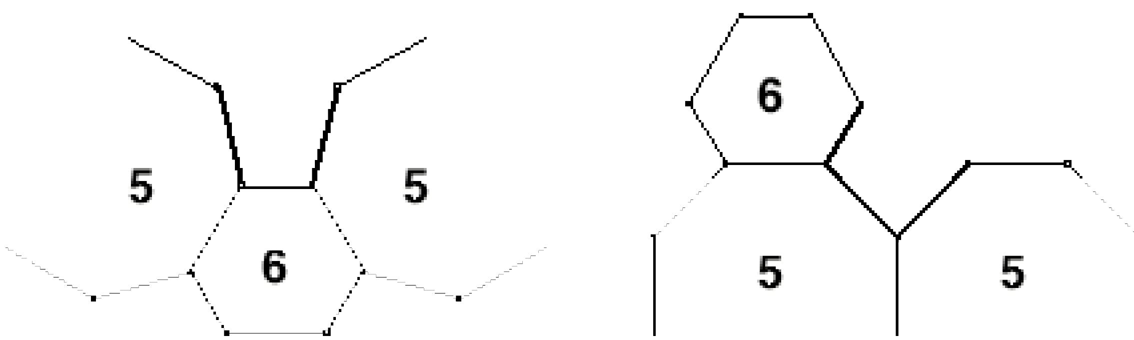


Figure 3. The direct connections of the hexagon and the pentagons are impossible as well.

References.

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