Levels of Orderliness: Global and Local Symmetry

Branko Grünbaum

Department of Mathematics

University of Washington

Seattle, WA 98195-4350

U.S.A.

e-mail: grunbaum@math.washington.edu

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Despite the great usefulness of the crystallographic (that is, discrete) groups, characterizing symmetry through group-theoretic criteria has shortcomings of several kinds. However, just as the developments in 20th century physics modified Newtonian mechanics without destroying its applicability to many everyday phenomena, so too the concept of symmetry as understood in light of group theory can and should be extended without being abolished. The modification should be made in appropriate ways, depending on the context. Clearly, not every extension is suitable in every case. To avoid confusion, instead of *symmetry* we shall speak of more general *orderliness*, which reduces to group-theoretical symmetry in many — but far from all — cases. Here we shall briefly discuss three possibilities. They all substitute local orderliness conditions instead of making assumptions about global symmetry. To simplify the exposition we shall, without further notice, restrict attention to the Euclidean plane; similar extensions might be possible in higher dimensions, but seem not to have been attempted so far.

The first generalization is obtained by the process of *inflation–and–decomposition*. In the simplest case this starts with a closed, simply connected region ("tile") which has such a shape that a copy of it, expanded in a suitable ratio, can be decomposed into several congruent copies of the original tile. Such tiles have been called *k-rep tiles* if the enlarged version can be decomposed into k tiles of the starting size. For example, a square is clearly a n^2-rep tile for every positive integer n. Less trivial is the case of a 1-by-$\sqrt{k}$ rectangle, which is a k-rep tile for every positive integer k. Another example is the L-tromino (Figure 1), which is a 4-rep tile. The definition of inflation–and–decomposition can be generalized by allowing, first, to start with a *family* of tiles which, upon magnification by a factor f, can each be decomposed into *families* of tiles congruent to some of the original ones. An example (due to the late Robert Ammann) with two tiles is shown in Figure 2. Here f is the "golden ratio" □ $= (1+\sqrt{5})/2 = 1.618...$, and the dimensions of the tiles are as indicated.

A variant of this process provides the starting tile or tiles with some markings or labels, and possibly with rules which impose restrictions on the placement of tiles of the decomposition. For example (see Figure 3), if the enlarged tile could be tiled by copies of the original in several ways, we may insist on the particular one shown.

The inflation–and–decomposition process can be applied in many ways. For example, the usual square tiling with a checkerboard coloring may be inflated by a factor $f = 2$, with the additional provision that all tiles in the decomposition of any tile receive the color of the original tile, see Figure 4; this kind of tiling is often seen on floors.

The inflation-and-decomposition can be used on any family of tiles, whether they constitute a tiling of the plane, or only of a part of it. It can be used once only, or several times, or, with some safeguards to guarantee the existence of a limit, indefinitely often. A simple example of a double use can be seen in Figure 5; pavements of this kind can be seen in many places in Morocco.

On the other hand, indefinite repetition can lead to non-periodic tilings, even starting from a single tile. An illustration using repeatedly the tromino inflation of Figure 1 is shown in Figure 6. With each inflation, the inflated tile is turned 90° counterclockwise, and made to have the original as one of the tiles of its decomposition. Another example is given in Figure 7, where four inflation-and-decomposition steps of the larger tile from Figure 2 have been carried out; the third and fourth stages of decomposition are shown. The tiles have been marked by the black marks as shown, and the requirement in decomposition is that the tiles be arranged in such a way that the black marks form diamonds. It can be shown that any tiling with copies of these two tiles, with this "matching" condition, has no translational symmetry. Hence this set of two tiles, with the matching condition indicated, is a so-called "aperiodic set". Another aperiodic example of two tiles, with an appropriate matching condition, are the well-known Penrose tiles. Aperiodic, and other non-periodic tilings, are of great interest in some modern crystallographic studies, as explained by Dr. Shechtman in his Aminoff lecture. In tilings with the Ammann tiles, as well as in those with the Penrose tiles, the two tiles occur with frequencies in ratio □: 1. However, for other sets of aperiodic tiles the ratio of frequencies does not involve □; for example, the ratio $\sqrt{2}:1$ appears in an aperiodic tiling described in [1]. The interested reader can find additional information in many of the published discussions of aperiodic or quasiperiodic tilings. We mention specifically [1 – 3]; an amusing recent presentation is [4].

A second kind of generalization of symmetry-by-group-action is derived from the observation that there are many instances in which a tiling, or a pattern formed by copies of some motif, exhibits a behaviour which is orderly in some sense although it is not well accounted for by any symmetry group. For example, the orderliness may consist of a tiling being *monohedral*. This means that all tiles have the same shape, that is, any two are related by an isometry, but this isometry need not be a symmetry of the whole tiling. A remarkable pattern of this kind appears on an embroidery made by people of the contemporary Hmong culture, see Figure 8. Its symmetry group is *pgg* , but the monohedral tiling has several interesting properties. Its tiles form two transitivity classes; it is one of only two tilings possible with this tile (the other tiling is very simple, with symmetry group *p2*). Moreover, while for most monohedral tilings it is possible to change the shape of the tiles in a consistent way without disturbing the arrangement of tiles in the tiling, this Hmong tiling is completely "rigid": no other shape admits a tiling with this arrangement of tiles !

Another example is derived from the pattern of "whirligigs" (circular discs with appropriate markings) shown in Figure 9(a). Its symmetry group *p6* acts transitively on its motifs (and even on halves of the motifs), and a translational fundamental domain consists of just three whirligigs. However, no group of color symmetries (that is, symmetries which induce consistent color changes) acts transitively on the 4-colored whirigigs in Figure 9(b); the translational fundamental domain consists of 12 of them. But the pattern is very *orderly*: each tine of one of two colors (white and dark) of each disk touches a tine of the same color of an adjacent disk, while tines of each of the remaining two colors touch tines of the other remaining color on adjacent disks. Thus every whirligig is in the same relation to its neighbors as every other. This pattern, and another orderly one of a similar kind, were first shown in [5].

Many patterns (especially tilings) with interesting properties arise through the use of "adjacency rules"; this is a systematic development of the example in Figure 9. While the idea can be extended easily to the case of several motifs or tiles, we shall discuss here only the case of monohedral tilings. The starting figure ("prototile") can be understood as a generalized "polygon" with curved or polygonal "sides"; the shapes of the sides impose restrictions on tiles that may be adjacent to any one tile, and other restrictions may be additionally imposed. The tilings arise by using copies of the prototile in accordance with all the restrictions. A few examples of such tilings appear in the last section of [2].

Four examples of "prototiles" are shown in Figure 10. It is easy to verify that the only way that any of these prototiles might lead to a tiling of the whole plane is by having the tiles fit in a square-tiling-like way; to emphasize this, the corners of the appropriate "squares" are marked by solid dots for tiles (c) and (d) (for the moment, we disregard the arrows on tile (b)). It is also almost obvious that the plane cannot be tiled by copies of the prototile in (a). However, with the other three tiles shown, many different tilings are possible.

In Figure 11 are shown three examples of such tilings with the prototile from 10(b); the patches shown can clearly be extended to tilings of the whole plane. In the tiling in Fig. 11(a) the arrows along common edges are aligned in the same direction, while in the other two they are directed oppositely. With oppositely directed arrows uncountably many distinct tilings possible; hence there are also uncountably many possible tilings if the arrows are disregarded. However, with the arrows and with the requirement that arrows along common edges be directed oppositely, it can be verified that the possible tilings are in one-to-one correspondence with tilings that are possible using the prototile in Fig. 10(c): the zigzag "sides" of the "square" necessarily match up in the same way as the arrows are required to do, so tilings in Figs. 11(b) and 11(c) could be redrawn with this prototile. A different tiling with the prototile in Fig. 10(c) is shown in Figure 12. A much more appealing version of this tiling, which was drawn by its discoverer, the artist Peter Raedschelders, replaces the tiles by Escher-style "rabbits"; the tiling is remarkable since all eight different *aspects* of the tile — the relative orientations, with or without reflection — appear in each row and in each column of the 8-by-8 repeat of the tiling. Hence it represents a nontrivial Latin square. In Figure 12, the different aspects are distinguished by numerals on each tile; the white tiles are mirror images of the original prototile and its rotated copies, which are shaded. The shading makes evident the existence of a 4-fold rotational symmetry, so that copies of the 8-by-8 patch shown tile the plane in a tiling with symmetry group *p4* and a translational fundamental region consisting of 64 tiles.

Prototile (d) in Figure 10, which was invented by Raedschelders as well (in the shape of a dog) also leads to an 8-by-8 patch of a tiling which represents a Latin square; this is shown in Figure 13. Here again the white tiles are reflected versions of the original prototile and its rotated images. For more details of Raedschelders' work see [6].

The development of a general theory of such adjacency rules has not even started. On the other hand, the application of analogous ideas to isohedral tilings — that is, tilings in which symmetries operate transitively on the tiles — is well known and quite thoroughly understood, see [2]. This is, in a sense, another way of saying that isohedral tilings are very special, and that one needs new ideas to appreciate the possibilities with more general tilings and patterns.

The last topic to be discussed in this brief survey deals with a much more restricted situation. It developed from an attempt to understand the structure of the patterns on textiles from ancient Peru, and to explain how it may have been possible to transmit between generations the knowledge of pattern construction in the Peruvian culture. Many of these fabrics have survived for more than two thousand years, with stunning decorations in still bright colors. Examples can be found in many books, such as [7, 8]. The problem that anthropologists have had with these fabrics is that their classification by the symmetry groups is useless, even if colors are disregarded. This is in fact natural, since the Euclidean geometry is not the appropriate one for these woven fabrics. For hand-woven fabric warp and weft play special roles, different from each other and from all other directions; hence patterns on these fabrics can be understood much better is one considers them in the *fabric plane*. By this is meant a plane in which the only isometries admitted are translations, reflections in horizontal or vertical mirrors, and halfturns (180° rotations about an axis perpendicular to the plane). In the fabric plane an asymmetric object such as that in to top row of Figure 14 can appear in only four aspects, shown in the bottom row of Figure 14. They arise from the original by parallel motion (P), halfturn (T), or by a vertical (V) or horizontal (H) flip. One other peculiarity of the Peruvian fabrics is that copies of any motif are, almost without exception, placed in either a rectangular or in a diamond pattern, see Figure 15. In such an arrangement each motif is adjacent to four other copies, situated in the quadrangles that share edges with the quadrangle containing the original.

A study of the Peruvian fabrics shows that very many of them exhibit the following kind of orderliness: The cyclic sequence of the aspects of the four motifs adjacent to the given one is the same for all motifs. This is illustrated by the diagram in Figure 16, in which the decorations are in the diamond pattern, and the cyclic sequence is characterized by THTV. Clearly, such a pattern can easily be explained in non-mathematical terms. A detailed description of this type of orderliness, together with an enumeration and illustration of all the possible patterns orderly in this sense, appears in [9].

In conclusion, it seems that the study of the different kinds of orderly structures may lead to a better understanding of designs and art in various ancient and modern cultures, as well as contributing to the utilization of orderly structures science and technology.

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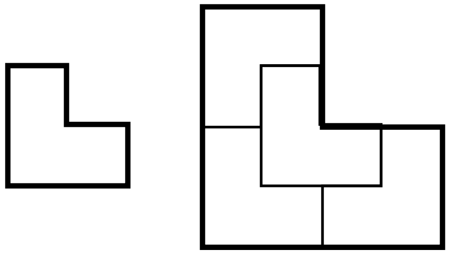


Figure 1.

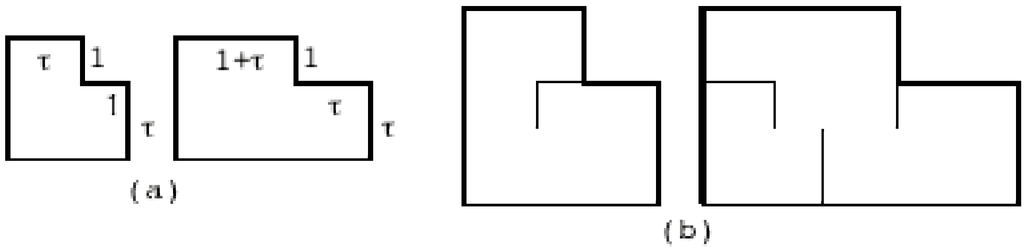
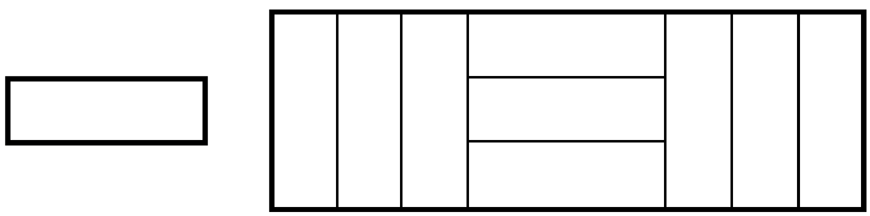


Figure 2.



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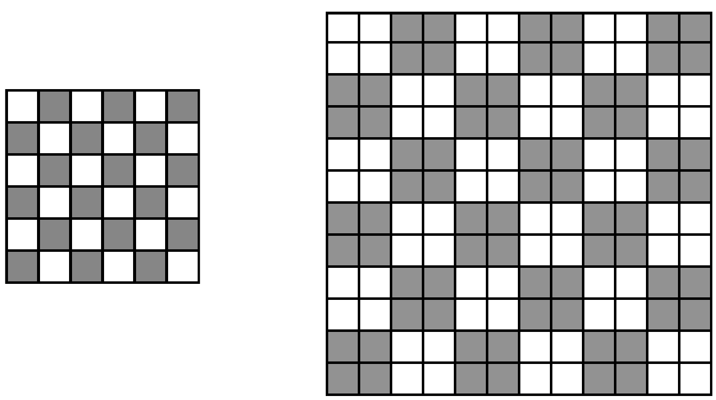

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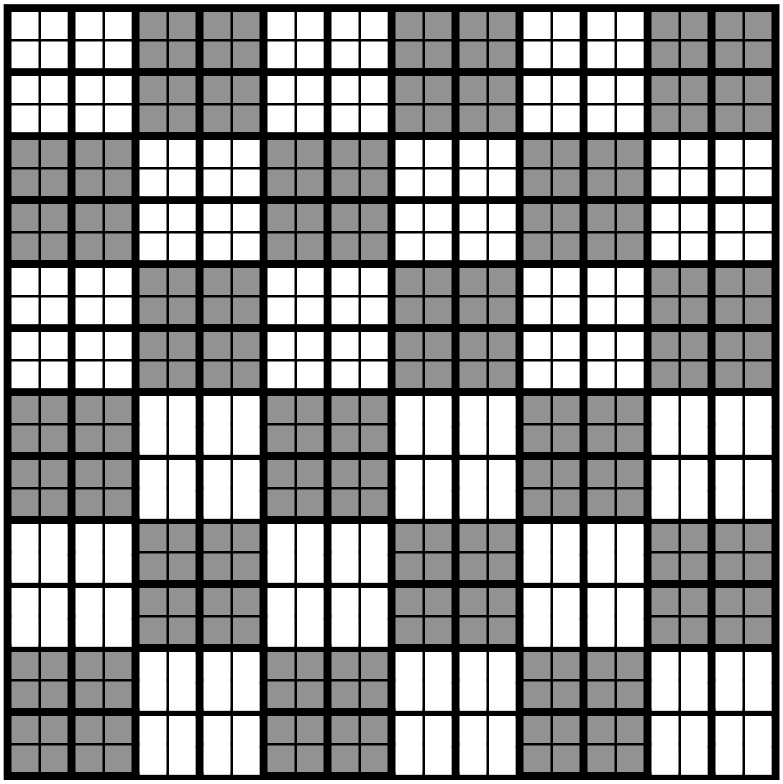


Figure 5.

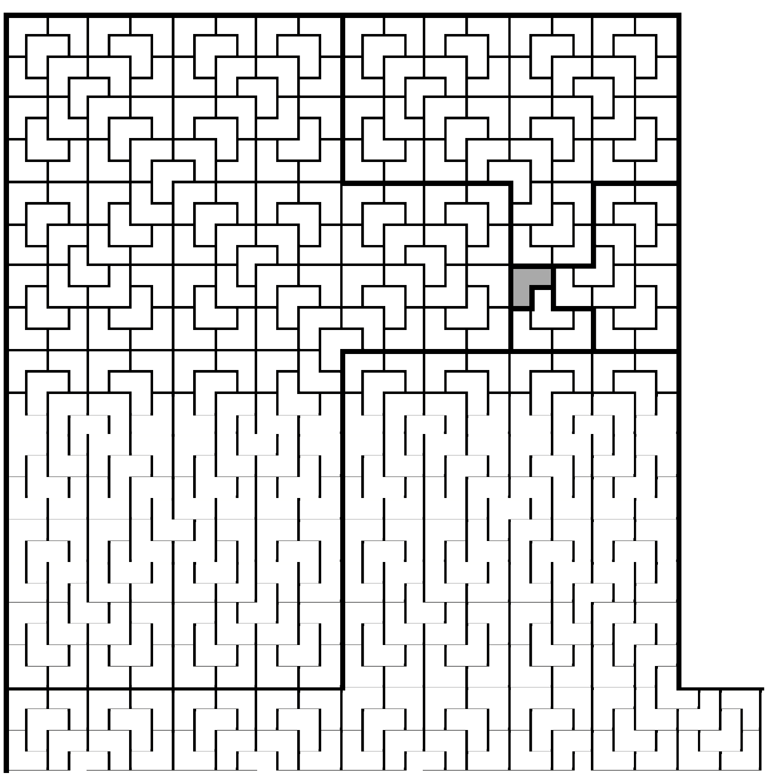


Figure 6.

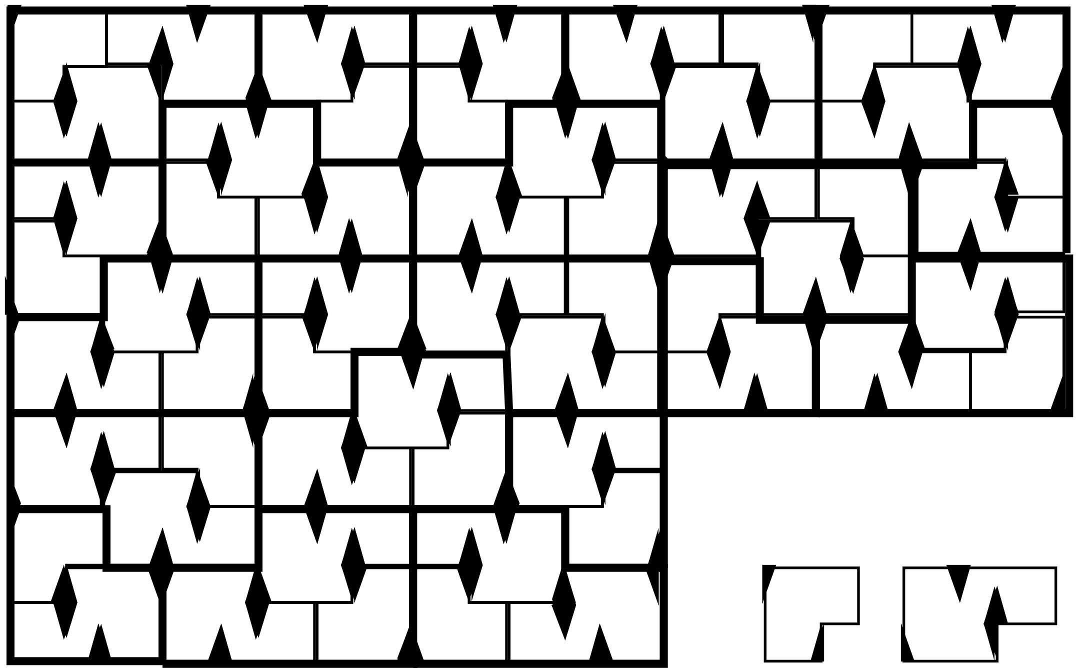


Figure 7.

Figure 8(a). A Hmong embroidery. (Photo attached on separate page)

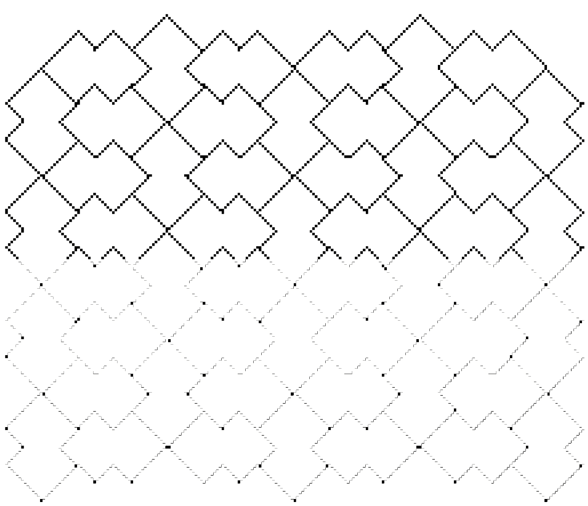
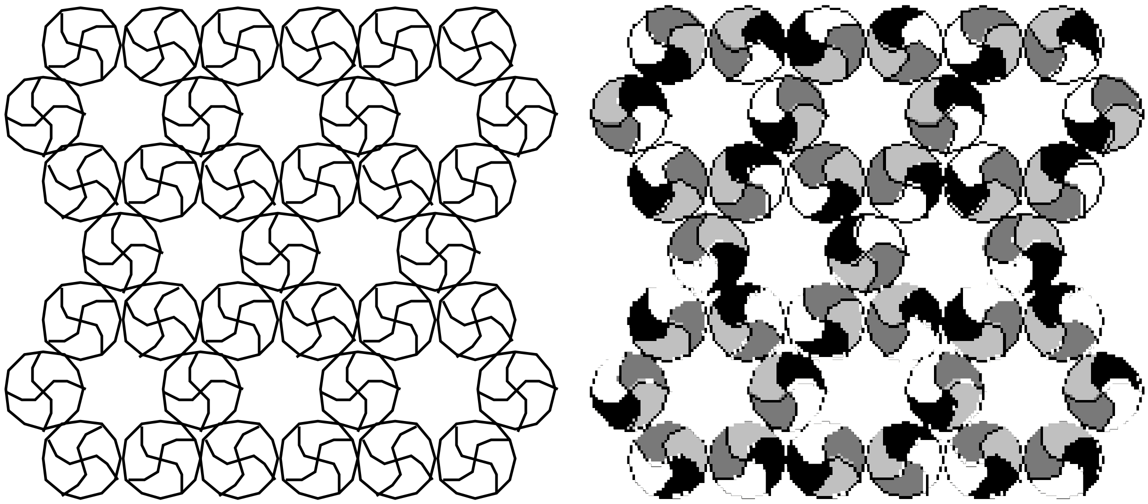
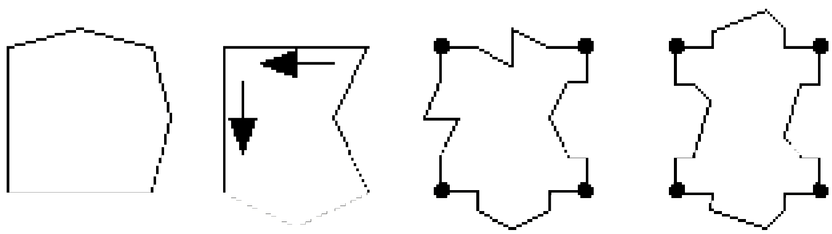


Figure 8(b).



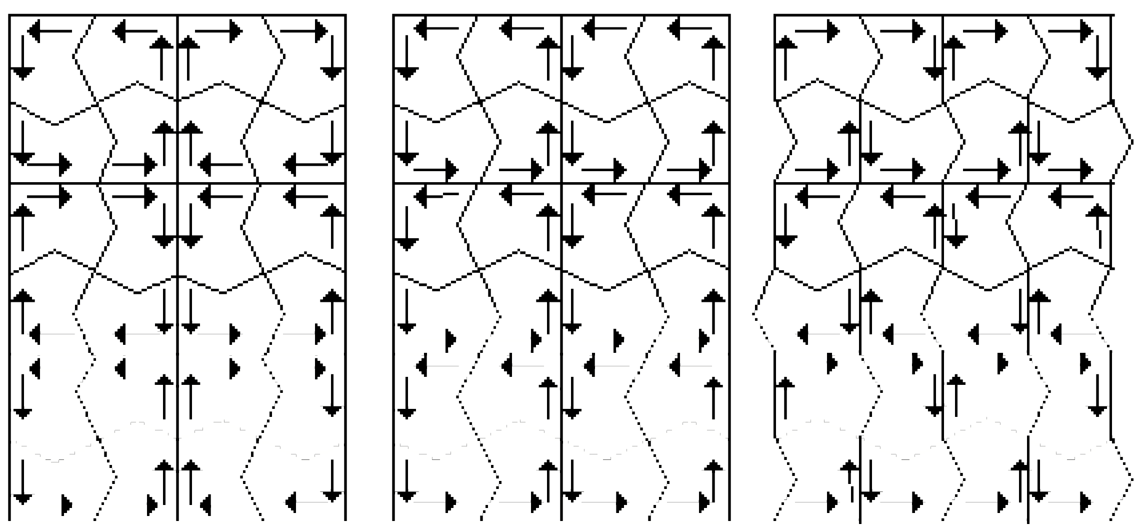
(a) (b)

Figure 9.



(a) (b) (c) (d)

Figure 10.



(a) (b) (c)

Figure 11.

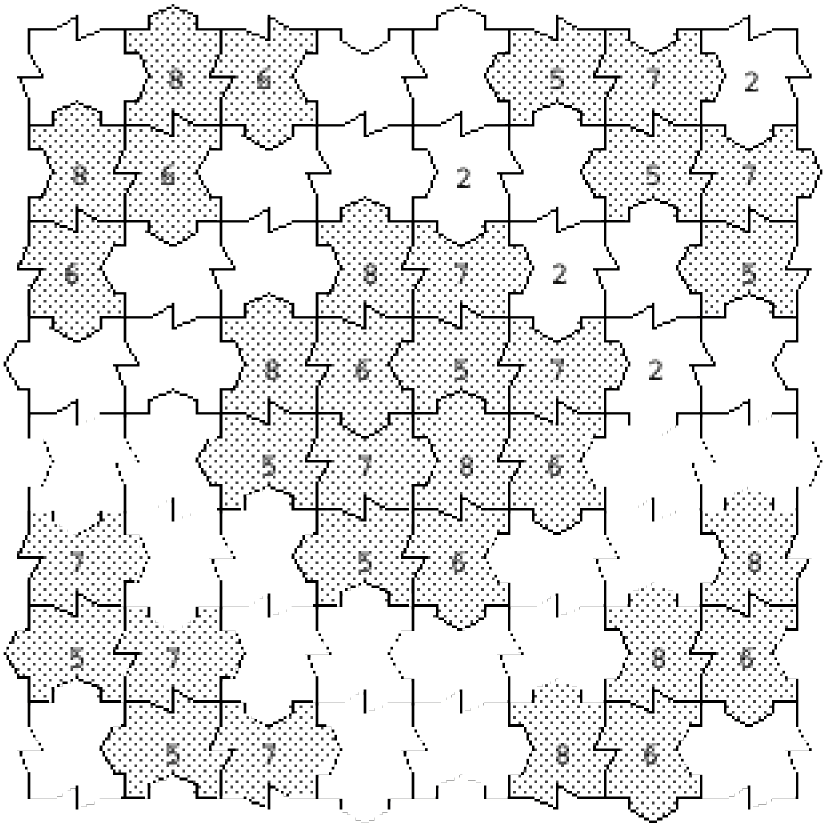


Figure 12.

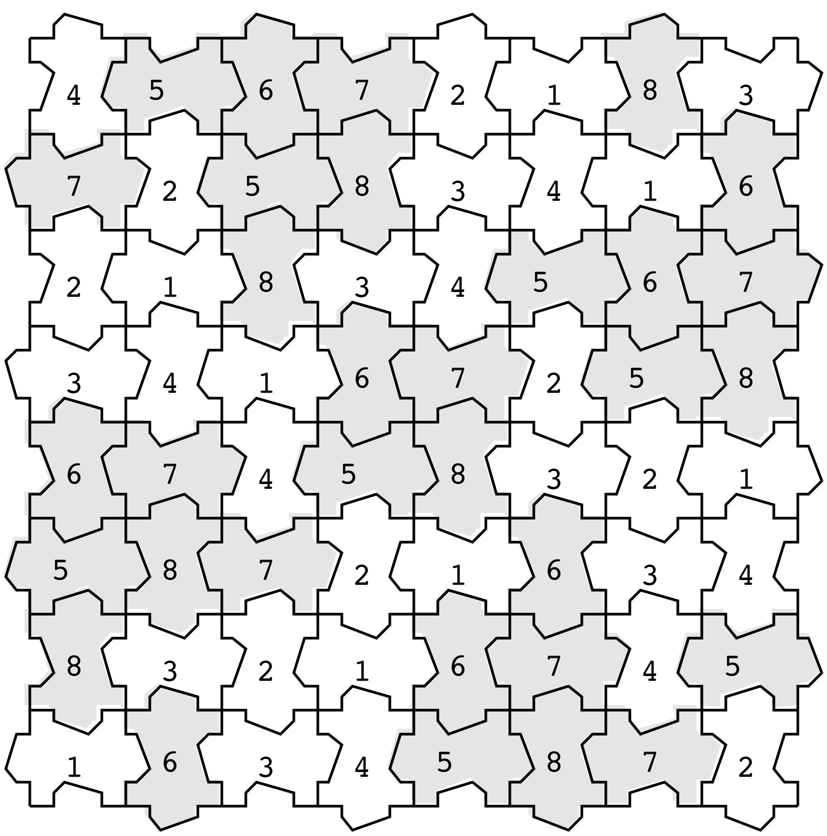
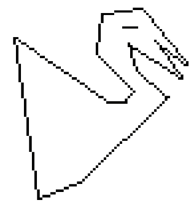


Figure 13.



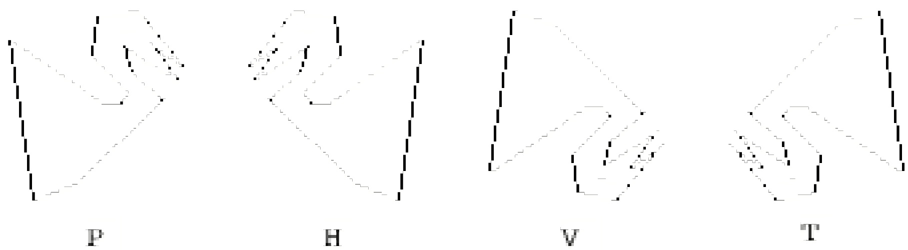


Figure 14.

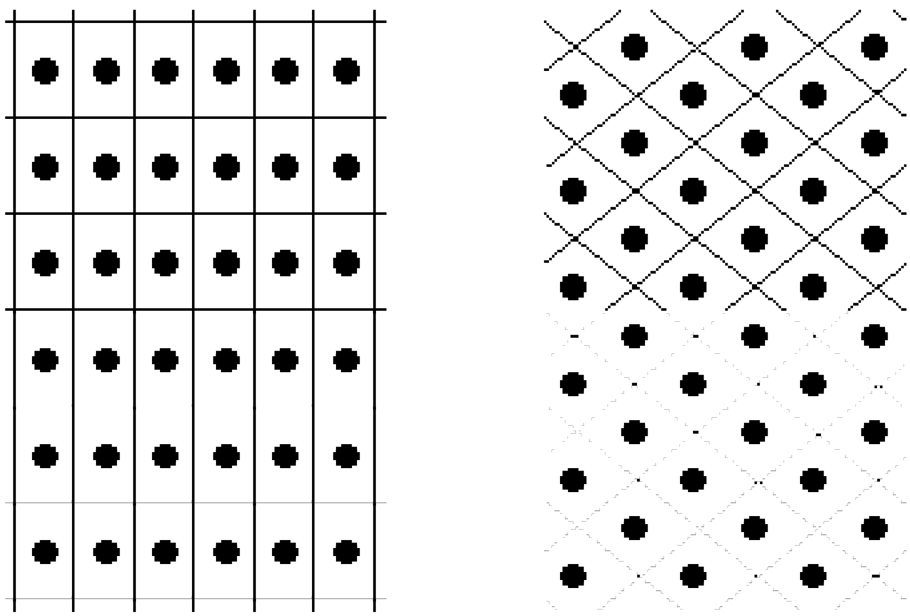


Figure 15.

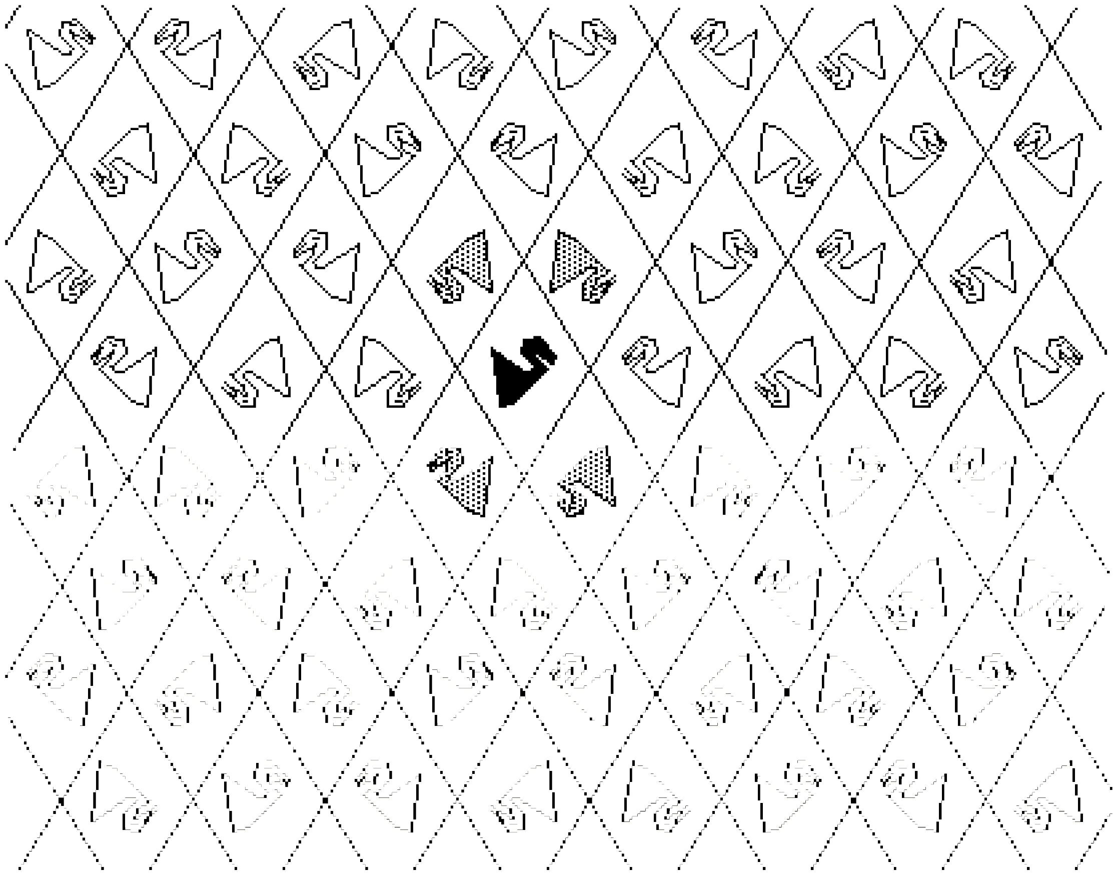


Figure 16.

Department of Mathematics

University of Washington

Seattle, Washington 98195-4350

e-mail: grunbaum@math.washington.edu

Captions for the figures.

Figure 1. The inflation-and-decomposition of the L-tromino.

Figure 2. A pair of tiles devised by Robert Ammann, and their inflation-and-decomposition.

Figure 3. A specified inflation-and-decomposition of the straight tromino.

Figure 4. Inflation-and-decomposition of a checkerboard-colored tiling by squares.

Figure 5. The second inflation-and-decomposition of a checkerboard-colored tiling by squares.

Figure 6. Part of a 5-step inflation-and-decomposition of the L-tromino.

Figure 7. The third and fourth inflation-and-decomposition steps of the larger tile from Figure 4, subject to restrictions concerning markings.

Figure 8. A photo (a) and the scheme (b) of the ornamentation of a Hmong embroidery.

Figure 9. An orderly pattern of 4-colored whirigigs.

Figure 10. Examples of prototiles that are "squares" with polygonal "sides".

Figure 11. Examples of tilings with the prototile in Figure 10(b), under different adjacency rules.

Figure 12. An 8-by-8 patch of a tiling with the prototile in Figure 10(c), which shows 4-fold rotational symmetry, and has all eight aspects of the prototile in every row and every column. It is a schematic representation of a tiling by P. Raedschelders.

Figure 13. An 8-by-8 patch of a tiling with the prototile in Figure 10(d), with 4-fold rotational symmetry and all eight aspect of the prototile in every row and every column. This is based on another tile of P. Raedschelders.

Figure 14. An asymmetric motif, and its four aspects possible in the fabric plane.

Figure 15. The two arrangements of motifs that are dominant in ancient Peruvian fabrics.

Figure 16. A orderly ornamentation, in the style of ancient Peruvian fabrics.