Geombinatorics 11(2002), 78 - 84

# THE GRUNERT POINT OF PENTAGONS

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**1. Introduction.** "Remarkable points" of triangles — for example, the centroid, the circumcenter and others — have been studied in detail; in fact, the recent book [3] by Clark Kimberling contains information on 400 such "centers". Although one can expect that in case of quadrangles there should be an even larger number of "remarkable points" — only few have been discussed in the literature. Similar, and even more pronounced, is the disparity between possibilities and available results concerning pentagons and polygons with larger number of sides.

The main aim of this short note is to present in Section 2 one such "remarkable point" for pentagons. We call it the *Grunert point* of the pentagon, after Johann August Grunert, who announced its existence in [1] and proved it in [2] — more than 170 years ago. We also show that Grunert's result can be slightly strengthened, and discuss some related developments. In Section 3 several comments, as well as the information available to me about the history of Grunert points are presented. Throughout, when speaking of a pentagon we shall assume that its five vertices are at distinct points of the plane.

**2. The main result.** Let L(P,Q) denote the line through the points P and Q. Then we have the following result, illustrated in Figure 1, and formulated using the notation shown.

**Grunert's theorem.** Given pentagon P = ABCDE, let us denote A* = L(B,C) ∩ L(D,E), B* = L(C,D) ∩ L(E,A), ..., and let A*A be the midpoint of A* and A, and BE be the midpoint of B and E, etc. Then the five lines L(A*A,BE), L(B*B,AC), etc. are concurrent.

We call the point of concurrency the *Grunert point* G = G(P) of the pentagon P.

In fact, the following stronger result holds (see Figure 2):

**Theorem.** In the notation of Grunert's theorem, let, in addition, B*C* be the midpoint of B* and C*, C*D* the midpoint of C* and D*, etc. Then C*D* is on the line L(A*A,BE), D*E* is on the line L(B*B,AC), etc. Thus each of the five lines concurrent at the Grunert point G carries three of the named points (besides G).

**Proof.** Let us describe the vertices A, B, C, D, E by the (column) vectors a, b, c, d, e. Then obviously AD is the point (a + d)/2, BE is the point (b + e)/2, and so on. With a little more calculation we find that A*, the intersection point of the lines L(B,C) and L(D,E), is given by

$$\frac{(b-c)\,|d, e| - (d-e)\,|b, c|}{|b-c, e-d|} ,$$

where |p, q| denotes the 2 by 2 determinant with columns p and q. Analogous expressions are obtained for the points B*, C*, D* and E*. Then it is easy to find the formulas for the remaining points of interest, A*A, B*B, ... , and A*B*, B*C*, ... . With this information in hand, it can be verified at once that A*A, BE and C*D* are collinear; by symmetry, this shows that all the appropriate triplets are collinear. This establishes part of the Theorem. The

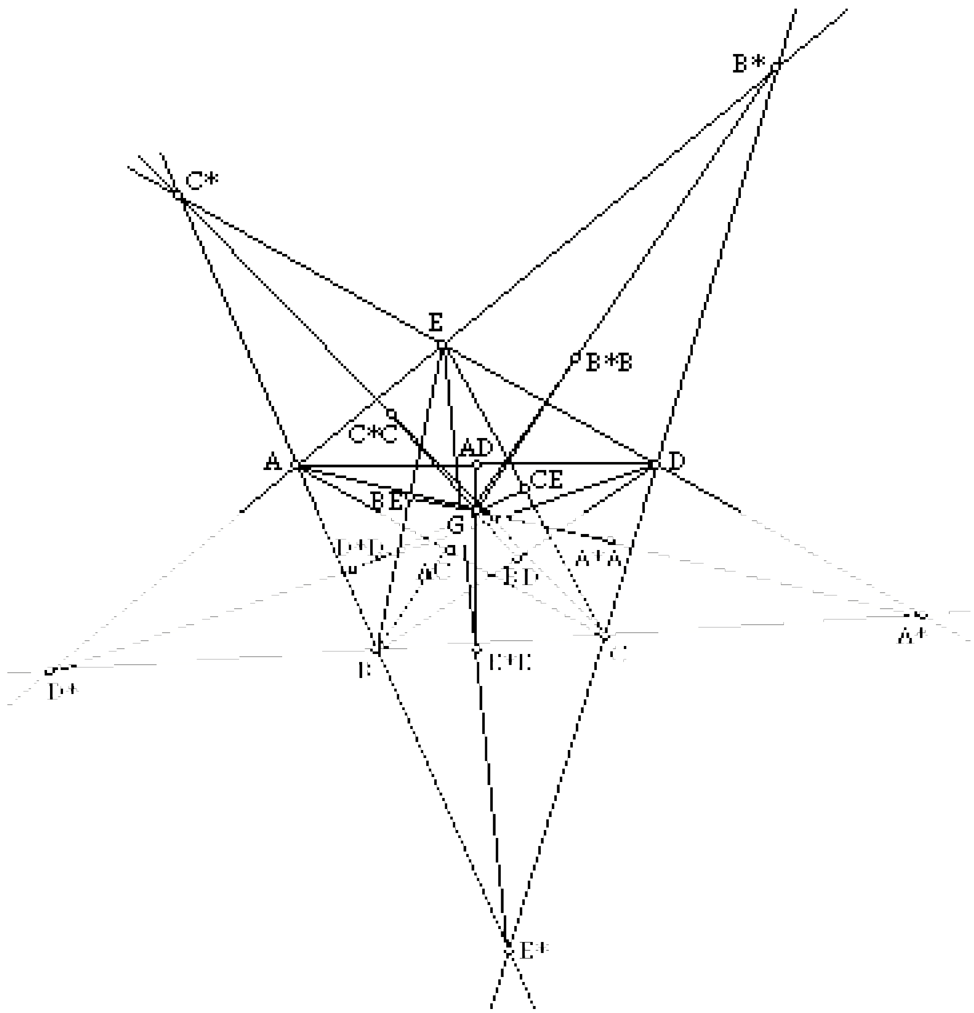


Figure 1. An illustrative example of the construction of the Grunert point

G = G(P) of a pentagon P = ABCDE.

assertion that the Grunert point G is on all five such lines requires — by symmetry — the checking of only one collinearity, for example, that of G, AD and E*E. This can be readily accomplished if one uses the coordinates of G, namely

$$\frac{a.(|e, b|-|c, d|) + b.(|a, c|-|d, e|) + c.(|b, d|-|e, a|) + d.(|c, e|-|a,b|) + e.(|d, a|-|b, c|)}{2(|a, c| + |b, d| + |c, e| + |d, a| + |e, b|)},$$

an expression which shows the required symmetry. If invoking this formula seems like a sleight of hand, one can, alternatively, find the coordinates of the intersection point of the diagonals of the quadrangle with vertices AD, BE, E*E, A*A, and show that it coincides with the intersection point of the diagonals of the quadrangle CE, AD, D*D, E*E. This proves the theorem since the common intersection points are, in fact, the point G. Its coordinates can therefore be found from either of the two quadrangles. ∧

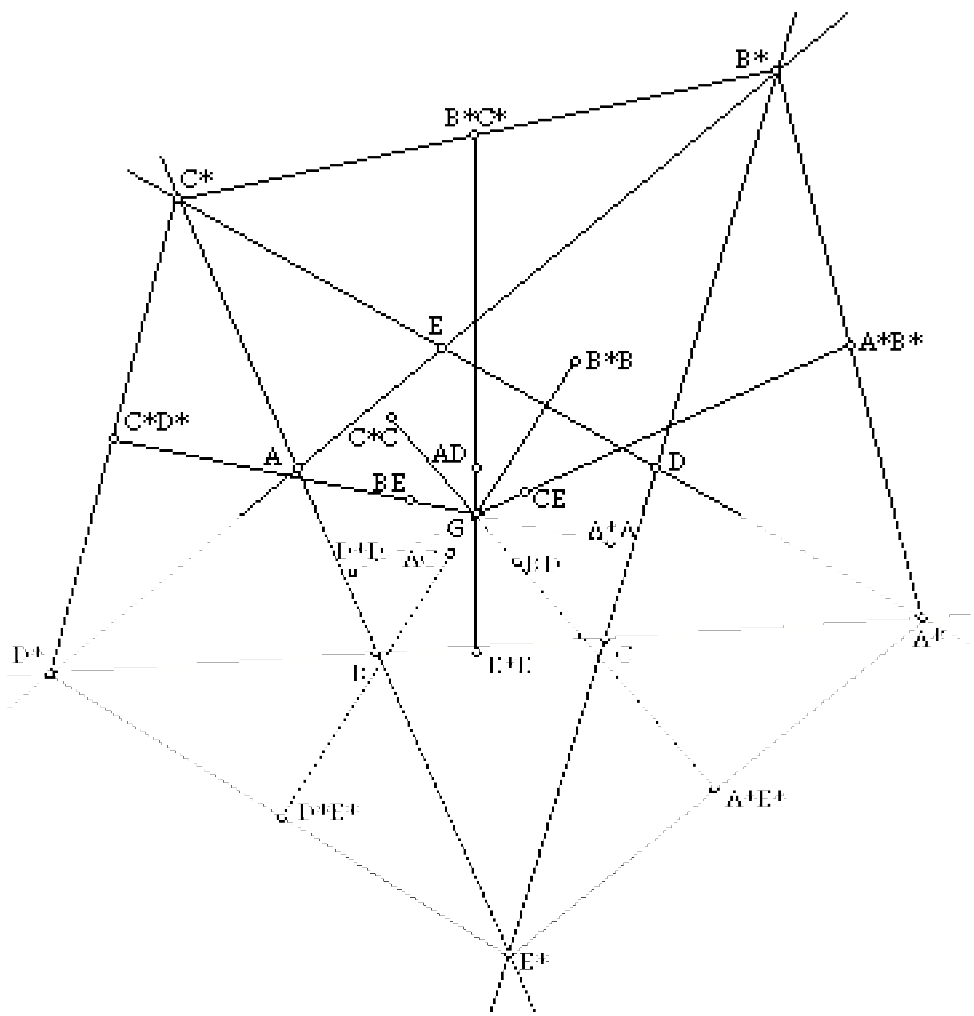


Figure 2. An illustration of Theorem 2. The Grunert point G of the pentagon ABCDE is the common point of five lines, each of which passes through three points determined by the vertices of the pentagon.

## 3. Comments.

(i) In the formulation of his result in [1] and [2] Grunert seems to have only convex pentagons in mind. This is visible by his determination of the points A*, B*, etc. by "... extending the sides of the pentagon till they intersect ...", and by the accompanying diagrams. On the other hand, such a restriction is not required for the validity of the result. In Figure 3 we show a convex pentagon (the vertices are indicated by hollow dots) together with its Grunert point (largest solid dot), the Grunert point of the pentagram with the same vertices (next-to-largest solid dot), and the Grunert points of the other ten different (unoriented) pentagons that have the same set of vertices. The existence of Grunert points in all these situations is easy to understand in view of the algebraic expressions used in the proof, which do not depend on convexity or betweenness considerations.

(ii) Having presented the Theorem and its proof in the previous section, it is now — unfortunately — the time to admit that as formulated, neither the proof, nor the theorem itself, are valid. As is the case with many other theorems of Euclidean geometry (for example, the theorem of Pappus), there are some exceptional cases in which the original formulation of the theorem becomes meaningless, even assuming that all the vertices are distinct. Such exceptions happen, among other situations, if some of the points used are intersections of lines which are not prevented from being parallel. In the case of Pappus' theorem, careful presentations either list the many cases which require special formulations, or else resolve the problem

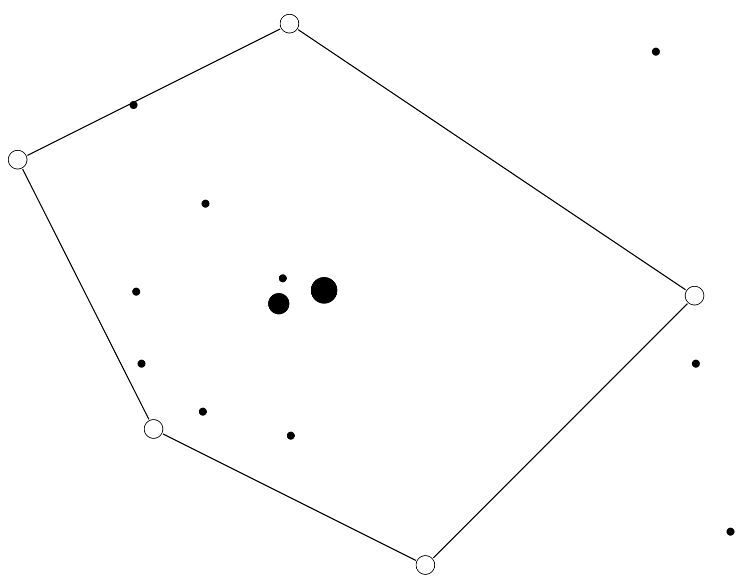

Figure 3. The solid dots are the Grunert points of the twelve distinct (unoriented) pentagons possible with the five vertices (hollow dots). The largest dot corresponds to the pentagon shown, the second largest to the corresponding pentagram.

by presenting the result in the setting of projective geometry. The The same approaches could be used in the case of Grunert points; Figure 4 shows the case in which one pair of sides of the pentagon are parallel. In the projective formulation, the midpoints appearing in the formulation of the theorem should be replaced by appropriate harmonic points corresponding to the "points at infinity", and the given proof would be valid if reformulated for homogeneous coordinates. Thus, after some skipped heartbeats, the Theorem is revived.

(iii) But there is still another problem. A contemplation of the expression for the Grunert point G reveals at once where a danger to the validity of the Theorem lies. The formula is meaningless if the denominator equals zero. For a pentagon ABCDE this denominator is the area of the "pentagram" ACEBD. Hence the polygon ABCDE in Figure 5 is an example of the unredeemable failure of our Theorem. The only rescue is to add a requirement to the effect that the polygon ABCDE is such that the area of the polygon ACEBD is nonzero. It is easy to verify that if points B, C, D, E in Figure 5 are kept fixed, the polygon ACEBD has zero area if and only in A has second coordinate equal -2. More generally, it can be shown

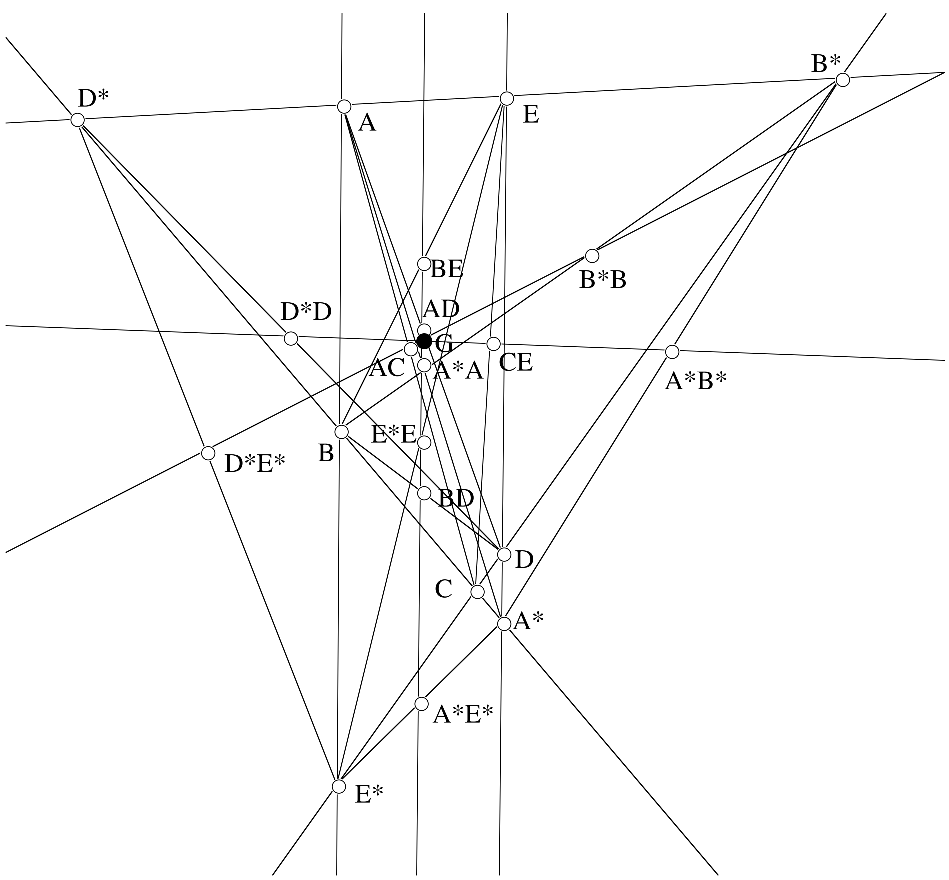


Figure 4. The variant of the Grunert point construction in case the pentagon has a pair of parallel sides.

that in the 2-dimensional manifold of classes of projectively equivalent pentagons, the ones with no Grunert point correspond to a 1-dimensional submanifold.

(iv) It is not clear to me what is the root of the imprecision in Grunert's own formulation of his theorem. Was he really thinking only of convex pentagons with no parallel sides? Was the failure to specify any condition due to sloppy thinking, or a naive point of view, or the expectation that a serious reader will independently figure out the details? What is certain is that a similar attitude was very wide-spread among geometers of that period — unfortunately, not only in that long-ago era. Books could be written about the failures of geometers to adequately define the concepts they are investigating, and fully describe the gaps in the formulations and proofs of many theorems. It is highly disturbing that such departures from our professed adherence to fully logical deductions continue to the present day.

(v) Grunert's point seem to be largely forgotten. The most recent mention of it that I am aware of is in the remarkable but little known survey [4] of elementary geometry in the nineteenth century, by Max Simon, published in 1906. Besides Grunert's papers, Simon mentions (on pages 163 and 166) two papers as having independently rediscovered Grunert's result. The first is by Paul Serret in 1847, the second by J. Mention in 1853. Both were published in the journal Nouvelles Annales de Mathématiques, but I have not been able to get a copy of either.

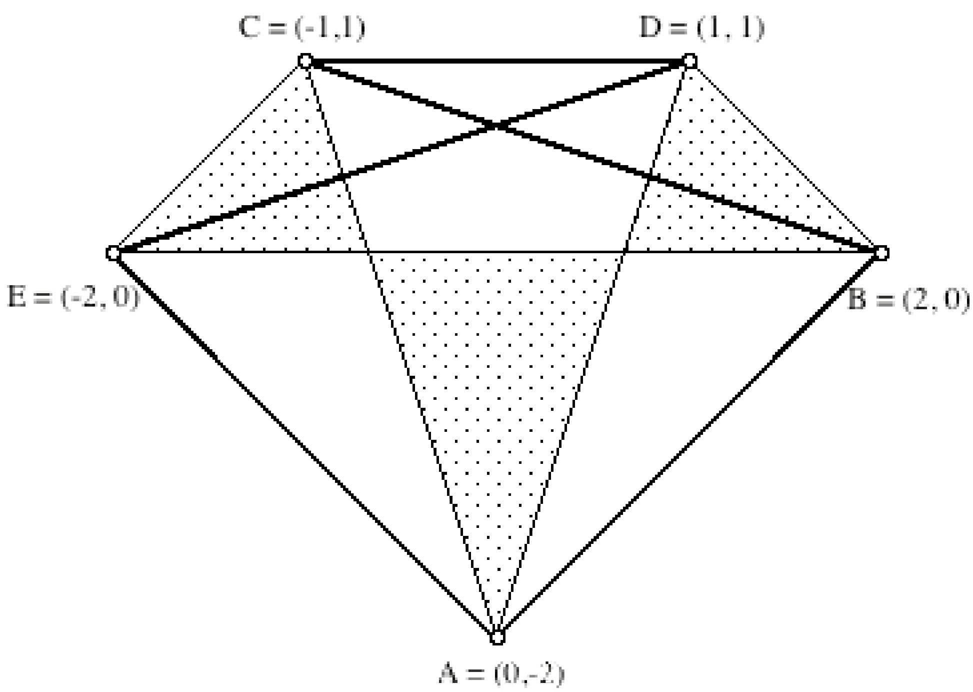


Figure 5. A pentagon ABCDE which has no Grunert point.

(vi) Like the proof given above, Grunert's proof is also, in essence, computational. But the elementary nature of the result points out a difference in the mathematical culture of that era compared to more recent times. The papers [1] and [2] were published in one of the leading mathematical journals of its time. The volume in which [2] appeared contains, among others, a collection of results of N. H. Abel prepared for publication after Abel's death by A. L. Crelle, as well as articles by Dirichlet, Gudermann, Gergonne, Jacobi, Minding, Plücker and others — very prominent mathematicians of the period.

(vii) It is somewhat remarkable that there seems to be no analog of the Grunert point for polygons with a number of sides different from five. Naturally, it is possible that only my lack of imagination and inventiveness is responsible for my failure to find such analogs, but certainly not any lack of trying.

(viii) The Grunert point $G(P)$ is an *affine invariant* for pentagons $P$ for which it is defined. By this is meant that if $T$ is a nonsingular affine transformation of the plane then $T(G(P)) = G(T(P))$. In that respect, the Grunert point is like the vertex centroid, and the area centroid — which, in contrast to the Grunert point — are meaningful for polygons with any number of sides. I venture the following:

**Conjecture.** The vertex centroid has is the unique affine invariant point-valued continuous function of polygons which, for any fixed $n \geq 3$, is defined for *all* n-gons.

## References

[1] J. A. Grunert, Lehrsatz. J. reine angew. Math. (Crelle) 4(1829), 396.

[2] J. A. Grunert, Beweis eines Lehrsatzes vom Fünfecke. J. reine angew. Math. (Crelle) 5(1830), 316 – 317.

[3] C. Kimberling, Triangle Centers and Central Triangles. Congressus Numerantium vol. 129(1998).

[4] M. Simon, Über die Entwicklung der Elementar-Geometrie im XIX. Jahrhundert. Jahresber. Deutsch. Math.-Verein., Ergänzungsband 1. Leipzig, Teubner 1906. 276 pages.