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OMITTABLE POINTS

by Branko Grünbaum
University of Washington, Box 354350, Seattle, WA 98195-4350
e-mail: grunbaum@math.washington.edu

One of the well known questions in elementary combinatorial geometry requires the proof of the assertion that for every finite set P of points in the Euclidean plane, not all on one line, there exists a line passing through precisely two points of P. Such a line if often called an *ordinary line* of P. First posed by J. J. Sylvester more than a century ago, the question was forgotten and revived only in the 1930's by Paul Erdös; the first proofs were found soon thereafter (Erdös 1943; for detailed accounts of Sylvester's problem, its history, and its many ramifications see Grünbaum 1972, Borwein & Moser 1990, Erdös & Purdy 1995). One of the many unsolved problems in this direction is finding the minimal number $\omega(n)$ of ordinary lines in sets P of n points. It is known that $\omega(n) \leq n/2$ for infinitely many values of n, but on the other hand, the best estimate from below is $\omega(n) \geq 6n/13$ (Csima & Sawyer 1993) for $n > 7$.

A different aspect of the same type of questions is the following. Since every finite set P of noncollinear points determines some ordinary lines, it follows that every such P contains points with the property that the omission of such a point would reduce the number of lines determined by the remaining points. We call such points non-omittable. We say that a point X of P is *omittable* if the set obtained from P by deleting X determines the same lines as the set P. For example the center of a square is an omittable point of the set P which consists of the vertices of the square and its center. A more elaborate example is shown in Figure 1; each of the four points marked by hollow circles is an omittable point of the set of seven point indicated by hollow and full circles.

Naturally, many sets P have no omittable points; in contrast, it is not easy to find large sets P with relatively many omittable points. The only published results seem to be in the old and little known (and not easily accessible) paper by Koutsky & Polak 1960. Theorems 1 and 2 below are slight extensions and strengthenings of their results with simplifications of their proofs. Before formulating these results and their proofs we note that instead of the Euclidean plane we may substitute the projective plane, and in particular, the extended Euclidean plane model of the projective plane. This is possible because all the concepts and properties considered here are invariant under projective transformations. The use of the extended Euclidean plane (that is, the Euclidean plane augmented by the "points at infinity", each of which is the common point of all straight lines parallel to a given direction, and the "line at infinity" which consists of

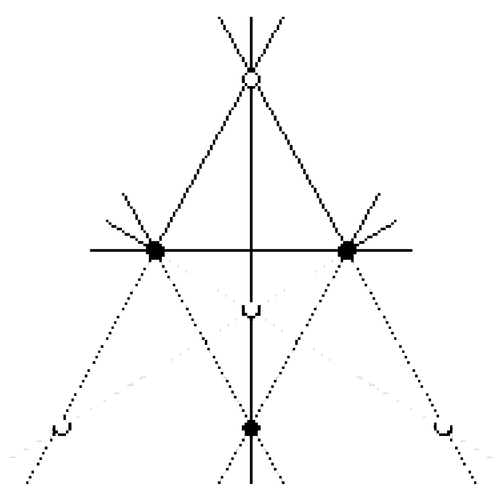

Figure 1. A set of seven points, four of which (shown by hollow circles) are omittable.

all points at infinity), enables one to consider such concepts as "convex hull" and "regular polygons".

Theorem 1. If a set P of n points, not all collinear, contains $k \geq 3$ omittable points then $n \geq 3k$. Moreover, for every $k \geq 3$ there exists a set of $3k$ points with k collinear omittable points.

Proof. Let the line containing the k omittable points $X_1, \dots,$ X_k be the line at infinity, and let C be the convex hull of the remaining points; clearly, C is a convex polygon. For each of the omittable points X_i consider the two supporting lines of C in direction of X_i. Because X_i is omittable, each of these lines contains at least two distinct points of P, and therefore an edge of C. Thus, P has at least $2k$ edges, hence at least $2k$ vertices, and so $n \geq k + 2k$, as claimed. To establish the second part of the theorem it is enough to consider as P the set consisting of the vertices a regular polygon with $2k$ edges, together with the k points at infinity in the directions of the edges of the polygon; these k points are omittable. ◊

Theorem 2. Given any set Q of k collinear points, there exists a set P such that the points of Q are the only omittable points of P.

Proof. Let the points $X_1, X_2, X_3, \dots$ of Q be at infinity. We start with points Y_1, Y_2 collinear with X_1, then construct translates Y_3, Y_4 of Y_1, Y_2 by a vector V_2 in the direction of X_2; the four points Y_j are translated by a vector V_3 in the direction of X_3, the choice of V_3 being such that no three of the eight points Y_j are collinear. Repeating the same procedure for the remaining X_i's we arrive at a set P consisting of $k + 2^k$ points and having precisely the set Q of omittable points. ◊

Let $\psi(n)$ denote the largest number of omittable points possible in any set of n points. The example given in the proof of Theorem 1 shows that $\psi(n) \geq [n/3]$, where the symbol $[t]$ denotes the largest integer not greater than t. This can be improved by observing that if k is odd, $k = 2s + 1$, then one can add the center of the regular polygon to the set P, thereby obtaining a set of $n = 6s + 4$ points with $2s + 2$ omittable points; hence $\psi(6s + 4) \geq 2s + 2 > [(6s + 4)/3]$. The example in Figure 1 shows that $\psi(7) \geq 4$, and it may be shown that equality holds. Six of the 13 points in Figure 2 are omittable, hence $\psi(13) \geq 6$; almost certainly equality holds.

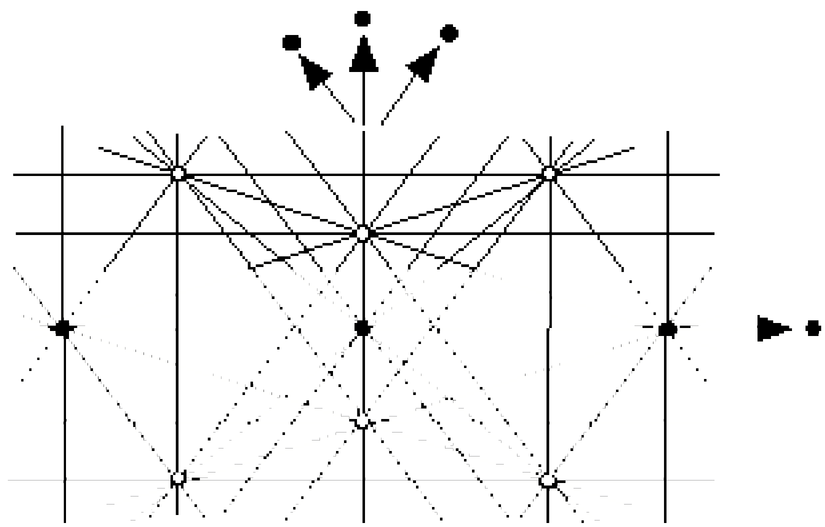

Figure 2. A set of 13 points (four at infinity, indicated by the directions of the arrows) which has six omittable points (shown by hollow circles.

Besides the examples in Figures 1 and 2, only one other set P of n points is known for which the number of omittable points is 2 + [n/3]. This is the set of n = 21 points shown in Figure 3, which has nine omittable points.

There are many open problems related to omittable points. Here are just a few for which I venture to guess an answer.

Conjecture 1. $\lim_{n \to \infty} \psi(n)/n = 1/3$.

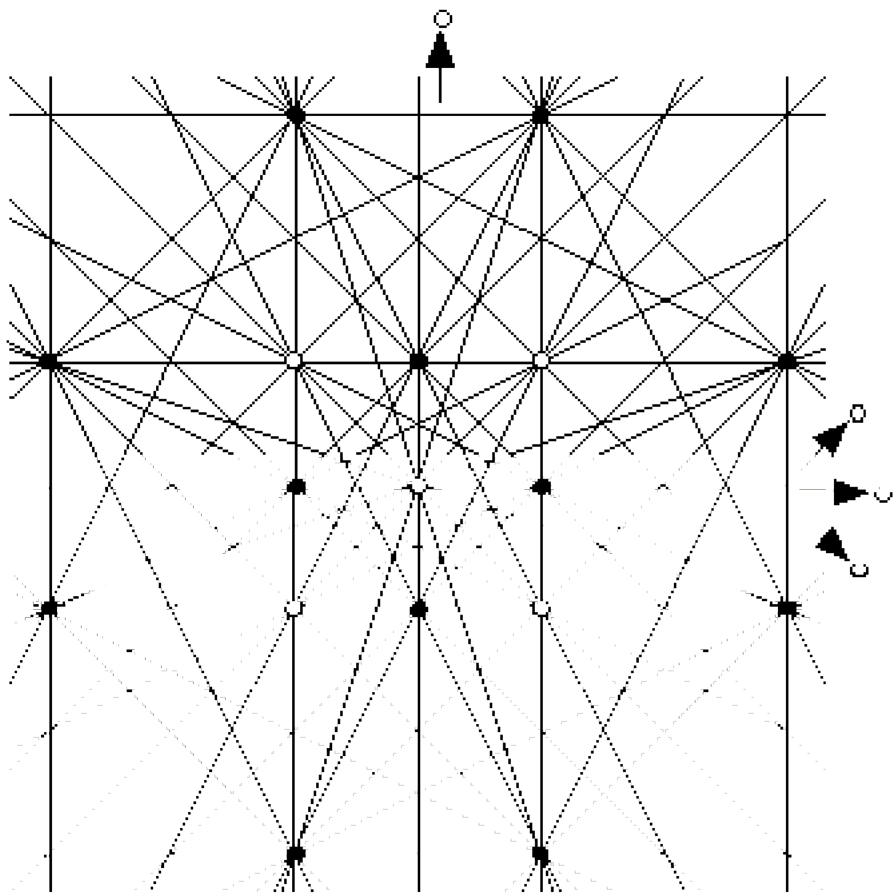

Figure 3. A set of 21 points, nine of which are omittable. These points are indicated by hollow circles, four of which are at infinity.

Conjecture 2. There exists an integer N such that for every set P of $n \geq N$ points , either all or all but one omittable points of P are collinear.

Conjecture 3. The construction in Theorem 2 is "essentially" best possible in the sense that there exists a constant $c > 0$ such that for each k there exist k collinear points such that any set for which the given points are omittable has at least $c\,2^k$ points.

Other questions worth investigating are: How many points can be omitted simultaneously ? What happens if pseudolines are admitted instead of lines ? What is the situation if the points generate planes in space instead of lines in the plane ?

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