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Monochromatic intersection points in families of colored lines

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Many questions about families of straight lines in the plane and their intersection points seem easy but the answers are hard to find. In this note we shall discuss one such question, the history of which goes back more than thirty years.

Let $\mathcal{L}$ be a family of straight lines, and let each line be *colored* either green or red; throughout, we shall assume that each of the colors actually appears on at least one line. The first question, which by all accounts seems to have been raised by Ron Graham around 1965 is whether each family $\mathcal{L}$ must have at least one "*monochromatic point*" — that is, whether some intersection point is contained only in lines of one of the colors. The question seems to have been widely disseminated by word of mouth, and D. Newman popularized it in various venues. Strangely, it remained out of the published literature until the appearance of an abstract (Motzkin [1967]) which contained both a formulation of Graham's question and a statement that the answer is affirmative. Michael Rabin arrived at the affirmative answer independently, and a joint paper of Motzkin and Rabin was supposed to appear; this was mentioned in the first proof that was published (Chakerian [1970]), and was confirmed to the present author by Prof. Rabin in a letter in 1975. However, this paper has not appeared so far, and it seems not likely that it will ever be published. A proof attributed to Motzkin (via private communications through a number of people) and presented by me in unpublished lecture notes 25 years ago appeared in Erdös & Purdy [1995], with mistaken attribution to S. K. Stein (who was a link in the transmission of Motzkin's proof).

Since there are now two published, easily accessible proofs of Motzkin's theorem, we shall not give a proof here; instead, we shall concentrate on investigating the situation in which it is given that there are no red monochromatic vertices; such families of lines are said to be *biased*. This topic has also been considered by Motzkin [1967], as well as in the abstract Grünbaum [1975]. The motivation for looking at biased families of lines is the common-sense feeling that if all monochromatic vertices are green, there cannot be too many red lines. However, this feeling is only partly justified — in fact, there is one class of families which behave in the opposite way. This is illustrated in Figure 1, which is typical of the situation in which there are two or more green lines and arbitrarily many red ones, but all red lines and at least one green line pass through one point, while at least one green lines does not contain that point. From now on, the term "biased family" will be used only for families in which the red lines have no common point.

To make the description of the available results simpler, let the *surplus* $s = s(\mathcal{L})$ of a biased family $\mathcal{L}$ be the difference between the numbers of red and green lines in $\mathcal{L}$. Motzkin [1967] stated that the

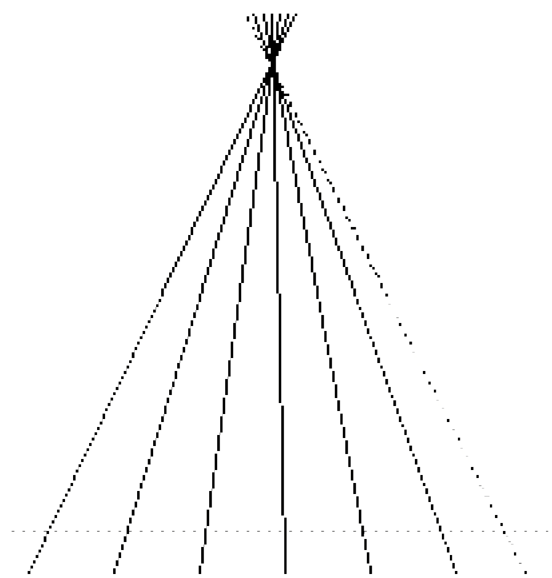

Figure 1. A typical example of a biased family. In this and all following illustrations, the solid lines are "red", and the dashed lines are "green".

largest known s is s = 4, which "occurs for the regular star-5-gon". He probably meant the family shown in Figure 2. Motzkin also stated that the only known biased families with arbitrarily large number of red lines are of 10 types, and that for them s = 1.

In Grünbaum [1975] I stated that there are (at least) two types of biased families (with arbitrarily large numbers of red lines) and s = 4; there exist many types of families with smaller s. The smallest members of the two families with s = 4 are shown Figure 3 and 4, from which the general construction can be immediately deduced. I also conjectured that s ≤ 4 for all biased families. However, this conjecture is not true, as is shown by the example, discovered very recently and shown in Figure 5. In this example s = 6.

We are left with many open questions, for which I dare not conjecture answers. Here are a few of the most immediate ones:

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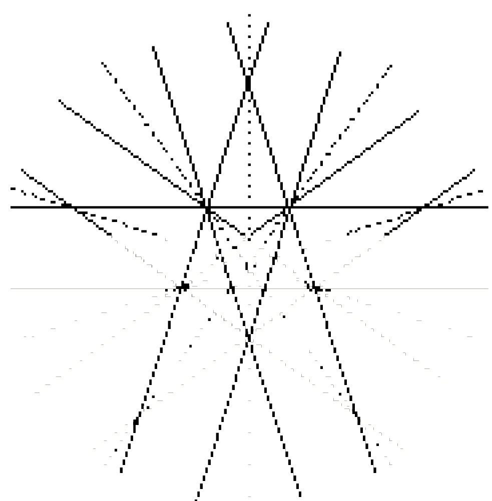

Figure 2. A biased family with s = 4; it consists of ten red lines and six green ones. In this and the following illustrations, the lines are in the extended Euclidean plane, see Remark 2. The presence of a dashed or solid horizontal line at the top of a diagram indicated that the "line at infinity" is included in the family.

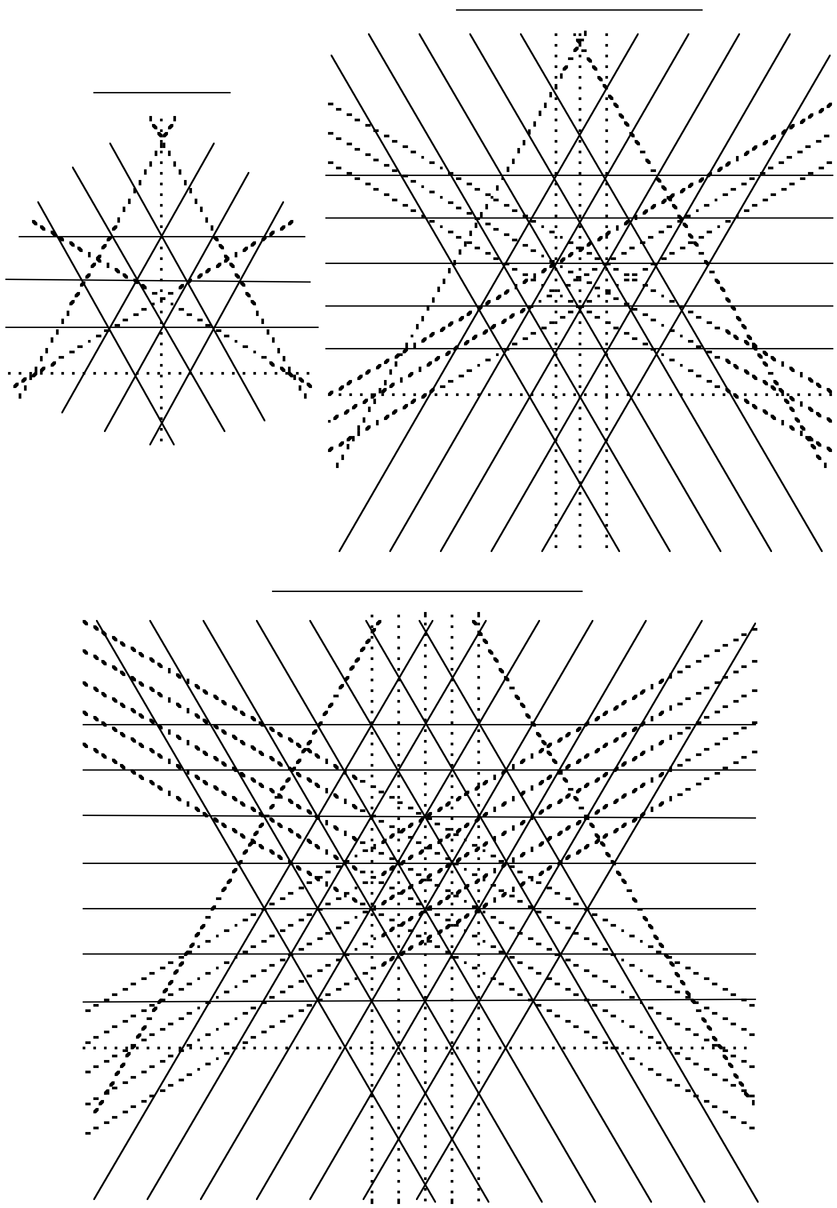

Figure 3. The first three members of an infinite sequence of biased families of lines with s = 4.

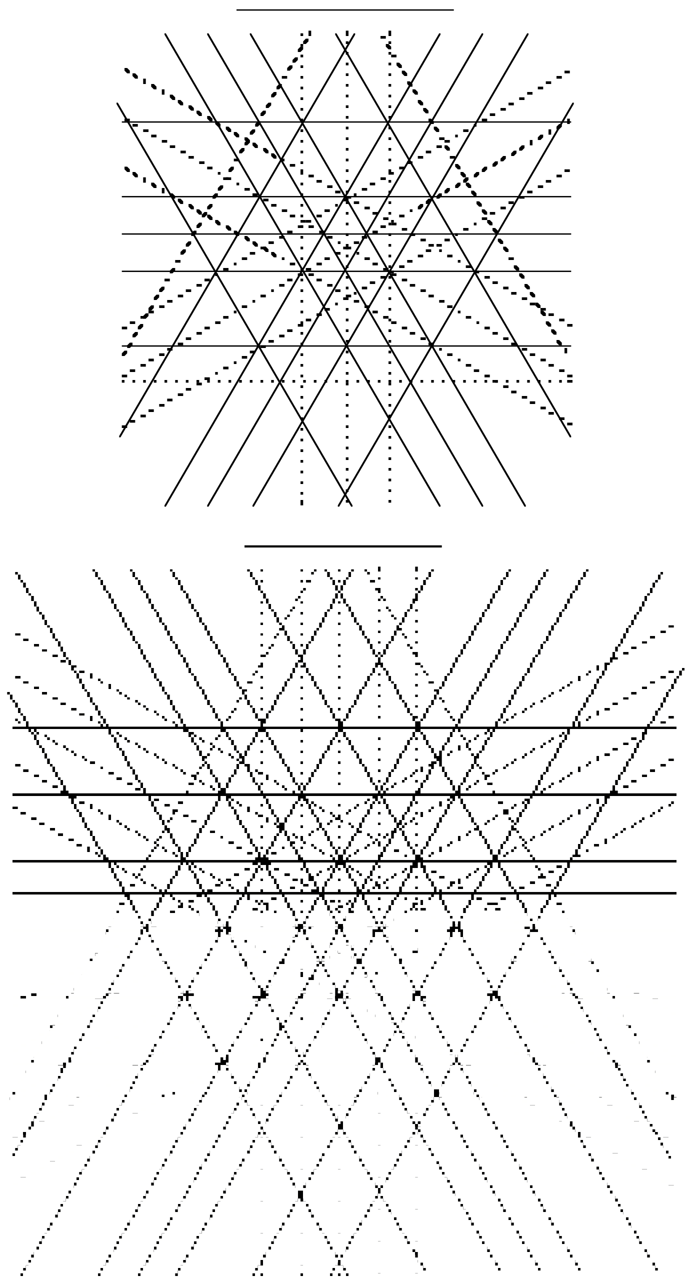

Figure 4. Two members of another infinite sequence of biased families of lines with s = 4.

(1) Is the biased family in Figure 5 unique — or do there exist other families with $s = 6$? If other families exist, is their number finite, or infinite ?

(2) Do there exist biased families with $s > 6$? If such families exist, is there a finite bound on s, or can one find biased families with arbitrarily large s ?

(3) Can one characterize biased families with a single monochromatic vertex ? The example in Figure 2 disproves the conjecture made in Grünbaum [1975] that for such families $s \leq 1$.

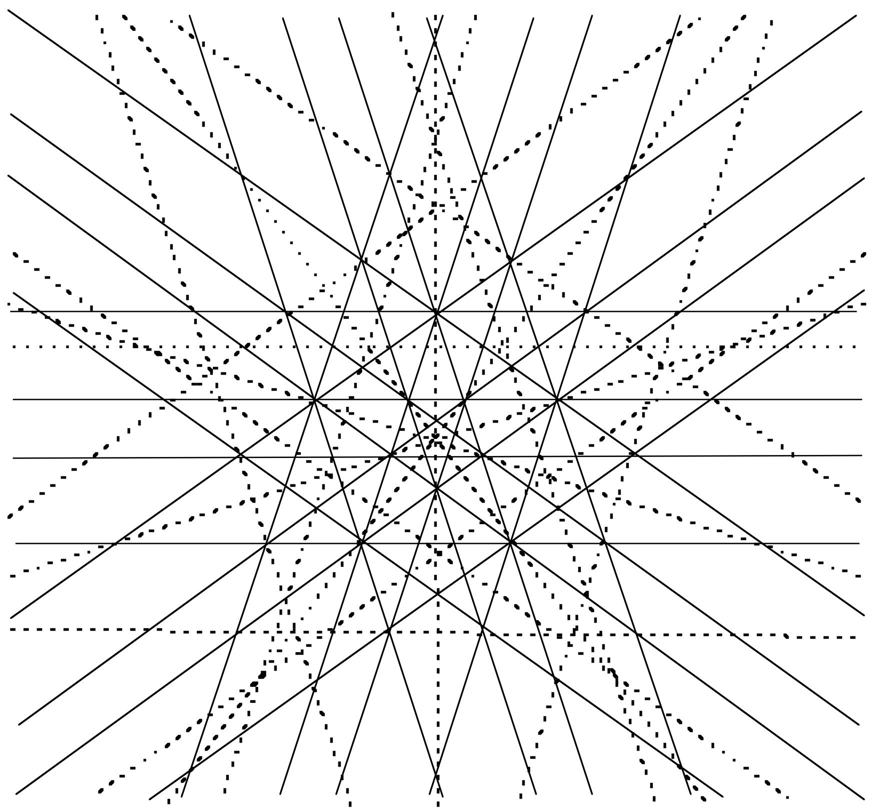

Figure 5. A biased family of lines with $s = 6$.

Remarks.

1. The presentation in Motzkin [1967] deals with the dual of the situation described here: given finite families of points, each point of one of two colors, are there monochromatic lines. While this formulation and the one used here are clearly equivalent, the examples in the illustrations seem to be more readily understandable in our version.

2. Our diagrams are understood to be situated in the projective plane, represented as the Euclidean plane extended by the "points at infinity" that correspond to pencils of parallel lines, and the "line at infinity" that consists of all the points at infinity. By taking a suitable projective image of these diagrams, equivalent examples could have been presented in the Euclidean plane; however, in doing so we would have lost the symmetry and simplicity visible in our illustrations.

References.

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