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ISOHEDRA WITH NONCONVEX FACES

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An isohedron is a 3-dimensional polyhedron all faces of which are equivalent under symmetries of the polyhedron. Many well known polyhedra are isohedra; among them are the Platonic solids, the polars of Archimedean polyhedra, and a variety of polyhedra important in crystallography. Less well known are isohedra with nonconvex faces. We establish that such polyhedra must be starshaped and hence of genus 0, that their faces must be star-shaped pentagons with one concave vertex, and that they are combinatorially equivalent to either the pentagonal dodecahedron, or to the polar of the snub cube or snub dodecahedron.

We shall use the word *polyhedron* in an elementary and familiar sense: a polyhedron P is a bounded three-dimensional solid in ordinary Euclidean space, whose boundary is the union of a finite number of (flat) polygonal regions called *faces*. The interiors of the faces are assumed to be disjoint, but their boundaries meet in pairs in line segments called *edges* or in a single point, or not at all. The endpoints of the edges are the *vertices* of P, and the faces meeting at a vertex V form a single circuit; hence the boundary of P is what is usually known as a manifold. The number of faces that meet at V (which is equal to the number of edges of which V is an endpoint) is called the *valence* of V, and the valence of any vertex is at least three.

We note that our interpretation of the word "polyhedron" does not include certain figures which are sometimes called polyhedra in the literature. For example, we exclude the four well-known Kepler-Poinsot polyhedra (see CUNDY & ROLLET [5], or WENNINGER [18] Photos 23 to 26; three are shown in Figure 6 below) since these either have faces with selfintersections, or faces that meet at interior points. The objects called "polyhedra" by topologists often depart in various ways from

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our definition. On the other hand, we do not preclude the possibility that two or more faces of a polyhedron may be coplanar, or that two or more edges of a face may be collinear. An example is shown in Figure 1 of a solid (whose pointset coincides with that of a cube) which we interpret as a polyhedron with 12 faces, 30 edges and 20 vertices; each face has a rectangular shape but since it contains five vertices and five edges it must be regarded as a pentagon. Our definition also does not exclude polyhedra of higher genus, such as the "toroidal" polyhedra shown in Figure 2(a)(b), though, as we shall soon see, these cannot arise in the problem we shall be considering here.

A *symmetry* of a polyhedron P is any isometry which maps P onto itself. It is easy to verify that every symmetry of P, apart from the identity, is either a reflection or a rotation. The set of all symmetries of P forms a group under composition, called the *symmetry group* S(P) of P. Clearly, the centroid O(P) of the vertices of P must be left invariant by all the symmetries of P; of course, there may also be other points with this property. In the case where O(P) is the *unique* invariant point we shall refer to it as the *center* of P. A polyhedron P is *isohedral* (or, as we shall

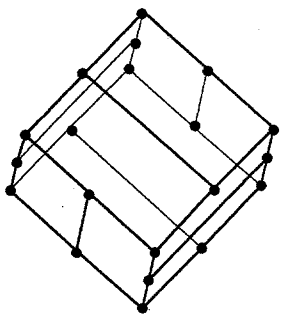


Figure 1. A polyhedron with 12 faces, coplanar in pairs; each face has the shape of a rectangle but is interpreted as a pentagon with two collinear edges. The vertices are indicated by solid dots. This polyhedron has the shape of a cube, but it is combinatorially equivalent to the regular pentagonal dodecahedron.

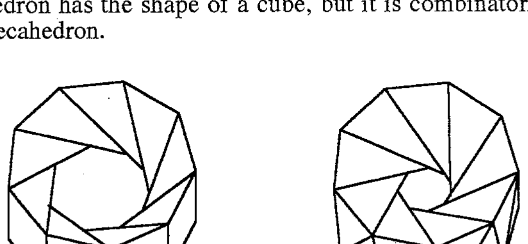


Figure 2. Two toroidal polyhedra, one (a) with 9 rectangles and 18 triangles as faces, the other (b) with 36 triangles.

say, is an *isohedron*) if, given any two faces F_1, F_2 of P, there exists a symmetry $\sigma \in S(P)$ such that $\sigma(F_1) = F_2$, that is, σ maps one of the given faces onto the other. This is equivalent to saying that S(P) is transitive on the faces of P, or that the faces of P belong to a single orbit under S(P). The polyhedron in Figure 1 is easily seen to be an isohedron, but those in Figure 2 are not. Clearly all the faces of an isohedron must be congruent polygons but the converse is not true, as can be seen from such well-known examples as some of the deltahedra (see Figure 3).

Many isohedra are quite familiar. Well known examples are the five regular (Platonic) solids, and the Catalan polyhedra (polars or duals of the Archimedean solids, named after the French mathematician E. C. Catalan), see Figure 4. All these polyhedra are convex, and they are described in detail in various publications; very convenient references are LINES [13] (where these polyhedra are said to be "vertically regular"), CUNDY & ROLLETT [5] and HAR'EL [12]. Convex isohedra combinatorially equivalent to the regular dodecahedron were investigated by BIGALKE [1]. However, even though less familiar, nonconvex isohedra are easily constructed. For example, to each face of a regular solid we may adjoin a pyramid (based on an equilateral triangle, square, or regular pentagon) of appropriate height (see Figure 5), or excavate such a pyramid. In fact, the usual representations of three of the four Kepler-Poinsot polyhedra can be considered as nonconvex isohedra (if we interpret the diagrams naively, as solids with triangular faces, and forget about stellations and selfintersections), see Figure 6. Other examples of nonconvex isohedra are shown in COXETER et al. [3] (see the "stellations" denoted B, De, Ef_1g_1, Fg_2) and WENNINGER [19] (see Photos 23, 24, 26, 33, 37, 41, 42, 44, 45, 46, 48, 51, 53, 55, 57, 58). However, in all these cases the illustrations must be given the appropriate *naive* interpretation and *not* that intended by their authors, as polyhedra with selfintersections.

It is worth remarking that whereas nonconvex isohedra with triangles or convex quadrangles as faces exist (see Figures 5, 6, 7) there are no such isohedra whose faces are convex pentagons. A proof of this will be given elsewhere. Of course, without the condition of isohedrality nonconvex

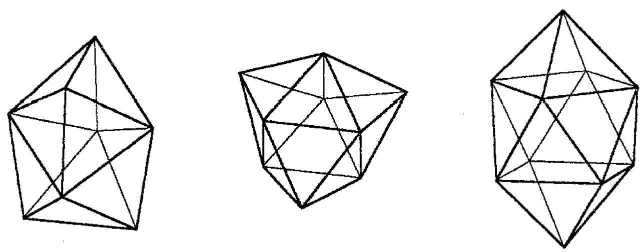


Figure 3. Three deltahedra, convex polyhedra all faces of which are congruent equilateral triangles. For each, the symmetry group is not transitive on the faces, so the polyhedron is not isohedral.

polyhedra bounded by convex (even regular) pentagons exist, as can be seen from the example obtained by glueing together two regular pentagonal dodecahedra at a face of each. It is also possible to construct nonconvex polyhedra in which all faces are convex hexagons. These polyhedra have genus greater than 1 (that is, their surfaces are homeomorphic neither to the sphere, nor to the torus), see WILLS [20] and, as we shall see, they cannot be isohedral.

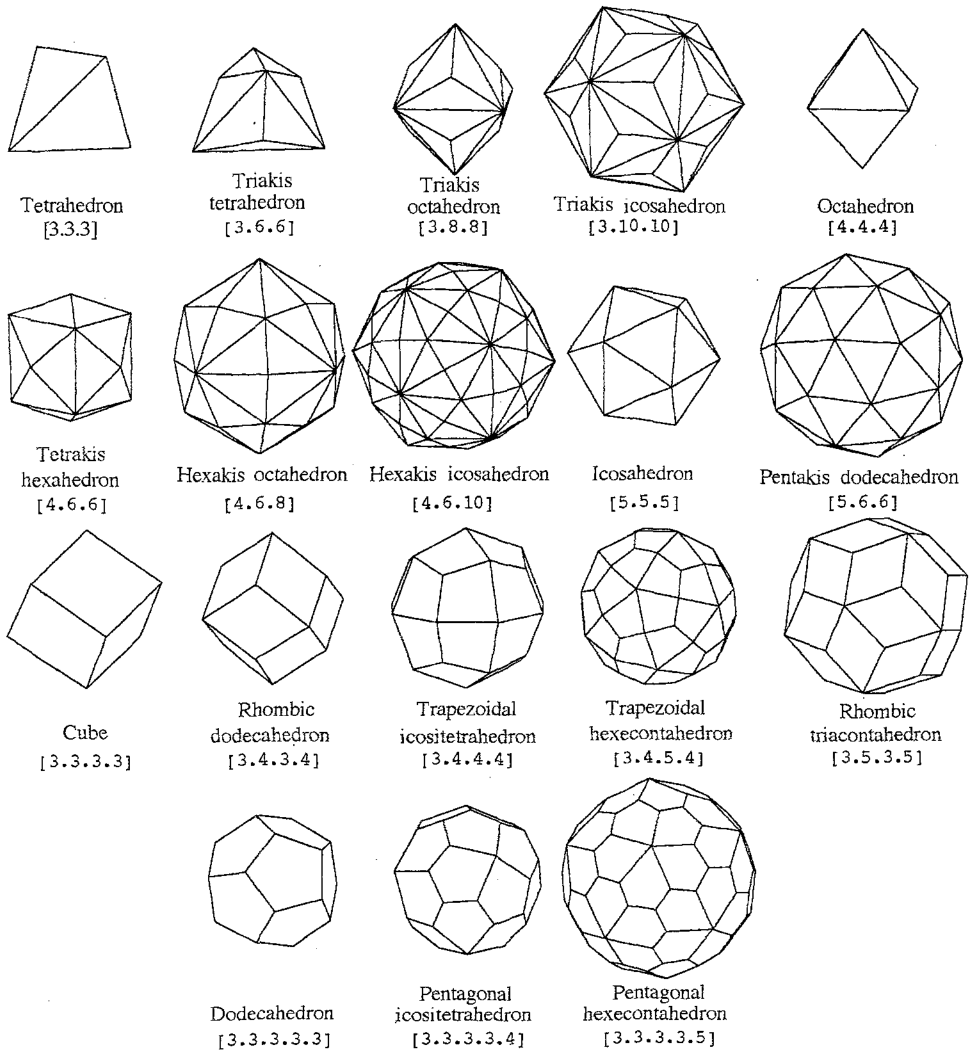


Figure 4. The five regular (Platonic) polyhedra, and the Catalan polyhedra. Near each is indicated its traditional name, as well its valence symbol (showing the valences of the vertices of each face).

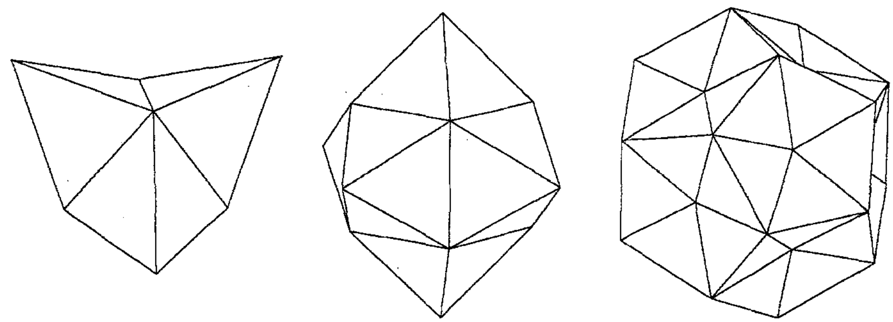


Figure 5. Examples of nonconvex isohedra with equilateral triangles as faces.

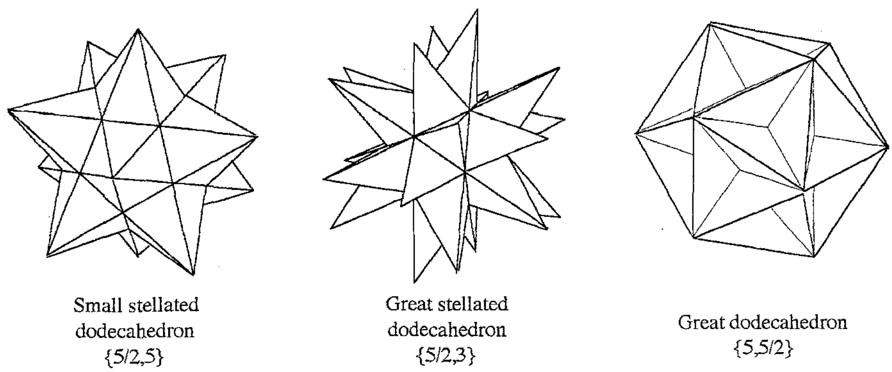


Figure 6. Models of three of the Kepler-Poinsot regular polyhedra can be interpreted as nonconvex isohedra with isosceles triangles as faces.

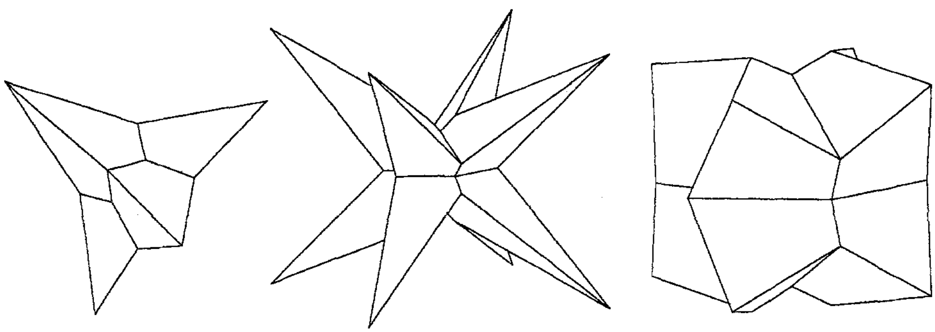


Figure 7. Examples of nonconvex isohedra with convex quadrangles as faces.

Much less familiar are isohedra with *nonconvex faces*. It is the purpose of this note to describe and illustrate some of these. Besides being of mathematical interest, they are aesthetically attractive and models of them are really beautiful. A complete classification of all these isohedra is outside the scope of this note, but we shall illustrate the various "types" (distinct in an intuitive sense) to show the great variety of possibilities.

We begin by stating some simple properties of isohedra.

(1) *An isohedron with nonconvex faces is nonconvex.* Let F_0 be a face of a convex isohedron P, and let $E(F_0)$ be the plane containing F_0. Then $E(F_0)$ is said to *support* P, which means that P lies entirely on, or to one side of, this plane. Now $E(F_0) \cap P$ is a convex polygon Q (because P is a convex polyhedron) and Q is the union of faces, say $F_0, F_1, \ldots, F_{n-1}$, of P. If $n = 1$, then $Q = F_0$ and so F_0 is convex. If $n = 2$, then $Q = F_0 \cup F_1$ and by isohedrality, there must exist either a reflection or a halfturn (a rotation of period 2) that maps F_0 onto F_1. In either case it is easily verified that F_0 and F_1 must be convex. Similarly, if $n > 2$, then $Q = F_0 \cup F_1 \cup \ldots \cup F_{n-1}$ and isohedrality implies that there exists a subgroup of order n of the symmetry group of P that maps F_0 onto each of the other F_i; this subgroup consists either of rotational symmetries alone, or of rotations and reflections. Again it is easily verified that this can only happen if F_0, and therefore all the F_i, are convex (see Figure 8(a)). Thus if P is convex all its faces are convex, from which we deduce that the existence of nonconvex faces implies the nonconvexity of P.

This result is only true on our assumption that every vertex of P has valence at least three. If vertices of valence two were allowed, that is, vertices where just two edges and faces meet, then it would be possible to express Q as a union of nonconvex faces F_i, for example as shown

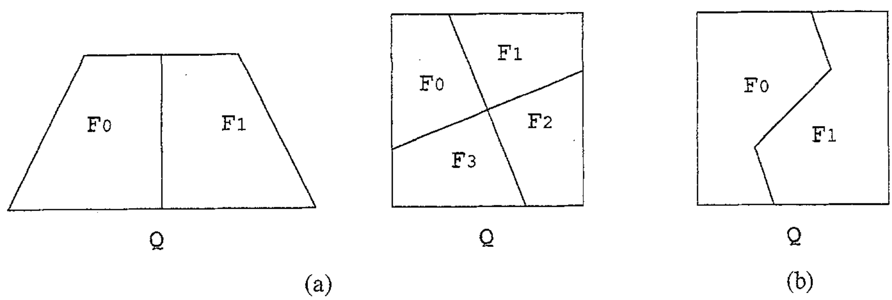


Figure 8. (a) If a convex polygon Q is partitioned into polygons $F_0, F_1, \ldots, F_n$ so that at least three meet at each vertex interior to Q, and all are equivalent under symmetries of Q, then each F_i must be convex. (b) If, on the other hand, one allows vertices interior to Q at which just two of the polygons F_i meet, then these polygons need not be convex.

in Figure 8(b). On the other hand, if we had insisted, in our definition of a polyhedron, that every face of P lies in a distinct plane, then we would have $F_0 = Q$ and so the property of isohedrality would not be required and the result would be true for polyhedra in general. We conjecture that the result continues to hold for all polyhedra (in the sense understood here).

(**2**) *An isohedron* P *necessarily has a center* O(P). In other words, there is a unique point, the centroid of P, which is invariant under all symmetries of P. To prove this, let us suppose that X is a point distinct from O(P) which is also invariant under every symmetry of P. Then it is easy to see that *every* point on the line L which joins O(P) to X also has this property. Consider the set of points in which the faces of P meet the line L. From this it must be possible to choose at least two distinct points; let F_1 and F_2 be any two distinct faces of P which contain these points. We have reached an impossible situation, for clearly no operation in S(P) can map F_1 onto F_2, contradicting the assumption that P is isohedral. Hence P has O(P) as its center.

(**3**) *An isohedron* P *is starshaped from* O(P). This means that if $Y \in P$, but $Y \neq O(P)$, then the closed line segment [Y, O(P)] lies entirely in P. This result is mentioned in [9], but no proof was published there. It may be established in the following way.

Let F be a face of an isohedron P, and let E(F) be the plane containing F. Since P need not be convex, it is possible that points of P lie on both sides of E(F). Even so, since F lies in the boundary of P, if X is any (relatively) interior point of F, the points of P that are sufficiently close to X must lie on E(F) or on *one* side of it. If they lie on the same side of E(F) as the center O(P), then we shall say that F is an *outside face* of P; if they lie on the opposite side then we say that F is an *inside face* of P. Clearly, by the isohedrality of P, since every symmetry of P preserves the center O(P), all faces of P must be either outside faces or all must be inside faces. If there were any inside faces then a ray from O(P) passing through a relatively interior point of one of these faces must necessarily meet the boundary of P again, and clearly it must do so in an outside face. We deduce that all faces of P are outside faces.

Now suppose that P is not starshaped from O(P). By definition it will be possible to find a point $Y \in P$ such that the line segment [Y, O(P)] is not contained in P. Since the same will then hold for all points of P sufficiently close to Y, no generality is list if we assume that Y is an interior point of P. Let Z be the point of the [Y, O(P)] nearest to O(P), such that the segment [Y, Z] lies in P. The point Z cannot coincide with O(P) -- if it did so then [Y, O(P)] would lie in P, contrary to assumption. Now Z lies on the boundary of P and must belong to a face F such that E(F) does not contain the line segment [Y, Z]. Since Y is on the opposite side of E(F) to O(P) we deduce that F must be an inside face of P. But this is impossible since all faces of P are outside faces. This contradiction shows that our original assumption was incorrect. Hence P contains the segment [Y, O(P)] for every point Y in P, and so P is starshaped from O(P).

Property (**3**) has many important consequences. One is that *every isohedron is homeomorphic to a three-dimensional ball.* In other words it cannot be toroidal (like the polyhedra in Figure 2(a)(b)) or, more generally, of genus greater than 0. Consequently, Euler's theorem

(**4**) $$v - e + f = 2$$

holds for every isohedron P, where v, e and f are the numbers of vertices, edges and faces of P, respectively.

Another consequence of property (3) is that radial projection from O(P) onto a sphere K centered at O(P) induces a bijection from the boundary of P onto K. The images of the faces of P under this bijection are spherical polygons which form an isohedral tiling T on K. That is, they cover K without gaps or overlaps, and the symmetry group S(T) of this tiling (which contains S(P) but is not necessarily equal to it) is transitive on the tiles of T. Such tilings have been investigated (see GRÜNBAUM & SHEPHARD [8]), and the results that have been obtained yield a powerful method of determining all kinds of isohedra. In particular, they can be classified, not only by their symmetry groups, but also by their *topological type*, in the following way. Suppose that each face of an isohedron P is an n-gon, and the valences of the vertices as we go round a face are $v_1, v_2, \ldots, v_n$. By isohedrality, these numbers must be the same for every face of P, up to cyclic permutations or reversal of order, so we may say P *is of type* $[v_1 \cdot v_2 \cdot \ldots \cdot v_n]$. For definiteness, amongst the possible orderings we shall usually choose that which is lexicographically first.

In [8] a result analogous to the following was used in the classification of spherical tilings on K.

(**5**) *Every isohedron* P *of type* $[v_1 \cdot v_2 \cdot \ldots \cdot v_n]$ *satisfies the relation*

(**6**) $$\frac{1}{v_1} + \frac{1}{v_2} + \ldots + \frac{1}{v_n} > \frac{n-2}{2}.$$

Moreover, if we denote by d *the difference between the two sides of the above inequality, then the number of faces of* P *is* 2/d.

To prove (**5**), suppose that each face of P has n_i vertices of valence i (for i = 3, 4, 5, ...), that is, i occurs n_i times in the list of valences $v_1, v_2, \ldots, v_n$. Then $n_i f$ is the total number of vertices of P of valence i, each counted i times (once for each of the i faces that contain it), so the actual number of such vertices is $\frac{n_i}{i}f$. Summing over all the different valences i that occur at vertices of P we deduce that

$$(7) \qquad \sum_i \frac{n_i}{i} f = v = 2 + e - f .$$

The right equality follows from Euler's theorem (**4**) which, as we have seen, holds for every isohedron. Since every face of P has n sides, and every edge of P is the side of two faces, we have $e = \frac{n}{2}f$. Substituting this value of e in (**7**), and dividing through by f, yields

$$\sum_i \frac{n_i}{i} = \frac{2}{f} + \frac{n}{2} - 1.$$

The left side can be written $\frac{1}{v_1} + \frac{1}{v_2} + \ldots + \frac{1}{v_n}$, and the right side is $d + \frac{n-2}{2}$; hence relation (**6**) holds and $f = \frac{2}{d}$.

From (**6**), since $v_k \geq 3$ (for $k = 1,2,\ldots,n$) and $n \geq 3$, we deduce that $n = 3$, 4, or 5, that is, every isohedron has triangular, quadrilateral, or pentagonal faces. Moreover, by examining the solutions of (6) in integers $v_1, v_2, \ldots, v_n$, it is straightforward (if time-consuming) to determine all possible types. The only difficulty that arises stems from the fact that not *every* solution of (**6**) in positive integers corresponds to a possible type. For example, $n = 3$, $v_1 = 3$, $v_2 = 5$, $v_3 = 5$ satisfy (**6**), yet no polyhedron of type [3·5·5] exists. These spurious solutions are quite easily eliminated using *ad hoc* arguments, and we can thus obtain the following complete list of topological types of isohedra:

[3·3·3], [3·3·3·3], [3·3·3·3·3], [3·3·3·3·4], [3·3·3·3·5], [3·3·3·k] where $k > 3$, [3·4·3·4], [3·4·4·4], [3·4·5·4], [3·5·3·5], [3·6·6], [3·8·8], [3·10·10], [4·4·4], [4·4·k] where $k \geq 3$ and $k \neq 4$, [4·6·6], [4·6·8], [4·6·10], [5·5·5], [5·6·6].

It is worth noticing that each of these types is represented by either a regular polyhedron or a Catalan polyhedron, see Figure 4.

After these preliminaries we can address our main problem: *which of these types can be represented by isohedra with nonconvex faces* ? Those types with triangular faces (that is, for $n = 3$) can be immediately rejected, so the next question is whether isohedra exist whose faces are nonconvex quadrangles; for brevity, we shall call such quadrangles *darts*. The negative answer to this question is given by the following result.

(**8**) *It is impossible for the faces of an isohedron* P *to be dart-shaped.*

This statement is phrased in the above manner since we do not wish to exclude the case where a vertex of P lies on the side of a face, so that although the face is shaped like a dart, it is a pentagon (in that its boundary contains five vertices of P, in exactly the same way as the polyhedron in Figure 1 must be regarded as having pentagonal faces, in spite of the fact that each is shaped like a rectangle).

We shall consider first the case where each face is a quadrangle (contains only four vertices of the polyhedron), see Figure 9(a). Let the number of faces of P be f; as each edge is the intersection of precisely two faces, the number of edges of P is $e = 2f$. Now let us estimate the number of vertices of P. For each face F of P, the points on its boundary which are its vertices (and also vertices of P) are of two kinds: three L-points at which the face angle of the dart is *less* than π, and one G-point at which the face angle is *greater* than π (that is, the angle is reflex).

Consider any vertex V of P. Several faces of P meet at V, and for each such face V will be either an L-point or a G-point. We assert that V *cannot be a* G*-point for two distinct faces.* For, by elementary geometry, if V coincided with the reflex angle of more than one face, P could not be starshaped from *any* point. This contradicts property (**3**) of isohedra given above. Hence the vertices of P are of two types:

(i) Vertices V which are a G-point for one face and an L-point for all the other faces meeting at V. Since each face has exactly one G-point, the number of such vertices is f.

(ii) Vertices V which are L-points for all the faces that meet there. Denote the number of such vertices by v_0.

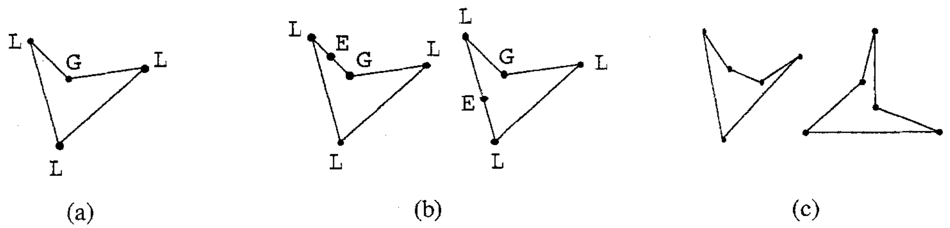


Figure 9. Examples of nonconvex polygons. In (a) and (b) are shown dart-shaped polygons referred to in statement (8). The solid dots represent possible positions of the vertices of P. The polygon in (a) must be regarded as a 4-gon, those in (b) as 5-gons. The meaning of the letters is explained in the text. In (c) two pentagons are shown, each with two reflex angles. As noted in the text such pentagons cannot be faces of isohedra.

The total number of vertices is therefore $v = f + v_0$, and if we substitute the values of v, e and f in Euler's theorem (**4**) we obtain

(**9**) $$2 = v - e + f = f + v_0 - 2f + f = v_0 .$$

We now make use of Descartes' Theorem which asserts that for any polyhedron P homeomorphic to a sphere

(**10**) $$\sum \delta(V) = 4\pi$$

where summation is over all vertices V of P, and $\delta(V)$ is the *deficiency* at vertex V, that is, $\delta(V) = 2\pi$ - (sum of the face angles at V). Descartes' Theorem is easily derivable from Euler's Theorem (and vice versa); for a thorough and critical discussion of the history of these results see Federico [6]. Notice that $\delta(V)$ may be positive, negative or zero, but $\delta(V) < 2\pi$ for every vertex V.

At vertices of type (i) it is easy to see that the sum of the face angles must exceed 2π, and so $\delta(V) \leq 0$. Hence the contributions of all vertices of this type (i) to the left side of (**10**) cannot be positive. Consequently the sum of the deficiencies of the v_0 vertices of type (ii) must be greater than or equal to 4π; but this is impossible since, by (**9**), $v_0 = 2$ and the deficiency at each such vertex is strictly less than 2π. This contradiction shows that the faces of an isohedron cannot be dart-shaped quadrangles.

For dart-shaped pentagons the above argument needs to be slightly modified. A vertex of P will lie on a side of each face, and we shall call this an E-point for the face because here the face angle is *equal* to π. Thus each face contains three L-points, one G-point and one E-point, see Figure 9(b). By elementary geometry, since P is starshaped, no vertex V can be a G-point of one face and an E-point of another, and since P is a polyhedron, no vertex V can be an E-point of more than one face. The latter follows from the fact that if V were an E-point of two or more faces (or an E-point for one face and a G-point for another face) then necessarily these faces or a third one at the same vertex would intersect at (relatively) interior points, a situation which is explicitly excluded by our definitions. Hence the vertices of P are of three types, namely (i) and (ii) as above and

(iii) Vertices V which are E-points of one face and L-points for all the other faces meeting at V. Since each face has at least one E-point, the total number of such vertices is either f.

Adding the numbers of vertices of the three types we obtain

$$v \geq f + f + v_0,$$

and so, since $e = \frac{5f}{2}$, by Euler's theorem (**4**),

$$2 = v - e + f \geq 2f + v_0 - \frac{5f}{2} + f > v_0.$$

Since the deficiency $d(V)$ at a vertex of type (iii) must be less than or equal to 0, we deduce from Descartes' Theorem (**10**), exactly as before, that the sum of the deficiencies at the $v_0 \leq 1$ vertices of type (ii) must be at least 4π. Thus we obtain a contradiction exactly as before, and it follows that an isohedron cannot have dart-shaped pentagons as faces. This concludes the proof of statement (**8**).

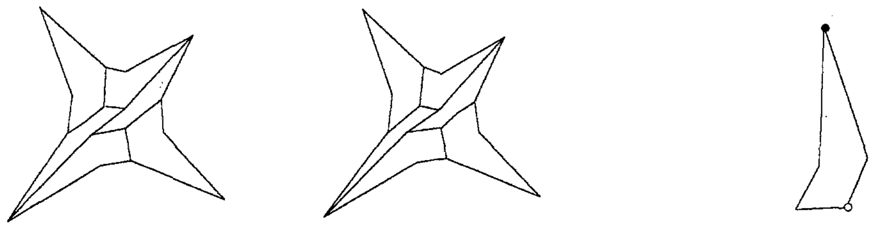
It is worth remarking that the second part of the proof can be modified in a simple manner to show that an isohedron cannot have faces which are pentagons with two reflex angles, see Figure 9(c). This implies the somewhat unexpected corollary: *Each isohedron has starshaped faces*.

The validity of Theorem (**8**) can probably be greatly expanded. Isohedrality plays no rôle in the proof beyond showing that P is starshaped (property (**3**)), and it seems likely that even this is not really required. Hence we may conjecture that Theorem (**8**) is true for all polyhedra. We know, however, that the theorem is not true if one allows P to have mutually intersecting faces.

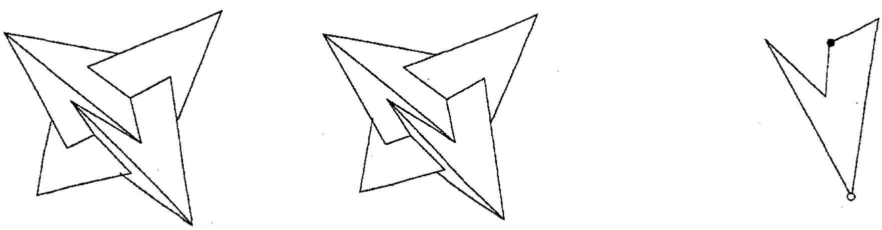
It is interesting to observe that it is possible to tile the plane by dart-shaped quadrilateral tiles in an isohedral manner (whatever shape of dart is chosen, see GRÜNBAUM & SHEPHARD [10], Section 9.2), but it is not possible to dissect a plane convex polygon into dart-shaped pieces, see SCHWENK [15], GALE [7].

Statement (**8**) shows that nonconvex faces of an isohedron can only be pentagons (with one reflex angle), and so an isohedron with nonconvex faces must be of one of the three types [3·3·3·3·3], [3·3·3·3·4] or [3·3·3·3·5]. For the first of these types there are three possibilities for the symmetry group, leading to a slightly finer classification of such isohedra. However, not all of these correspond to isohedra with nonconvex faces. The exception is that an isohedron of type [3·3·3·3·3] with symmetry group [3,5] (in the notation of COXETER & MOSER [4]) must be a regular dodecahedron, and consequently its faces are convex. In the other four cases isohedra with nonconvex faces exist, and in Figures 10, 11, 12 and 13 we show a number of examples of such isohedra. These examples were selected so as to show the great variability of the shapes of nonconvex isohedra, and represent what we consider to be the main types in a complete classification of nonconvex isohedra. To aid visualization, for each example we show a stereoscopic pair of views of the polyhedron, as well as the shape of a single face.

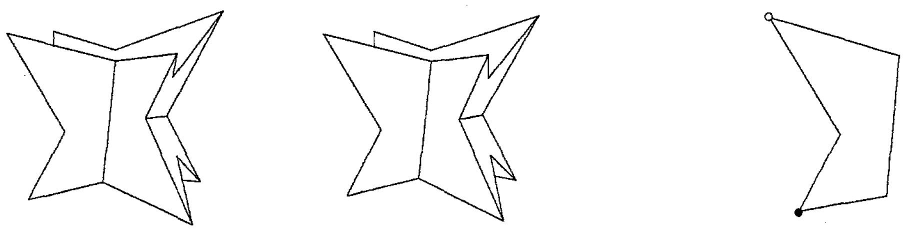
The expressions $f = \frac{2}{d}$ (given in statement (**5**)), $e = \frac{5f}{2}$ and $v = \frac{3f+4}{2}$ enable us to determine the numbers of elements of the polyhedron of each of the three types. Those of type [3·3·3·3·3] are combinatorially equivalent to the regular pentagonal dodecahedron: they have 12 faces, 30 edges and 20 vertices. Polyhedra of type [3·3·3·3·4] are combinatorially equivalent to a dual of the snub



$a = 1; b = 1; c = 1.15$



$a = 1; b = 1.1; c = 5$



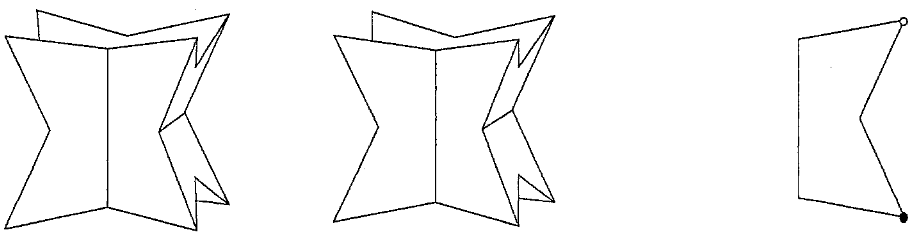
$a = 1; b = 0.5; c = -0.3$

Figure 10. Examples of the various kinds of isohedra of topological type [3·3·3·3·3] and symmetry group $[3,3]^+$, with nonconvex pentagons as faces. In this and the following illustrations we show for each example a stereoscopic pair of views of the isohedron, together with an orthogonal view of a single face, and the value of the parameters a, b and c. The z vertex (in the notation used in the APPENDIX) of the single face is indicated by a solid dot, and the y vertex by a hollow one.

cube, and have 24 faces, 60 edges and 38 vertices. Polyhedra of type [3·3·3·3·5] are combinatorially equivalent to a dual of the snub dodecahedron and have 60 faces, 150 edges and 92 vertices. Data determining these isohedra are collected in the Appendix. We believe that many of these isohedra have not been previously described or illustrated. The only examples of isohedra with nonconvex pentagonal faces we found in the literature are the following ones (in all except the last two, the authors intended their diagrams to represent selfintersecting polyhedra, and *not* isohedra as the term is used here): Coxeter *et al.* [3], stellations Ef_1, Ef_1f_2 (the former is the "compound of five tetrahedra", and appears in many places; see, for example, BRÜCKNER [2], Plate IX, #11, CUNDY & ROLLETT [5], Figure 173, WENNINGER [18], #24); WENNINGER [19], Photo 71; OUNSTED [14] and STEWART [16], p. 255.

A detailed description of the mutual relations between the various polyhedra shown here, and of other isohedra, is in preparation.

We conclude with some observations concerning *isogonal polyhedra* (or *isogons*), that is, polyhedra for which the symmetry group acts transitively on their vertices. In view of properties of duality that are often asserted or assumed, it may be thought that the behaviour of isogons should be very similar to that of isohedra. In fact, this is not so (see GRÜNBAUM & SHEPHARD [11]), and the differences between isohedra and isogons are much more striking than the similarities between them. For example, isogons necessarily have convex faces but need not be starshaped, and the boundary of an isogon can be a manifold of genus 1, 3, 5, 7, 11 or 19. Such isogons are described in [9]; two are shown above, in Figure 2.



a = 1; b = 0.7; c = -0.3

Figure 11. Example of an isohedron of topological type [3·3·3·3·3] and symmetry group $[3^+,4]$, with nonconvex pentagons as faces.

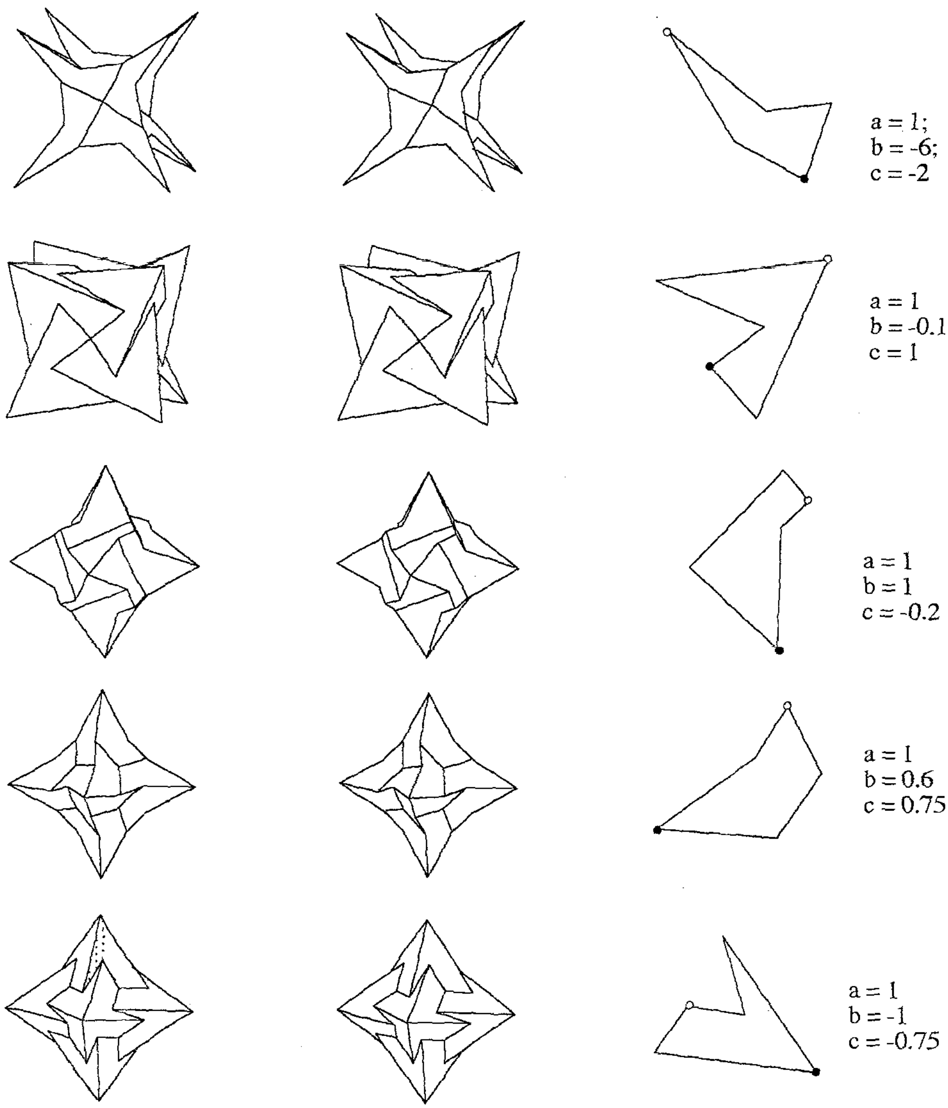


Figure 12. Examples of isohedra of topological type [3·3·3·3·4] and symmetry group $[3,4]^+$, with nonconvex pentagons as faces.

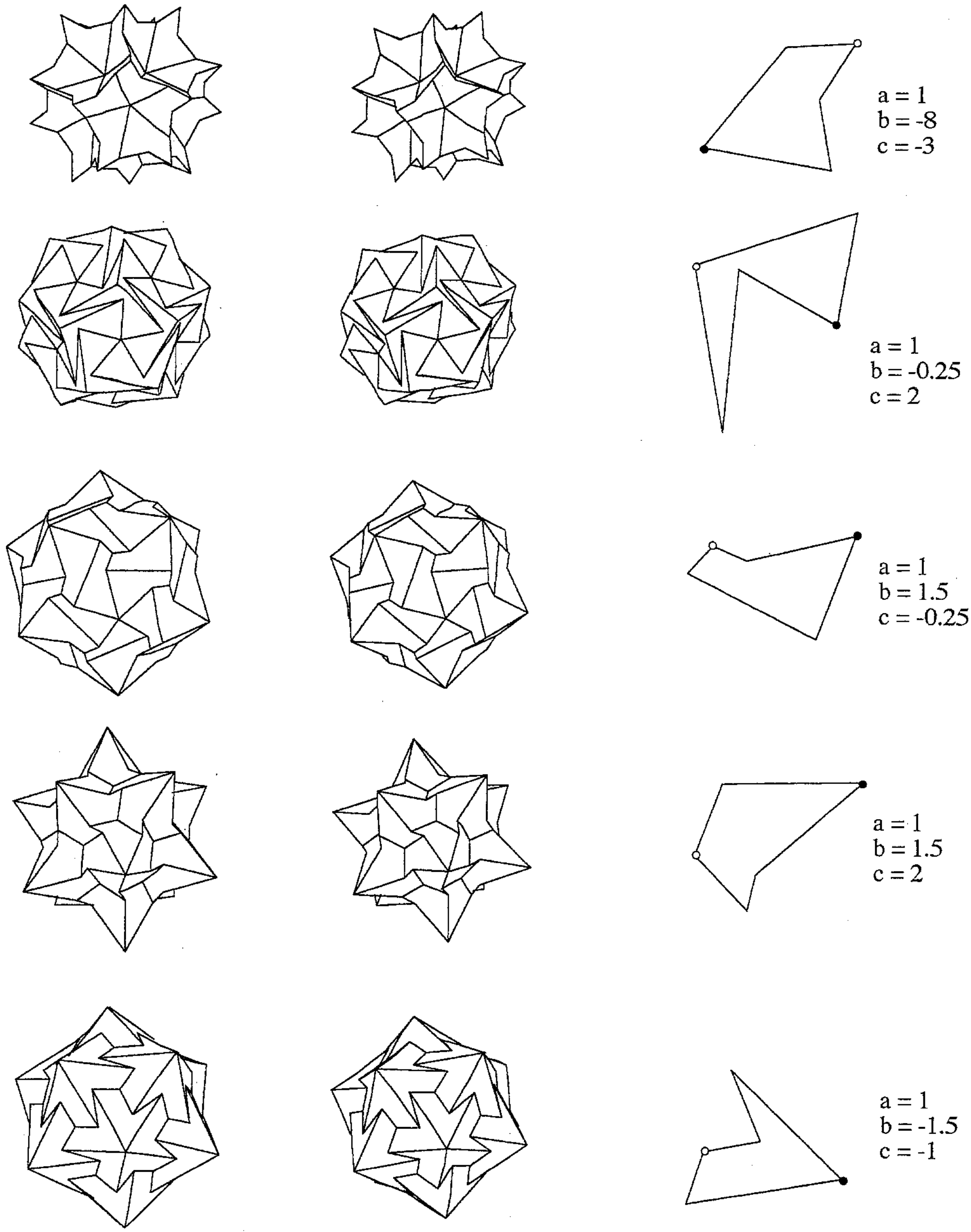


Figure 13. Examples of isohedra of topological type [3·3·3·3·5] and symmetry group $[3,5]^+$, with nonconvex pentagons as faces.

APPENDIX

For isohedra with non-convex faces (of which examples are shown in Figures 10, 11, 12 and 13) we explain here how to calculate the coordinates of the vertices, the shape and convexity character of the faces, and derive other information of a metric nature.

In each case we start by considering a regular polyhedron P (tetrahedron, octahedron or icosahedron) whose vertices are denoted by D_i, and a dual polyhedron P' (tetrahedron, cube or dodecahedron) whose vertices are denoted by E_i. Then, in terms of the parameters a, b and c indicated in the diagrams, the five vertices of a face F can be given, in order, by

$$(*)\quad \begin{cases} X_1 = aD_1+bE_1+cE_2 \\ X_2 = aD_2+bE_2+cE_1 \\ Y_2 = eE_2 \\ X_3 = aD_1+bE_2+cE_3 \\ Z_1 = dD_1 \end{cases}$$

Here d and e are functions of a, b, c, determined so as to make the five points (*) coplanar; the explicit results in the various cases are indicated below. Clearly, uniform scaling of the parameters corresponds to homothetic polyhedra; hence the parameters can be interpreted as homogeneous coordinates in a "space of isohedra" of a given topological type.

For each of the three topological types of isohedra, equations (*) describe not only the non-convex polyhedra, but also the convex ones, and "polyhedra" with selfintersecting faces or with faces that are simple polygons but in which faces intersect in various ways; these selfintersecting "polyhedra" are not included in our discussion. The full determination of the dependence of the shapes of polyhedra on the parameters a, b, c is quite complicated, and will be presented in detail in a separate publication. Here we show only examples of the main kinds of nonconvexities that can occur.

From the coordinates in (*) the shape of a face of the isohedra in the diagrams is easily determined, and hence models can be constructed out of cardboard or similar material. Some details concerning the three types of polyhedra are as follows.

(A) For isohedra of type [3·3·3·3·3] (Figures 10 and 11) we may take P as the tetrahedron whose four vertices have coordinates

$(\pm 1, \pm 1, \pm 1)$ with an even number of negative signs,

and as P' a dual tetrahedron with vertices at points with coordinates

$(\pm 1, \pm 1, \pm 1)$ with an odd number of negative signs.

In (*) we take $D_1 = (1, 1, 1)$, $D_2 = (-1, -1, 1)$, $E_1 = (-1, 1, 1)$, $E_2 = (1, -1, 1)$, $E_3 = (1, 1, -1)$, and then

$$d = \frac{a^2c-2abc+2ac^2-b^2c+c^3-2a^2b-2ab^2}{3c^2+ac-3bc-2ab},$$

$$e = \frac{a^2c-2abc+2ac^2-b^2c+c^3-2a^2b-2ab^2}{c^2-ac-bc-2ab}.$$

The remaining eleven faces can be obtained from F by applying the rotations of the symmetry group of the tetrahedron P, that is, the group $[3,3]^+$. If $d \neq e$ this is also the symmetry group of the isohedron; examples are shown in Figure 10. In this case the radial projection of the faces of the isohedron from O(P) onto a sphere centered at O(P) is an isohedral spherical tiling of type SIH50 un the notation of GRÜNBAUM & SHEPHARD [8]. The coordinates of the 20 vertices are

$(\pm e, \pm e, \pm e)$ with an odd number of negative signs,

$(\pm d, \pm d, \pm d)$ with an even number of negative signs,

even permutations of $(\pm a, \pm b, \pm g)$ with an even number of negative signs,

where $a = a - b + c$, $b = a + b - c$, $g = a + b + c$.

If $d = e$ the symmetry group is $[3^+,4]$. This occurs when either $c = 0$ and then $d = e = a \neq b$, or when $b = a + c$ and then $d = e = 2(a + c)^2/(a + 2c)$. For these isohedra there are symmetry operations interchanging the roles of the tetrahedra P and P' (see Figure 11), and the radial projection onto a sphere centered at O(P) is an isohedral spherical tiling of type SIH55.

(B) For isohedra of type [3·3·3·3·4] (see Figure 12) we may take P as the octahedron whose six vertices have coordinates

$(\pm 1, 0, 0)$ (all permutations of coordinates),

and then the coordinates of the eight vertices of a dual cube P' can be taken as

$(\pm 1, \pm 1, \pm 1)$ (all choices of signs).

In (*) we take $D_1 = (1, 0, 0)$, $D_2 = (0, 1, 0)$, $E_1 = (1, 1, 1)$, $E_2 = (1, 1, -1)$, $E_3 = (1, -1, -1)$, and

$$d = \frac{ac^2-3abc-2b^2c+2c^3-a^2b-2ab^2}{2c^2-2bc-ab},$$

$$e = \frac{ac^2-3abc-2b^2c+2c^3-a^2b-2ab^2}{2c^2-ac-2bc-2ab}.$$

The remaining 23 faces can be obtained from F by applying the rotations of the symmetry group of the octahedron P, that is, the group $[3,4]^+$, which is also the symmetry group of the isohedron. Radial projection of the faces of the isohedron from O(P) onto a sphere centered at O(P) is an isohedral spherical tiling of type SIH54, and the coordinates of the 38 vertices are

$(\pm e, 0, 0)$ (all permutations of coordinates),

$(\pm d, \pm d, \pm d)$ (all choices of signs),

even permutations of $(\pm a, \pm b, \pm g)$ with an even number of negative signs, or odd permutations with an odd number of negative signs.

Here $a = a + b + c$, $b = b + c$, $g = b - c$.

(C) For isohedra of type [3·3·3·3·5] (see Figure 13) we may take P as the icosahedron whose twelve vertices have coordinates

$(\pm\tau, \pm 1, 0)$, $(0, \pm\tau, \pm 1)$, $(\pm 1, 0, \pm\tau)$

and then the coordinates of the twenty vertices of a dual dodecahedron P' are

$(\pm\tau, \pm\tau, \pm\tau)$, $(\pm 1, \pm\tau^2, 0)$, $(0, \pm 1, \pm\tau^2)$, $(\pm\tau^2, 0, \pm 1)$,

where $\tau = (1 + \sqrt{5})/2$. In (*) we take $D_1 = (0, \tau, 1)$, $D_2 = (\tau, 1, 0)$, $E_1 = (1\ \tau^2, 0)$, $E_2 = (\tau, \tau, \tau)$, $E_3 = (0, 1, \tau^2)$, and

$$d = \frac{(a^2c+3abc+b^2c-c^3-ac^2+2ab^2)+\tau(2a^2b+2ab^2+4abc+2b^2c-2c^3)}{(ac+2bc-2c^2)+\tau(2ab+bc-c^2)},$$

$$e = \frac{(a^2c+3abc+b^2c-c^3-ac^2+2ab^2)+\tau(2a^2b+2ab^2+4abc+2b^2c-2c^3)}{(2ab+ac+bc-c^2)+\tau(2ab+ac+2bc-2c^2)}.$$

The remaining 59 faces can be obtained from F by applying the rotations of the symmetry group of the icosahedron P, that is, the group $[3,5]^+$, which is also the symmetry group of the isohedron. Radial projection of the faces of the isohedron from O(P) onto a sphere centered at O(P) is an isohedral spherical tiling of type SIH55.

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