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Cyclic ratio sums and products

The well known classical theorems of Menelaus and Ceva deal with certain properties of triangles by relating them to the products of three ratios of directed lengths of collinear segments. Less well known is a theorem of Euler [2] which states, in the notation of Figure 1, that $\sum_{j=1}^{3} \|QB_j/A_jB_j\| = 1$ for every triangle $T = [A_1A_2A_3]$. However, while the theorems of Menelaus and Ceva have been generalized to arbitrary polygons, and in many other ways — see, for example, [4] [5] [6] — until very recently there have been no analogous generalizations of Euler's result. One explanation for this situation may be that attempts at straightforward generalizations lead to invalid statements. An example of such a failed "theorem" is given by the question whether, in the notation of Figure 2, $\sum_{j=1}^{n} \|QB_j/A_jB_j\|$ equals 1 or some other constant independent of Q and the polygon. Recently, Shephard [9] had the idea, apparently not considered previously, of attaching to the ratios $r_j = \|(QB_j)/(A_jB_j)\|$ certain weights w_j, which depend on the polygon $\rceil = [A_1, A_2, \ldots , A_n]$ but not on the point Q, such that $\sum_{j=1}^{n} w_j r_j = 1$. (In fact, Shephard established a much more general result in this spirit; its complete formulation would lead us too far from the present aims.)

By sheer chance, the same day I received from Shephard a preprint of [1], I happened to read [7], in which two different sums of ratios appear, one in Bradley's solution, the other in Konečný's comments. This coincidence lead me to consider whether these results could be generalized along Shephard's idea. As it turns out, the answer is affirmative, and leads to a number of other results.

Let $\rceil = [A_1, A_2, \ldots , A_n]$ be an arbitrary n-gon, and Q an arbitrary point, subject

only to the condition that all the points B_j mentioned below are well determined. On

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each side A_jA_{j+1} of $\rceil$ (understood as the unbounded line) the point B_j is the intersection with the line through Q parallel to $A_{j+1}A_{j+2}$. (Here, and throughout the present note, subscripts are understood mod n). This is illustrated in Figure 3 by an example with $n = 5$. We are interested in the ratios $r_j = \|B_jA_{j+1}/A_jA_{j+1}\|$. We denote by $\Delta(UVW)$ the signed area of the triangle UVW with respect to an arbitrary orientation of the plane and, more generally, by $\Delta(\rceil)$ the signed area of any polygon $\rceil$, calculated with appropriate multiplicities for the different parts if $\rceil$ has self-intersections.

Theorem 1. For each polygon $\rceil$ we have $\sum_{j=1}^{n} w_j\, r_j = \Delta(\rceil)$ for all Q, where $w_j = \Delta(A_jA_{j+1}A_{j+2})$ are weights that depend on the polygon $\rceil$ but are independent of the point Q.

For a proof it is sufficient to note that

(i) by straightforward calculations or by easy geometric arguments it can be shown that $r_j = \Delta(QA_{j+1}A_{j+2})/\Delta(A_jA_{j+1}A_{j+2})$; and

(ii) therefore the sum $\sum_{j=1}^{n} w_j\, r_j$ is equal to $\sum_{j=1}^{n} \Delta(QA_{j+1}A_{j+2}) = \Delta(\rceil)$, since the triangles with vertex Q triangulate the polygon $\rceil$.

As a corollary we deduce at once that $\sum_{j=1}^{n} w_j\, s_j = \left(\sum_{j=1}^{n} w_j\right) - \Delta(\rceil)$, where $s_j = \|A_jB_j/A_jA_{j+1}\| = 1 - r_j$.

In the special case that ⌉ is a *regular* (n/d)-*gon*, all the weights w_j are equal to the value $w = 4\sin^3(d\pi/n)\cos(d\pi/n)$. (The regular (n/d)-gon has n vertices and surrounds its center d times. Successive vertices are obtained by rotation through $2\pi d/n$, see [1]. It is usually assumed that n and d are coprime, but this is a restriction that is unnecessary here and in most other contexts, and downright harmful in some cases, see, for example, [3]). Hence, in this case one can divide throughout by w, and the result becomes

$$\sum_{j=1}^{n} r_j = \Delta(⌉)/w = \frac{n}{4\sin^2(d\pi/n)} \quad . \tag{*}$$

Since the ratios r_j involve only collinear lengths, the sum is invariant under affinities, and so the result (*) remains valid for all *affine-regular* (n/d)-*gons* ⌉. (An (n/d)-gon is affine-regular if it is the image of a regular (n/d)-gon under a nonsingular affinity.) Thus in this special case we actually achieve the analogue of the generally invalid statement mentioned above. Since all triangles are affine-regular, this establishes the condition for concurrency found by Václav Konečný, mentioned in [7]. (We note that Shephard obtains in [9] the analogous generalization of Euler's result to affine-regular n-gons.) In the affine case, the above corollary can be simplified in the same way. For $n = 3$ this yields the condition for concurrency obtained by Bradley in [7].

From the above it follows that in the case of affine-regular polygons (but not for general polygons) we have

$$\sum_{j=1}^{n} \|B_jC_j/A_jA_{j+1}\| = -\frac{n\cos(2d\pi/n)}{2\sin^2(d\pi/n)} \quad , \tag{**}$$

where the C_j is the intersection of the line A_jA_{j+1} with the parallel through Q to the line $A_{j-1}A_j$ (see Figure 4). For $n = 3$ the right-hand side of (**) equals 1, and the result coincides with Problem 16 in [8].

It may be observed that for $n = 3$ and $d = 1$ the right-hand side of condition (*) equals 1, and the equality to 1 of the ratio sum is necessary and sufficient for the three parallels to the sides of the triangle to be concurrent, just as the equality to 1 of the product in Ceva's theorem for triangles is necessary and sufficient for theconcurrence of the Cevians. However, for $n > 3$ it is not obvious that the weights given above are the only ones which yield the right-hand constants for all Q, although one may conjecture that this is the case. Naturally, for particular choices of ⌉ and Q other weights may be used.

The expression for r_j obtained in (i), together with the analogous formula for the ratio $t_j = \|A_jC_j/A_jA_{j+1}\|$ (in the notation of Figure 4) leads at once to the following:

Theorem 2. For each polygon ⌉ with we have $\prod_{j=1}^{n} \frac{r_j}{t_j} = \prod_{j=1}^{n} \|B_jA_{j+1}/ A_jC_j\| = 1$ for all Q.

Finally, since $QB_jA_{j+1}C_{j+1}$ is a parallelogram for every j, we also have $\prod_{j=1}^{n} \|B_jQ/QC_{j+2}\| = 1.$

This last is a Ceva-type result which seems not to have been noticed previously.

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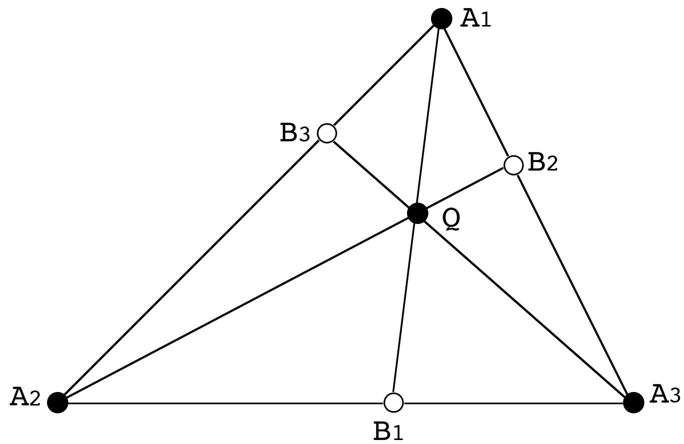

Figure 1. A theorem of Euler states that if B_j is the intersection of the line A_jQ with the side of the triangle $A_1A_2A_3$ opposite to A_j, then $\sum_{j=1}^{3} \|QB_j/A_jB_j\| = 1$. Here and throughout the note, $\|MN/RS\|$ stands for the ratio of signed lengths of the collinear segments MN and RS.

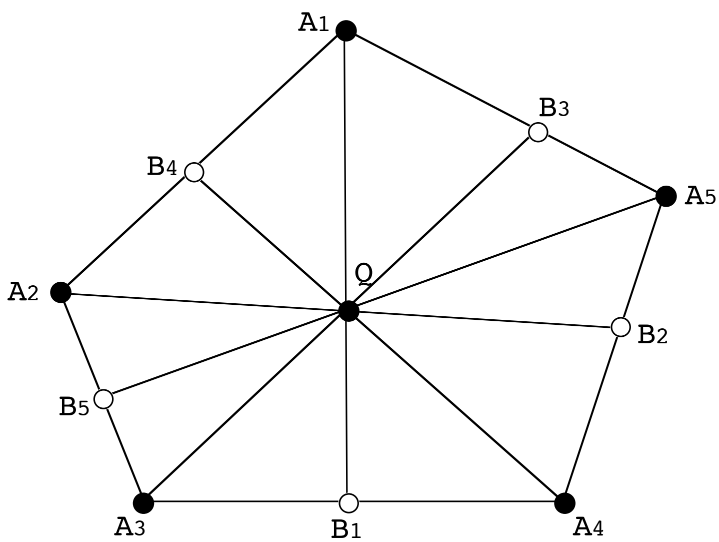

Figure 2. Attempts to generalize Euler's theorem in the form $\sum_{j=1}^{n} \|QB_j/A_jB_j\| = \text{const.}$ necessarily fail for $n > 3$ (here $n = 5$). However, as shown by Shephard [9], it is possible to find weights w_j which depend on the polygon but not on the position of Q, such that $\sum_{j=1}^{n} w_j \|QB_j/A_jB_j\| = 1$.

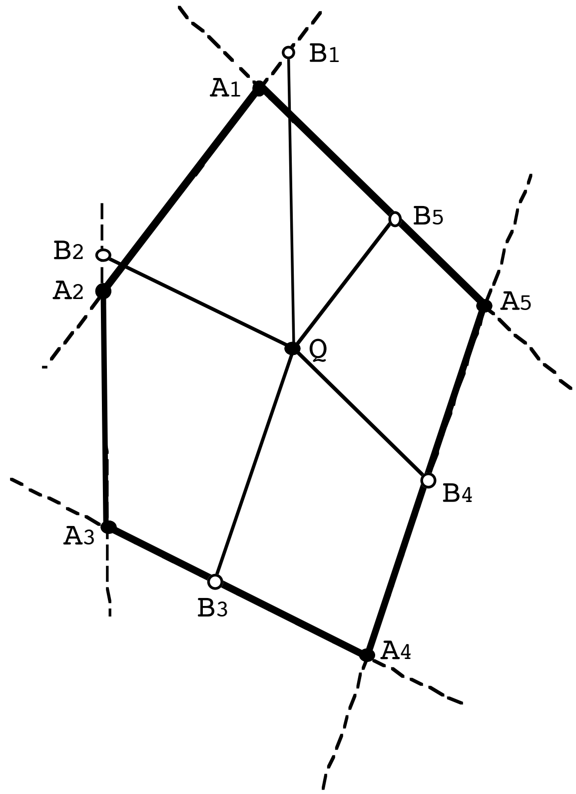

Figure 3. The point B_j is the intersection of the line A_jA_{j+1} with the parallel through Q to the line $A_{j+1}A_{j+2}$. The ratios $r_j = \|B_jA_{j+1}/A_jA_{j+1}\|$ of directed segments are considered in Theorem 1.

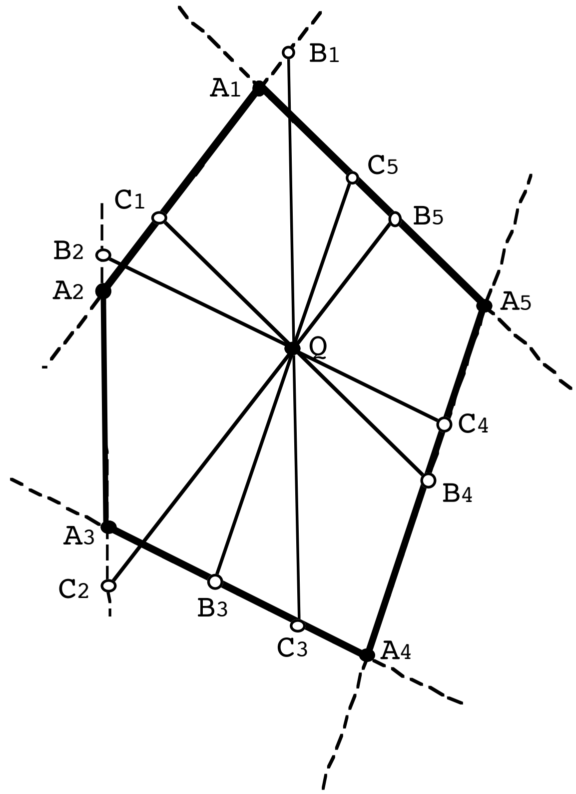

Figure 4. The point B_j is obtained as in Figure 3, while the point C_j is the intersection of the line A_jA_{j+1} with the parallel through Q to the line $A_{j-1}A_j$. The ratios $r_j = \|B_jA_{j+1}/A_jC_j\|$ of directed segments are considered in Theorem 2.