ON QUADRANGLES DERIVED FROM QUADRANGLES -- PART 3

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The first two parts of this series ([6], [7]) dealt with properties of quadrangles obtained by taking as vertices the incenters or the orthocenters of the triangles determined by triplets of vertices of given quadrangles. Here we shall consider an analogous construction where for every quadrangle $Q = V_1V_2V_3V_4$ a new quadrangle $\mathcal{C}(Q) = C_1C_2C_3C_4$ is formed by taking as vertices the circumcenters C_j (intersection points of perpendicular bisectors of the sides) of the triangles $T_j = V_{j-1}V_jV_{j+1}$, for $j = 1, 2, 3, 4$; throughout, subscripts should be reduced mod 4. This "circumcenter map" is illustrated in Figure 1; in slightly different terminology and notation this construction was considered in [4] and [5].

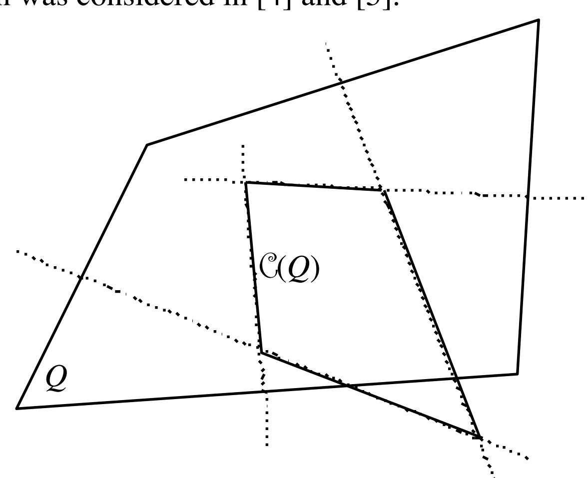

Figure 1. An illustration of the circumcenter map leading from a polygon Q to the polygon $\mathcal{C}(Q)$.

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In the present note we shall establish some apparently new properties of the circumcenter map. We also report on the discovery of a 150-years old solution of the problem posed by Langr [8] and discussed in [4] and [5], and present a solution of this problem that is simpler than the published ones. Throughout, we shall assume that the vertices of the quadrangles considered are in sufficiently general position for the constructions considered to be possible and have uniquely determined outcomes. However, the quadrangles need not be convex or simple.

The main result is the following fact about the circumcenter map, the proof of which will also lead to a description of the map in terms of geometric properties of the starting quadrangle:

Theorem 1. For every quadrangle Q, the quadrangle $O(Q)$ is affinely equivalent to Q under an affinity $\alpha = \alpha_Q$.

We recall that an *affinity* α is a linear transformation λ of the Euclidean plane onto itself, followed possibly by a translation τ. Theorem 1 was discovered experimentally, using "Geometer's Sketchpad"® and "Mathematica"® on a Macintosh computer.

The proof is a straightforward exercise in analytical geometry. It is preferably carried out using some symbolic algebra software (I used Mathematica® V.3 on a Macintosh PowerBook 1400), but the steps can be given easily enough.

We start by observing that (as easily verified) the circumcenter of a triangle $T = ABC$, where $A = (a_1, a_2)$, $B = (b_1, b_2)$, $C = (c_1, c_2)$, has coordinates

$$\frac{1}{4 \text{ area } T}\left(\det\begin{bmatrix} a_1^2 + a_2^2 & a_2 & 1 \\ b_1^2 + b_2^2 & b_2 & 1 \\ c_1^2 + c_2^2 & c_2 & 1 \end{bmatrix}, \det\begin{bmatrix} a_1^2 + a_2^2 & a_2 & 1 \\ b_1^2 + b_2^2 & b_2 & 1 \\ c_1^2 + c_2^2 & c_2 & 1 \end{bmatrix}\right).$$

Next we note that in order to show that two quadrangles are affinely equivalent it is sufficient to establish that the intersection points of the diagonals divide corresponding diagonals in equal ratios. Without loss of generality we assume that the vertices of Q are given by $V_1 = (a, b)$, $V_2 = (c, d)$, $V_3 = e(a, b) = (ea, eb)$, $V_4 = f(c, d) = (fc\, fd)$, and therefore the intersection point of the diagonals is at the origin O; see the notation in Figure 2. Then the intersection point X of the diagonals of $\mathcal{C}(Q) .= C_1C_2C_3C_4$ has coordinates given by the rather unwieldy expression

$$X = \left(\frac{(a^2 + b^2)(1 + e)d - (c^2 + d^2)(1 + f)b}{2(ad\text{-}bc)} \ , \ \frac{-(a^2 + b^2)(1 + e)c + (c^2 + d^2)(1 + f)a}{2(ad\text{-}bc)} \right)$$

.

The radius-vector OX is the translational part τ of the affinity α. The coordinates of the vertices C_j are complicated as well, but for the points $C_j - X$ we have the much more revealing expressions:

$$C_1 - X = \left(\frac{(-(a^2 + b^2)e + (c^2 + d^2)f)}{2(ad\text{-}bc)} d \ , \ \frac{((a^2 + b^2)e - (c^2 + d^2)f)}{2(ad\text{-}bc)} \ c \right)$$

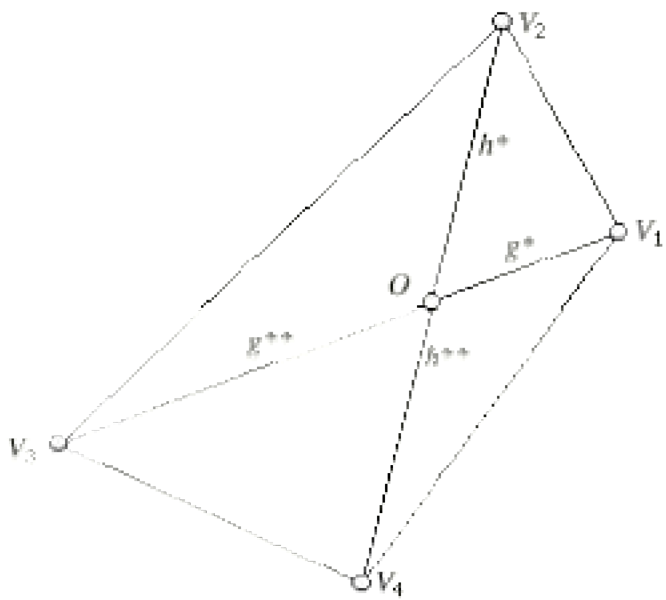

Figure 2. Illustration of the notation used in the text. The lengths g^* and g^{**} are signed, that is, they have opposite signs if O is between V_1 and V_3; similarly for h^* and h^{**}.

$$C_2 - X = \left(\frac{(-(a^2+b^2)e + (c^2+d^2)f)}{2(ad-bc)}b \quad , \quad \frac{((a^2+b^2)e - (c^2+d^2)f)}{2(ad-bc)}a\right)$$

$$C_3 - X = \left(\frac{(-(a^2+b^2)e + (c^2+d^2)f)}{2(ad-bc)e}d \quad , \quad \frac{((a^2+b^2)e - (c^2+d^2)f)}{2(ad-bc)e}c\right)$$

$$C_4 - X = \left(\frac{(-(a^2+b^2)e + (c^2+d^2)f)}{2(ad-bc)f}b \quad , \quad \frac{((a^2+b^2)e - (c^2+d^2)f)}{2(ad-bc)f}a\right) .$$

Using the notation indicated in Figure 2 this can be written as

$$C_1 - X = k\,(-d, c) = \frac{k}{f}\ V_4{}^{\perp}, \qquad C_2 - X = k\,(-b, a) = \frac{k}{e}\ V_3{}^{\perp}$$

$$C_3 - X = \frac{k}{e}\ (-d, c)\ = \frac{k}{e}\ V_2{}^{\perp}, \qquad C_4 - X = \frac{k}{f}\ (-b, a)\ = \frac{k}{f}\ V_1{}^{\perp}$$

where $k = \dfrac{(a^2+b^2)e - (c^2+d^2)f}{2(ad-bc)} = \dfrac{g^*g^{**} - h^*h^{**}}{4 \text{ area } OV_1V_2}$ and $V_j{}^{\perp}$ is the point obtained from V_j by a $90°$ counterclockwise rotation about the origin.

These are our main formulas. From them follows that $C_1 - X = e(C_3 - X)$ and $C_2 - X = f(C_4 - X)$, thus proving that $\mathbb{C}(Q)$ is an affine image of Q under an affinity α such that $\alpha(V_1) = C_3$, $\alpha(V_2) = C_4$, $\alpha(V_3) = C_1$, $\alpha(V_4) = C_2$. Moreover, we see that the diagonals of $\mathbb{C}(Q)$.are perpendicular to those of Q . Since the determinant Λ of the linear part λ of α equals the ratio of areas of $\mathbb{C}(Q)$.and Q we have

$$\Lambda = -\frac{k^2}{ef} = -\frac{(g^*g^{**} - h^*h^{**})^2}{16(\text{area } OV_1V_2)^2 ef} = -\frac{(g^*g^{**} - h^*h^{**})^2}{16(\text{area } OV_1V_2)(\text{area } OV_3V_4)}$$

This is of considerable interest in connection with Langr's problem discussed below. ◊

The simplicity of the main formulas allows to find the linear part of the second iteration of the circumcenter map. Let $\mathbb{C}(\mathbb{C}(Q)) = D_1D_2D_3D_4$ and let Y be the intersection point of the diagonals of $\mathbb{C}(\mathbb{C}(Q))$. In order to apply the main formulas for the quadrangle with vertices $C_j - X$, $j = 1,2,3,4$, we need to replace a, b, c, d, e, f, by $-kd$, kc, $-kb$, ka, $1/e$, $1/f$, respectively, and hence k by k/ef.

Then the main formulas imply that

$$D_1 - Y = \frac{k}{ef}\,\frac{1}{1/f}\,(C_4 - X)^{\perp} = \frac{k}{e}\,\frac{k}{f}\;V_1^{\perp\perp} = -\frac{k^2}{ef}\;V_1 = \Lambda V_1.$$

Similarly we find

$$D_2 - Y = \Lambda V_2,\; D_3 - Y = \Lambda V_3,\;\; D_4 - Y = \Lambda V_4,$$

which establishes the following

Theorem 2. The iteration of the circumcenter map $\mathbb{C}$ leads from a quadrangle Q to a quadrangle $\mathbb{C}(\mathbb{C}(Q))$ which is homothetic to Q in ratio Λ, the determinant of the linear part of the affinity which maps Q onto $\mathbb{C}(Q)$.

The value of Λ in terms of geometric parameters of Q can be easily determined using the expressions given above.

We conclude with some historical details; additional information can be found in [4], [5].

Langr [8] posed the problem of

(i) showing that $\mathbb{C}(\mathbb{C}(Q))$ is similar to Q; and

(ii) finding the ratio of similarity (which by the above is Λ).

Chou [3, Example 65] established part (i), and Shephard [9] gave for the solution of (ii) an expression which can be rendered in a more symmetric form as

$$-8\Lambda = \sum_j \frac{1}{\sin^2\theta_j} \;+\; \frac{\sin\theta_1\,\sin\theta_3 + \sin\theta_2\,\sin\theta_4}{\sin(\theta_1+\theta_3)\,\sin(\theta_1+\theta_4)} \cdot \sum_j\,(-1)^j\,\sin^2\theta;$$

here θ_j is the "deflection" at vertex V_j of Q, see Figure 3.

However, Langr's problem has already been solved more than 150 years ago! In a pair of papers, Bretschneider [1], [2] develops a long series of trigonometric and other formulas dealing with all sorts of entities that can be associated with four points. Among the (about one hundred) formulas, some of them so long that they had to be printed sideways on the pages, is an expression for Λ. To formulate Bretschneider's result, let us denote by d_{ij} the distance between vertices V_i and V_j of Q. Then Bretschneider first proves that the numbers $p = d_{01}\, d_{23}$, $q = d_{02}\, d_{13}$, and $r = d_{03}\, d_{12}$ (which involve the sides and the diagonals of the quadrangle) satisfy the triangle inequality. Then he considers the quantity (which corresponds to the Heron formula for the area of a triangle)

$$e = (p + q + r)\,(p + q - r)\,(p - q + r)\,(-p + q + r)/16$$

and defines a_j as the area of the triangle with vertices $V_{j\text{-}1}, V_j, V_{j+1}$. With this notation Λ is given by $\Lambda = \dfrac{e}{a_0\, a_1\, a_2\, a_3}$.

It is interesting that there seems to be no easy way of transforming one of the three expressions for Λ into another.

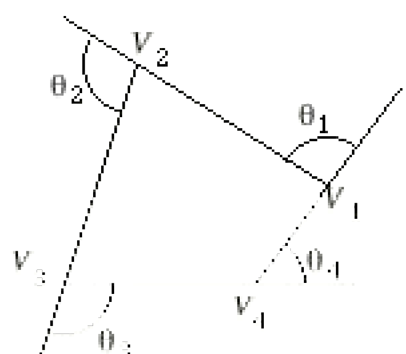

Figure 3. An illustration of the "deflection" θ_j at the vertex V_j.

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