ON QUADRANGLES DERIVED FROM QUADRANGLES -- PART 2

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In [2] we discussed some properties of quadrangles obtained by taking as vertices the incenters of the triangles determined by triplets of vertices of given quadrangles. Here we shall consider an analogous construction, using orthocenters instead of incenters.

Given a quadrangle $Q = V_1V_2V_3V_4$ a new quadrangle $O(Q) = H_1H_2H_3H_4$ is formed by the orthocenters (intersection point of altitudes) H_j of the triangles $T_j = V_{j-1}V_jV_{j+1}$, for $j = 1, 2, 3, 4$; throughout, subscripts should be reduced mod 4. This "orthocenter map" is illustrated in Figure 1.

Although this construction appears quite natural, it seems not to have been considered in the literature. Only one well–known result (see, for example, Coxeter [1, Section 1.7]), concerning the orthocenter of an arbitrary triangle, can be interpreted as dealing with the orthocenter map. This result is the following: If D is the orthocenter of the triangle $T = ABC$, then A is the orthocenter of BCD, B is the orthocenter of ACD, and C is the orthocenter of ABD; see illustration in Figure 2. This clearly implies that the quadrangle $Q = ABCD$ coincides (with a permutation of the vertices) with its image $O(Q) = CDAB$ under the orthocenter map.

Our main result, illustrated by the examples in Figure 3, concerns an unexpected aspect of the action of the orthocenter map on quadrangles. It was discovered experimentally, using computer software "Geometer's Sketchpad"® and "Mathematica"® on a Macintosh computer. To formulate this result we recall that an affinity is a linear transformation of the Euclidean plane onto itself, followed possibly by

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a translation; an equiaffinity is an area-preserving affinity. We have:

Theorem 1. For every quadrangle Q, the quadrangle $O(Q)$ is affinely equivalent to Q under an equiaffinity $\alpha = \alpha_Q$.

Proof. The proof is a straightforward exercise in analytical geometry. It is preferably carried out using some symbolic algebra software, but the steps can be given easily enough. Starting, for example, with vertices of Q given as $V_1 = (p,q)$, $V_2 = (1,0)$, $V_3 = (r,s)$, $V_4 = (0,1)$, we find that the vertices of $O(Q)$ have coordinates as follows:

$$H_1 = \left(\frac{q + pq - q^2}{-1 + p + q} , \frac{p - p^2 + pq}{-1 + p + q} \right),$$

$$H_2 = \left(\frac{-pq + pqr + q^2s + rs - prs - qs^2}{-q + qr + s - ps} , \frac{-p + p^2 + r - p^2r - r^2 + pr^2 - pqs + qrs}{-q + qr + s - ps} \right),$$

$$H_3 = \left(\frac{s + rs - s^2}{-1 + r + s} , \frac{r - r^2 + rs}{-1 + r + s} \right),$$

$$H_4 = \left(\frac{q - q^2 + pqr - s + q^2s - prs + s^2 - qs^2}{p - r + qr - ps} , \frac{pq - p^2r + pr^2 - pqs - rs + qrs}{p - r + qr - ps} \right).$$

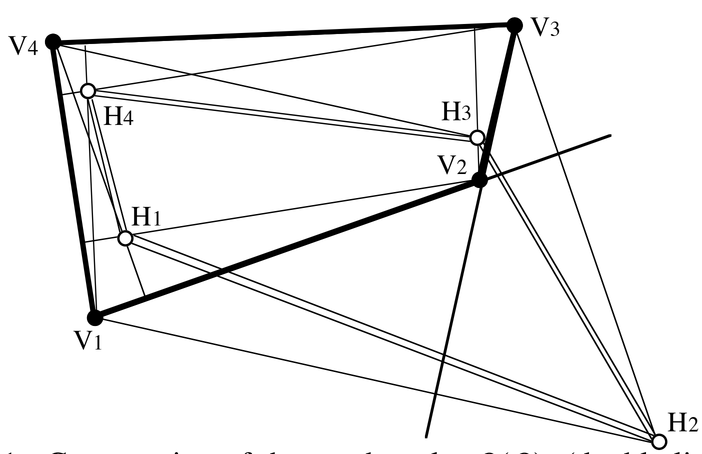

Figure 1. Construction of the quadrangle $O(Q)$ (double lines) from the quadrangle Q (heavy lines). Only two altitudes (thin lines) are shown for each of the four triangles, each of which is determined by two adjacent sides of the quadrangle and one diagonal (not shown).

The intersection points of the diagonals of the two quadrangles are

$$D_Q = \left(\frac{p - r + qr - ps}{p + q - r - s} \, , \frac{q - qr - s + ps}{p + q - r - s} \right) \text{ and}$$

$$D_{O(Q)} = \left(\frac{q + pq - q^2 - s - rs + s^2}{p + q - r - s} \, , \frac{p - p^2 + pq - r + r^2 - rs}{p + q - r - s} \right) .$$

From this it follows that $\dfrac{|V_3 - V_1|}{|V_3 - D_Q|} = \dfrac{|H_1 - H_3|}{|H_1 - D_{O(Q)}|} = \dfrac{-p - q + r + s}{-1 + r + s}$ and $\dfrac{|V_4 - V_2|}{|V_4 - D_Q|} = \dfrac{|H_2 - H_4|}{|H_2 - D_{O(Q)}|} = \dfrac{-p - q + r + s}{-p + r - qr + ps}$, which shows the affine equivalence of Q and $O(Q)$. A calculation of areas shows that $\text{Area}\,(V_1V_2V_3V_4) = \text{Area}\,(H_1H_2H_3H_4) = -p - q + r + s$, thus completing the proof of the fact that α_Q is an equiaffinity.

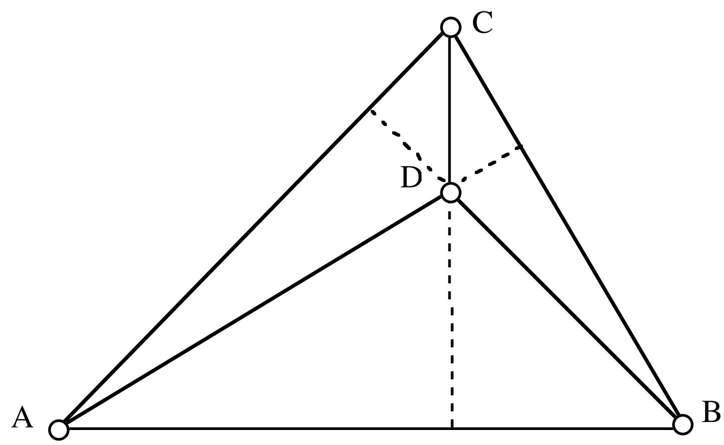

Figure 2. If D is the orthocenter of the triangle ABC then each of the four points A, B, C, D is the orthocenter of the triangle formed by the other three. This implies that the quadrangle $Q = ABCD$ is mapped onto itself by the orthocenter map.

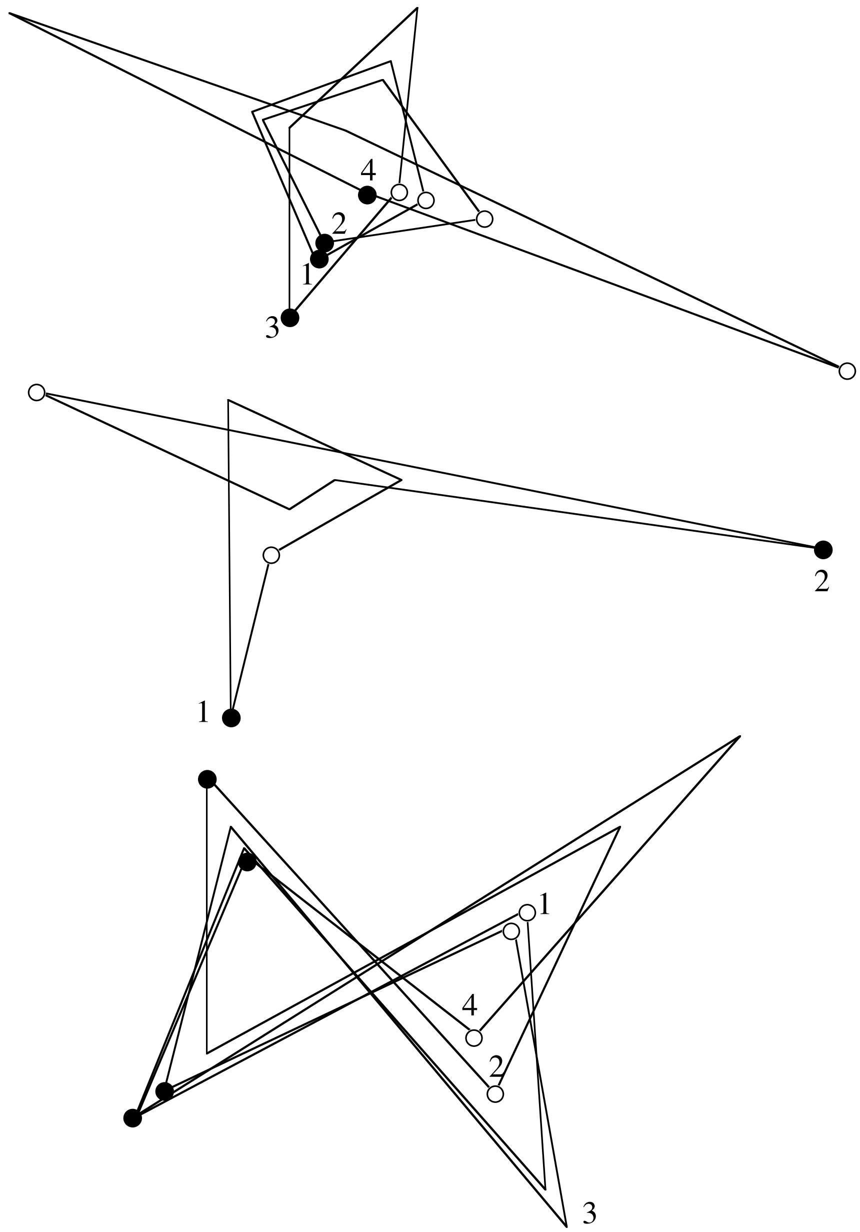

Figure 3. Three examples of iterations of the orthocenter map O. The starting quadrangle is labelled 1, and the other numerals indicate the iterates. Examples like these led to the idea of affine equivalence of the quadrangles Q and $O(Q)$.

It is easily seen that if Q is a rectangle, or a selfintersecting quadrangle whose vertices coincide with those of a rectangle, then $Q = O(Q)$ (although in the selfintersecting case the vertices are permuted by a reflection of the quadrangle). Thus the quadrangles illustrated in Figure 2 are not the only ones with the property of coinciding with their image under the orthocenter map. We conjecture that the examples mentioned so far are the only quadrangles Q such that $Q = O(Q)$. On the other hand, there are other quadrangles Q which are congruent to their image $O(Q)$ without coinciding with it. One family of these quadrangles is characterized by the following result:

Theorem 2. A quadrangle Q is congruent by a half-turn (that is, homothetic in ratio -1) to its image $O(Q)$ under the orthocenter map if and only if Q is cyclic.

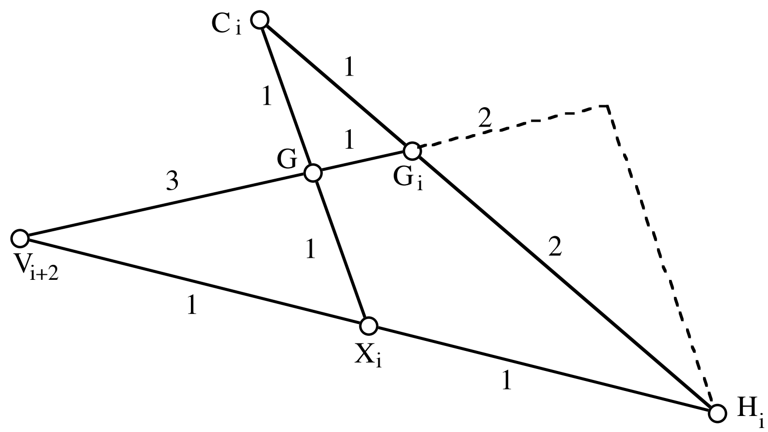

Figure 4. Relationships of various distances in an arbitrary quadrangle $Q = V_1V_2V_3V_4$. For the triangle $T_i = V_{i-1}V_iV_{i+1}$ the points H_i, G_i, C_i are, respectively, the orthocenter, the centroid and the circumcenter. The centroid of the quadrangle Q is G. By Euler's theorem (see [1, Section 1.6] on the mutual positions of H_i, G_i, C_i the ratio of distances on the segments they determine is as indicated, and the ratio of segments determined by V_{i+2}, G, G_i is as given because of the numbers of points involved. By elementary consideration of ratios of segments determined by parallels it follows that X_i is the midpoint of H_iV_{i+2}, and that G is the midpoint of C_iX_i.

Proof. With the notation introduced in Figure 4 we see that if Q is cyclic then all four points C_i coincide, hence all points X_i coincide. Therefore $O(Q)$ is congruent to Q by a half-turn. Conversely, if $O(Q)$ is congruent to Q by a half-turn, then all points X_i coincide, hence all points C_i coincide, and so Q is cyclic.

Figure 5 gives illustrations of the situation described in Theorem 2.

It may be conjectured that every Q which is congruent to $O(Q)$ is related to $O(Q)$ by a translation or a half-turn.

Among other open problems is the question of characterizing the area-preserving affinity α_P in terms of the quadrangle P. Also, nothing seems to be known about properties of $O(P)$ for n-gons P with $n \geq 5$.

Helpful comments by Prof. Coxeter are gratefully acknowledged.

References.

[1] H. S. M. Coxeter, Introduction to Geometry. 2nd ed. Wiley, New York 1969.

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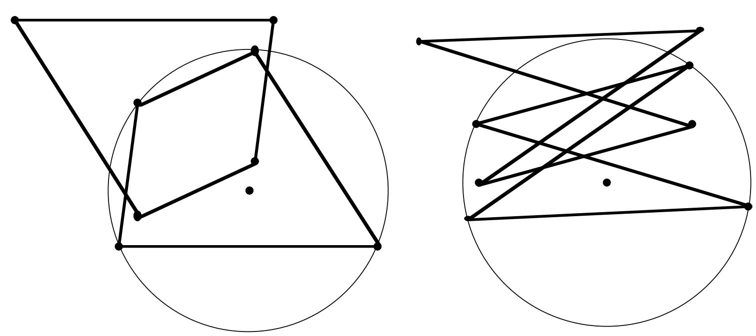

Figure 5. Examples of cyclic quadrangles. Each such quadrangle Q is congruent by a half-turn to its images $O(Q)$ under the orthocenter map.