ON QUADRANGLES DERIVED FROM QUADRANGLES

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In elementary geometry there are innumerable results on various points, lines, triangles, circles, and other figures associated with every given triangle. Everybody is familiar with objects such as centroids, circumcircles (centered at circumcenters), incircles (centered at incenters), orthocenters (intersection points of altitudes), and with some of their properties. One could expect that there is an even larger collection of known results on quadrangles, or pentagons, or polygons in general — but as far as it is possible to infer from the literature, this is not the case. Several questions of this general character were discussed in the notes [1] and [2], with references to additional literature. (The paper [3] has been published in the meantime, and the complete reference is given below.) In this note and a few to follow I shall present some apparently new facts concerning quadrangles, as well as some open questions. In each case we shall be concerned with quadrangles derived from given quadrangles by appropriate systematic procedures.

Given a quadrangle $Q = [V_1, V_2, V_3, V_4]$, which may be nonconvex or selfintersecting, a new quadrangle $I(Q)$ is formed by the incenters V_i^* of the triangles $T_j = [V_{j-1}, V_j, V_{j+1}]$, for $j = 1, 2, 3, 4$. (Here and throughout, subscripts are understood mod 4.) This is illustrated in Figure 1(a), and in Figure 2.

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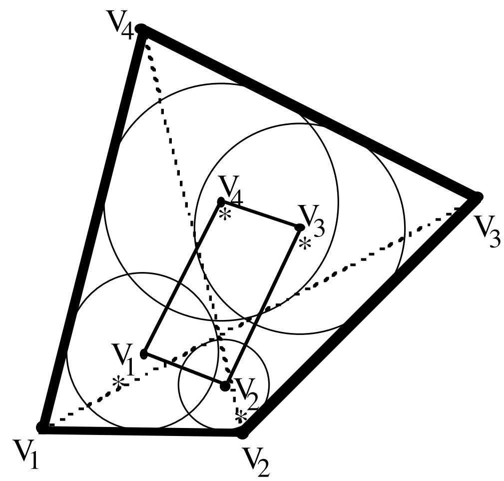
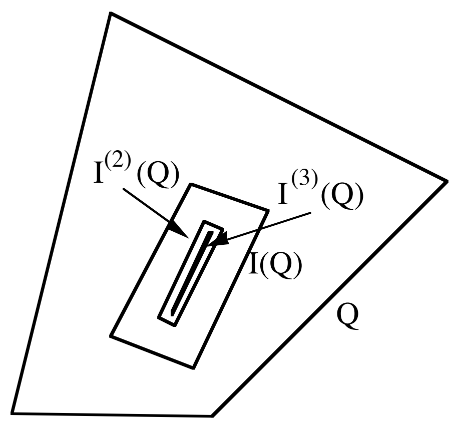
The sequence of quadrangles $I^{(n)}(Q)$, where $I^{(1)}(Q) = I(Q)$ and $I^{(n)}(Q) = I(I^{(n-1)}(Q)$ for $n > 1$, is convergent since, as is not hard to show, every term is contained in the interior of the convex hull



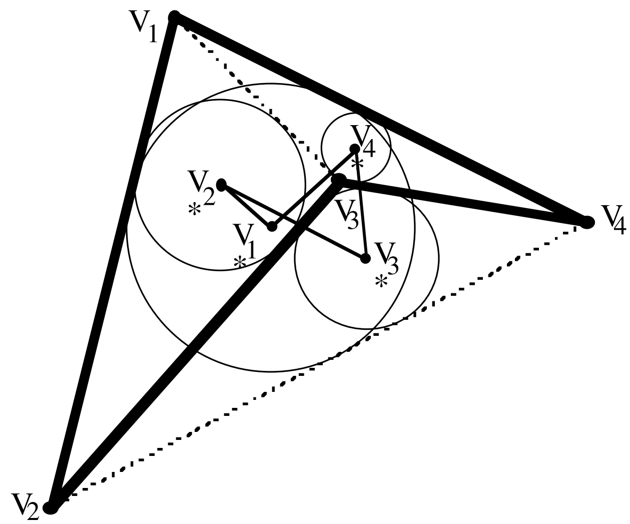
(a)



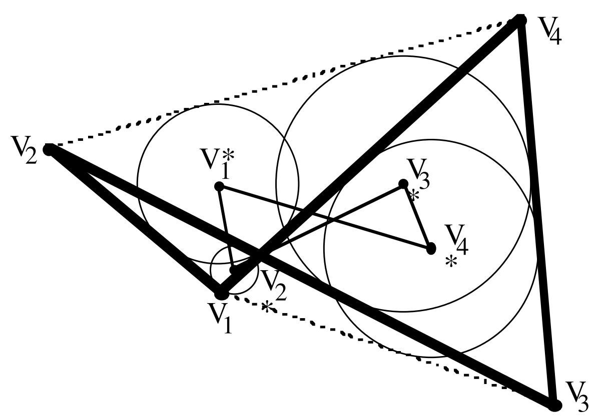
(b)

Figure 1. In (a) the original quadrangle Q is shown with heavy lines, its diagonals with dotted lines, and the derived quadrangle with thin lines. The four circles inscribed into the triangles T_j are also shown. In (b) the same quadrangle Q is shown, together with the first three derived quadrangles $I(Q)$, $I^{(2)}(Q)$, $I^{(3)}(Q)$.

of the preceding one. For the same reason, the limit must be either a single point or a segment. This is the same situation as can be found in various other constructions (see, for an example, the problem of



(a)



(b)

Figure 2. Two additional examples of the construction of the derived quadrangle I(Q) from the given quadrangle Q. The explanations are the same as for Figure 1(a). Notice that the quadrangle Q in (a) is simple but nonconvex, while the other quadrangles are selfintersecting. The starting quadrangle in (b) is similar to the quadrangle I(Q) in (a).

Schoenberg discussed in [2]). However, in most of these cases the limit is, in fact, always a point. This contrasts to the situation considered here, where the limit may happen to be a segment, see the illustration in Figure 1(b).

The easiest example in which this behaviour can be proved (instead of only suggested by diagrams) is that of rectangles. Using elementary results on angle bisectors, it is not hard to calculate that if Q is an a by b rectangle, with $a > b$, then $Q^* = I(Q)$ is an a^* by b^* rectangle, concentric with Q and having sides parallel to those of Q, where

$$a^* = \frac{a(a - b + c)}{(a + b + c)} \quad \text{and} \quad b^* = \frac{b(-a + b + c)}{(a + b + c)} \quad ,$$

and $c = \sqrt{a^2 + b^2}$ is the length of the diagonal of Q. This implies the unexpected relation $a^* - b^* = a - b$, which shows that the sequence $I^{(n)}(Q)$ converges to a segment of length $a - b$. For squares, and some other quadrangles, the limit is a point, while for the remaining quadrangles it is a segment.

It is not known how to characterize the quadrangles Q for which $I^{(n)}(Q)$ converges to a point, or how to determine the length of the limit segment for nonrectangular quadrangles for which $I^{(n)}(Q)$ converges to a segment.

References.

[1] B. Grünbaum, Quadrangles, pentagons and computers. Geombinatorics 3(1993), 4 – 9.

[2] B. Grünbaum, Quadrangles, pentagons and computers, revisited. Geombinatorics 4(1994), 11 – 16.

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