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Isogonal Prismatoids*

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Abstract. The study of the polyhedra (in Euclidean 3-space) in which faces may be self-intersecting polygons, and distinct faces may intersect in various ways, was quite fashionable about a century ago. The Kepler–Poinsot regular polyhedra, and several of their generalizations, were investigated about that time by Cayley, Wiener, Badoureau, Fedorov, Hess, Pitsch, and others; the accumulated wisdom was presented in Max Brückner's well-known book *Vielecke und Vielflache* in 1900. Despite the intrinsic interest of the topic, and its relations to various other disciplines, there have been very few additional investigations during the intervening century, except for discussions of uniform polyhedra. In particular, there has been no mention or clarification of the many errors and other shortcomings of Brückner's book. One of our aims is to point out the extent of these inadequacies; they are illustrated by a discussion of *isogonal prismatoids*, the investigation of which is our main objective. A *prismatoid* is a polyhedron having all its vertices in two parallel planes. Familiar examples are prisms and antiprisms. A polyhedron P is *isogonal* if all its vertices form one transitivity class under isometric symmetries of P. Although these restrictions appear very severe, there exist many different kinds of isogonal prismatoids. Some general concepts concerning polyhedra with possible self-intersections are presented, and several classes of isogonal prismatoids are discussed in some detail.

1. Introduction

According to Webster [18] a *prismatoid* is "a polyhedron having all of its vertices in two parallel planes," which we always take as distinct. Not every text agrees completely with this definition, but it is precisely the one we need. This is a very wide class of polyhedra—it includes prisms, antiprisms, pyramids, and many other polyhedra. Therefore we restrict attention to *isogonal prismatoids*, that is, prismatoids in which all vertices are equivalent

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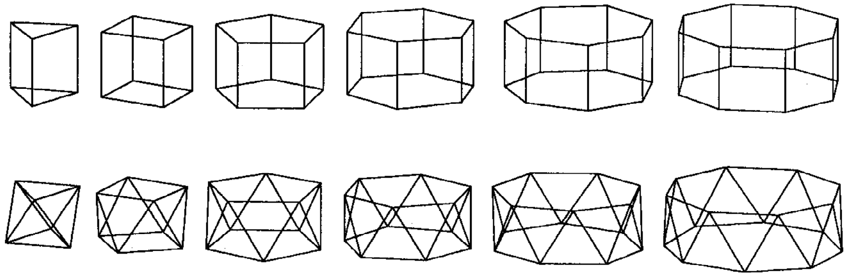


Fig. 1. Examples of Archimedean prisms and antiprisms.

under isometric symmetries of the polyhedron. We note that we understand the isogonality restriction in a strong sense: since prismatoids have two distinguished planes, only symmetries which map the pair of planes onto itself are considered. Two classes of examples of isogonal prismatoids are known since antiquity: certain prisms and antiprisms. The *prisms* and *antiprisms* have two congruent *bases* (which are polygons contained in the parallel planes mentioned in the definition) together with a *mantle* connecting the two bases; the mantle consists of quadrangles in the case of prisms and triangles in the case of antiprisms. Figure 1 shows examples of *Archimedean prisms*, in which the bases are regular convex polygons and the mantle faces are squares, as well as examples of *Archimedean antiprisms*, in which the bases are regular convex polygons and the mantle consists of equilateral triangles. It is well known that every convex isogonal prismatoid is *isomorphic* (or *combinatorially equivalent*) to an Archimedean prism or antiprism.

The aim of this paper is to investigate more general isogonal prismatoids, in which faces may be self-intersecting polygons, and different faces may have intersections that are not common edges or vertices. As we shall see, this enlarges the family of isogonal prismatoids much beyond the convex prisms and antiprisms. In fact, it turns out that this family contains orientable polyhedra of positive genus, as well as nonorientable polyhedra.

The only source of which we are aware, in which prismatoids other than Archimedean prisms and antiprisms are discussed in some detail, is Sections 114 and 140 of [2]. Uniform prismatoids (that is, prisms and antiprisms with regular polygons as faces) have been discussed by several authors, as part of more inclusive investigations of uniform polyhedra in general. Coxeter *et al*. [3] give a list of uniform prisms and antiprisms, as well as references to the earlier literature. The enumeration of uniform polyhedra given by Coxeter *et al*. was shown to be complete by Sopov [16] and Skilling [15]. Har’El [9] gives skeletal illustrations of all nonprismatic uniform polyhedra, but illustrates only pentagonal uniform prisms and antiprisms. In the more general case considered by Brückner [2] some of the unusual nonuniform isogonal prisms and antiprisms can be glimpsed. However, Brückner’s presentation is completely *ad hoc*, with no particular guiding ideas and no clear classification or description principles; moreover, it is very incomplete and misses some of the most interesting polyhedra of the types it purports to enumerate. We shall show that it is possible to describe many additional prismatoids in a satisfactory and natural manner, and illustrate how varied are the shapes that prismatoids can have.

In order to do this, we have to face another curious shortcoming regarding polyhedra—namely, the fact that the literature contains no satisfactory methods or ideas for a classification of polyhedra of the general kind with which we are concerned. We attempt to give usable definitions of polyhedra, and of ways of deciding whether two polyhedra are "of the same type," and apply them to investigate isogonal prismatoids.

This paper is organized as follows. In Section 2 we give precise definitions of the concepts we need. In Section 3 we present a complete classification of prisms and antiprisms (not necessarily convex or uniform), while Sections 4 and 5 discuss other isogonal prismatoids. Additional comments are collected in Section 6.

2. Polygons and Polyhedra

There is no generally accepted terminology for polyhedra in which faces may self-intersect, or intersect each other in various ways. Below we give the definitions which seem appropriate to the topic at hand; however, since the polyhedra are built up of polygons, we first supply and illustrate the corresponding definitions for planar polygons. The exposition here follows the one in [7], simplified as appropriate for the restricted classes of polygons and polyhedra under investigation. Specifically, we are concerned here only with polygons and polyhedra that are called *unicursal* in [7], and only with *epipedal* realizations of such polyhedra.

An *abstract polygon* is a fixed simple *circuit* $\mathcal{C}$, that is, a system consisting of a finite, cyclically ordered set $\mathcal{V}$ of distinct elements, and the set $\mathcal{E}$ of distinct unordered pairs of adjacent elements of $\mathcal{V}$. The elements of $\mathcal{V}$ are the "vertices" of the polygon, and the pairs represent the "edges" of the abstract polygon.

A *geometric polygon* (or *polygon* for short) P is the image of an abstract polygon $\mathcal{C}$ under a map φ which associates with each element of $\mathcal{V}$ a point (*vertex* of P) in a Euclidean plane E^2, and with each pair from $\mathcal{E}$ the line segment (*edge* of P) having as endpoints the images of the elements of $\mathcal{V}$ that constitute the pair. If P has n edges we call it an *n-gon*. We note that different vertices of $\mathcal{V}$ may be represented by the same point of the plane; this does not affect the incidences of the vertices of P with its edges, although it entails the possibility of edges that have coinciding vertices and are therefore represented by single-point line segments. Also, edges may cross or overlap in various ways, or even coincide.

In this paper we are interested mainly in triangles, quadrangles, and isogonal polygons, the latter including, in particular, regular polygons. Since the literature on isogonal polygons is meager, and that on regular polygons contains a considerable amount of misleading statements, we present some details concerning these concepts. In analogy to the definition for polyhedra, a polygon is called *isogonal* provided its isometric symmetries act transitively on its vertices. A polygon P is called *regular* if its isometric symmetries act transitively on the *flags* of P, where a flag is the pair consisting of an edge and one of its endpoints. (Many other definitions of regular polygons, equivalent to the one given here, are possible.) If $k = [n/2]$, then there are k different regular n-gons, denoted by $\{n/d\}$, where $d = 1, 2, \ldots, k$. (Throughout, we consider as equal all polygons which

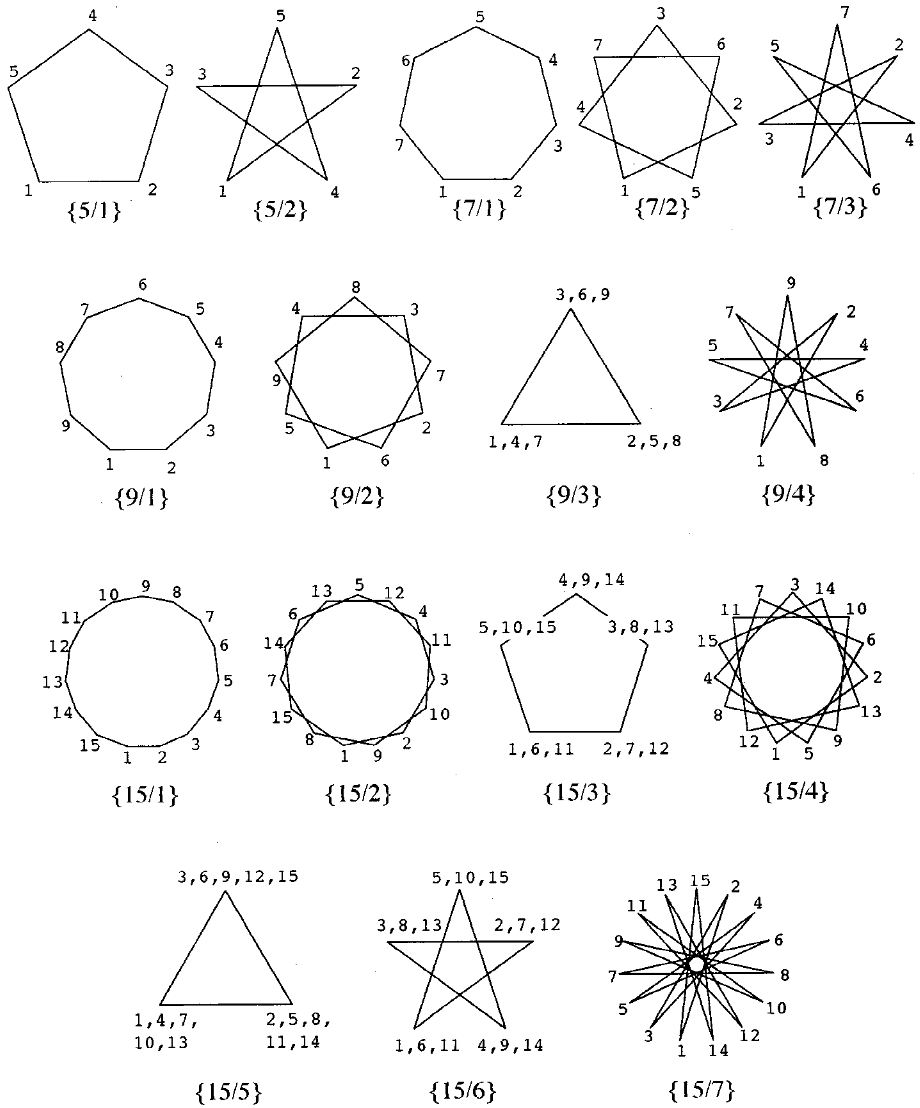


Fig. 2. The different isogonal n-gons with $n = 5, 7, 9$, or 15, and their symbols. All isogonal n-gons with odd n are regular. The vertices are denoted by the labels $1, 2, \dots, n$. Several labels near a single point indicate that these vertices (although different as vertices of the polygon) are all represented by one point.

can be mapped onto each other by similarity transformations.) We note that, contrary to frequently encountered assertions, a regular n-gon exists, and is a well-defined geometric object, even if the integers n and d **are not** relatively prime; see Fig. 2 for examples. (Concerning this topic and its history see [5] and [7].)

Isogonal polygons seem to have been first investigated by Hess [10]; however, due to the lack of a consistent point of view and disregard of the deeper insights of Meister [11]

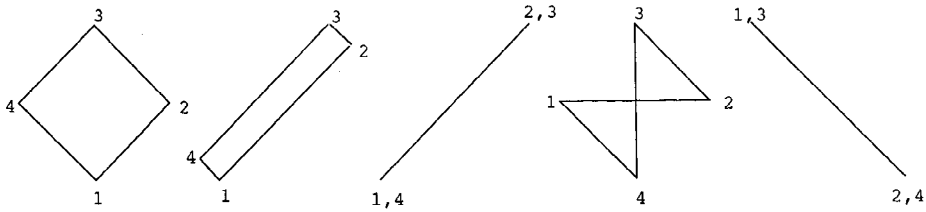


Fig. 3. The different types of isogonal quadrangles. The types denoted 4-2 and 4-4 occur in continuous families, whose shape varies from the square 4-1 to the quadrangle 4-3 in which two opposite sides are represented by segments of length 0, and the other two are represented by the same segment, and from that quadrangle to the quadrangle 4-5 which has four sides represented by the same segment. The quadrangles 4-1 and 4-5 are the only regular quadrangles.

and Wiener [19], this long work is unusable for any further investigations. A systematic approach to isogonal polygons, including a full description and classification, appears in [6]; here we only briefly recall the results. In order to avoid trivialities, we exclude from further consideration isogonal polygons where all vertices coincide.

For odd $n = 2k+1$, the only isogonal polygons are the regular polygons $\{n/d\}$ where $d = 1, 2, \ldots, k$. In Fig. 2 we show all the types of isogonal n-gons for $n = 5, 7, 9$, and 15; these examples are typical and should be sufficient to show what happens if n and d are not relatively prime.

For even n several cases need to be distinguished. A schematic illustration of the somewhat special case $n = 4$ is shown in Fig. 3. For even $n \geq 6$, the isogonal n-gons form $[(n+2)/4]$ families, of which $[n/4]$ are continuous and can be parametrized by one real parameter each; an exceptional family occurs for $n = 4k + 2$, and consists of the single (regular) polygon $\{n/(2k+1)\}$. If $n = 4k + 2$ with $k \geq 1$, then, for each $d = 1, 2, \ldots, k$, there is a continuous family denoted n/d of isogonal n-gons that starts with the regular polygon $\{n/d\}$ and ends with the regular polygon $\{n/e\}$, where $e = 2k + 1 - d$. The cases $n = 6$ and $n = 10$ are illustrated in Figs. 4 and 5; the three continuous families that occur for $n = 14$ are shown in [6]. Similarly, for $n = 4k$ with $k \geq 2$ there are k continuous families n/d, where $d = 1, 2, \ldots, k$; each family n/d starts at the regular polygon $\{n/d\}$, and—except for the family n/k—ends at the regular polygon $\{n/e\}$, where $e = 2k - d$. The family n/k reaches from $\{n/k\}$ to $\{n/(2k)\}$. The case $n = 8$ is illustrated in Fig. 6.

Before starting the detailed discussion of isogonal prismatoids we have to make explicit what we understand as polyhedra, and how we distinguish between "types" of polyhedra. This definition turns out to be applicable to polygons as well, and to be an extension of the classification of polygons proposed long ago by Steinitz [17]. We begin by defining *abstract polyhedra*.

A finite family of abstract polygons is an *abstract polyhedron* provided:

(i) Each "edge" of each of the polygons (which are the "faces" of the abstract polyhedron) is an "edge" of precisely one other "face."
(ii) All "faces" that contain a "vertex" form a single simple circuit.

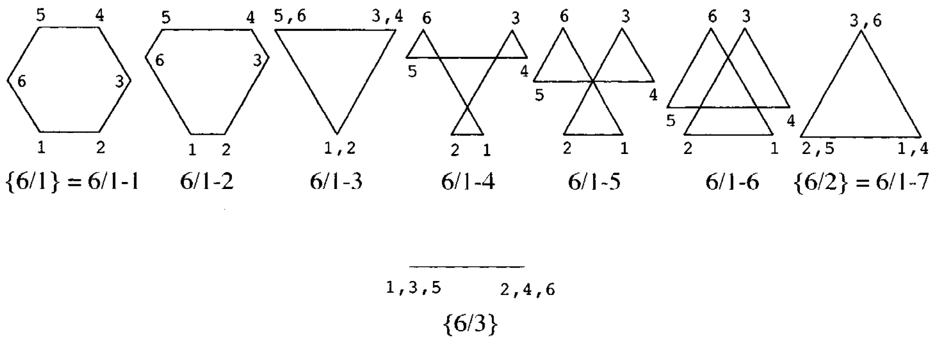


Fig. 4. The different types of isogonal hexagons. The polygons of types 6/1-1 to 6/1-7 form one continuous family, which starts at the regular polygon {6/1} and ends at the regular polygon {6/2}. The types denoted 6/1-2, 6/1-4, and 6/1-6 depend on a real-valued parameter, and their shape varies between the shapes of the hexagons adjacent to them in the diagram. The regular hexagon {6/3} is isolated.

(iii) The family of "faces" is connected in the sense that any two belong to a chain of "faces" in which adjacent "faces" share an "edge."

Since each "face" of an abstract polyhedron can be considered as the boundary of a topological disk, an abstract polyhedron can be interpreted as a cell complex decomposition of a 2-manifold. Two abstract polyhedra are *isomorphic* if there is an incidence-preserving bijection between their "vertices," their "edges," and their "faces." An *abstract prismatoid* has two disjoint "faces" which together comprise all "vertices"; it is *isogonal* if the group of automorphisms acts transitively on the set of "vertices." For any abstract isogonal polyhedron one can define its vertex-symbol, the cyclic list of sizes of the polygons at one (hence every) vertex of the polyhedron; of the different possible symbols, the one lexicographically first is usually chosen. Abstract prisms and antiprisms have vertex-symbols $(4.4.n)$ and $(3.3.3.n)$, respectively, where n indicates the number of sides of the basis.

A *geometric polyhedron*, or *polyhedron* P for short, is a representation of an abstract polyhedron (said to be the *underlying abstract polyhedron* of P) in the Euclidean 3-space E^3, such that "vertices" are represented by points, "edges" by segments, and "faces" by (planar) polygons, giving the *vertices*, *edges*, and *faces* of the polyhedron.

We say that a polyhedron is *acoptic* (from the Greek $\kappa o\pi\tau\omega$, to cut) if:

(i) All its faces are simple polygons, so that they can be unambiguously represented (or replaced) by *simply-connected polygonal regions*.
(ii) The intersection of any two such regions consists of a union (possibly empty) of vertices and edges of each.

Clearly, convex polyhedra are acoptic, but so are many nonconvex polyhedra. Acoptic polyhedra are the ones for which cardboard models give a faithful representation; such a model is, in fact, an embedding in 3- space of the 2-manifold determined by the abstract polyhedron.

A polyhedron is called *aploic* (from the Greek $\alpha\pi\lambda oo\sigma$, onefold, simple) if, whenever X and Y are two distinct "vertices," distinct "edges," or distinct "faces" of the underlying

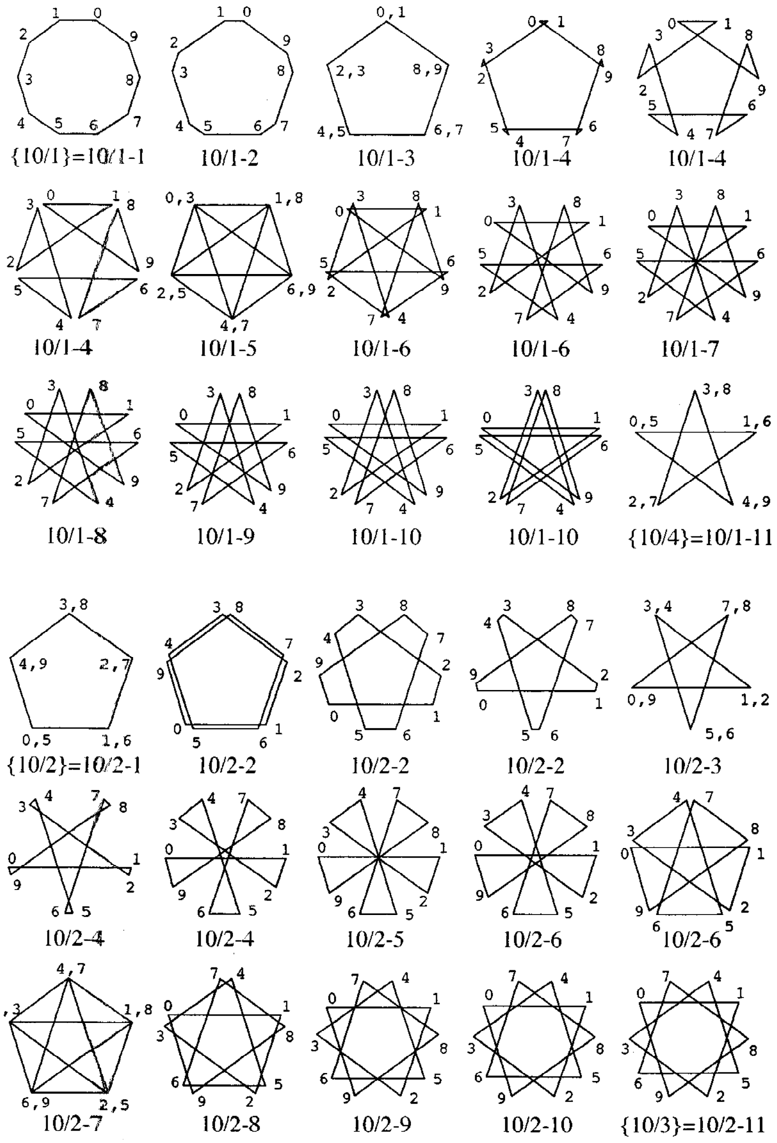


Fig. 5. The two continuous families of isogonal decagons. The types with odd suffix contain a single polygon, the others contain a continuum of polygons.

polyhedron, then the affine hulls aff X and aff Y of X and Y are distinct. The analogous definition applies to polygons as well. Aploic polyhedra can be considered as that generalization of acoptic polyhedra to polyhedra with self-intersections which is closest to the “naive” understanding. However, it should be pointed out that the tradition which attempts to present aploic polyhedra (such as the Kepler–Poinsot regular polyhedra) by cardboard models is in many cases misguided and misleading; the most instructive mod-

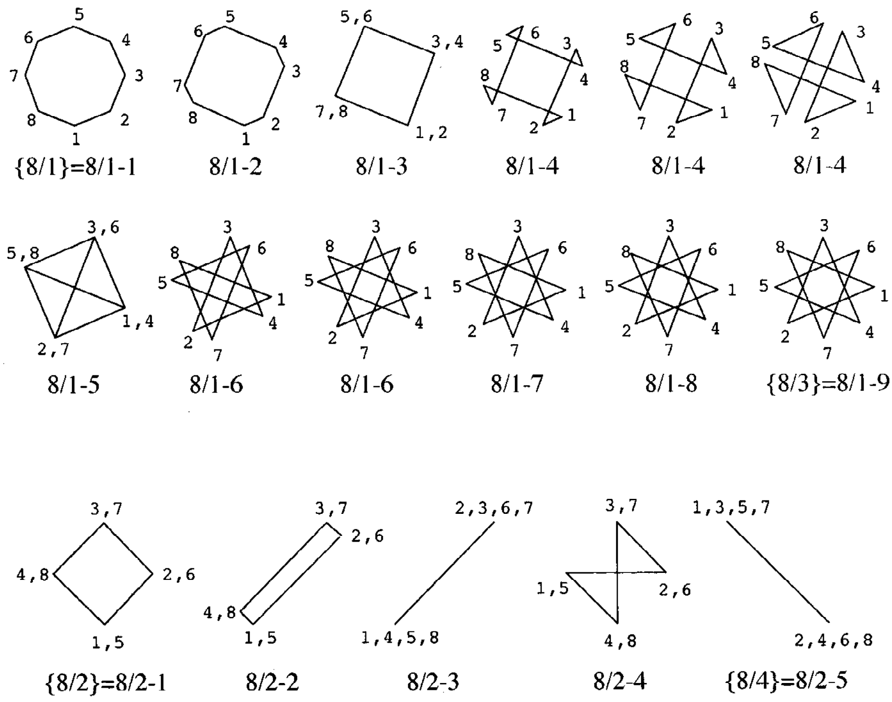


Fig. 6. The two continuous families of isogonal octagons.

els of nonacoptic aploic polyhedra are the skeletal ones, or those cardboard models in which the "hidden" parts are included and made visible through appropriate openings.

Each of the two planes that appear in the definition of a prismatoid may contain one or more faces, or may not contain any faces; for isogonal prismatoids the figures in the two planes must be congruent, and in each plane the symmetries of the polyhedron must act transitively on the vertices contained in that plane. The polygons contained in each of the two planes are said to form the *basis* of the prismatoid; if no polygon is a basis, we say that the prismatoid is *basis-free*. An obvious consequence of the isogonality condition is that the bases of any prismatoid which is not basis-free must be congruent *isogonal polygons* or *isogonal compounds* of polygons. (An "isogonal compound" of polygons is a collection of polygons such that the isometric symmetries of the plane act transitively on the vertices of the collection.)

Since all vertices of any isogonal polyhedron are cospherical, and all vertices of any prismatoid lie on two parallel planes, it is immediate that any aploic isogonal prismatoid can have only triangles and quadrangles as faces of its mantle, and that it is either basis-free or each basis consists of a single isogonal polygon. The prisms and antiprisms (that is, realizations of abstract prisms or antiprisms) are examples of the latter possibility, and we present their classification in the next section.

Before we can proceed with the classification, we need to define what constitutes a "type." While this is a staple in the theory of convex polyhedra, and even for more general

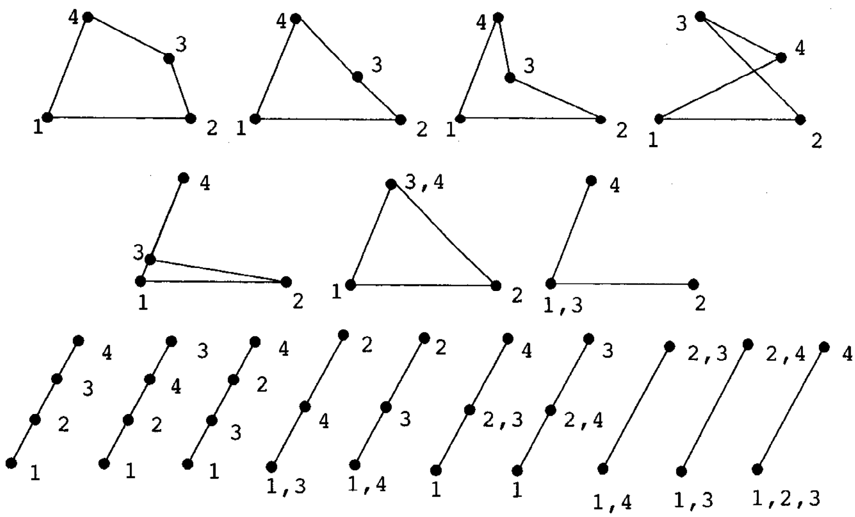


Fig. 7. Representatives of all the nontrivial geometric types of quadrangles. Only the first three are acoptic, and only the ones in the first row are aploic.

acoptic polyhedra the problem of definition is not severe, in the present situation there seems to be no reasonable definition to be found in the literature. Here by "reasonable" is meant a definition that would be applicable to polyhedra of some degree of generality, and which reflects some natural properties we may wish to see preserved among polyhedra we assign to the same type—but which, at the same time, is finer than the combinatorial classification which is given by the abstract polyhedra. (We note that we find it appropriate not to distinguish between two isogonal prismatoids—or more general polyhedra—if one can be mapped onto the other by a nonsingular affinity that is compatible with all their isometric symmetries.) It is not clear whether our definition is reasonable for very general polyhedra, but its application to aploic isogonal prismatoids appears to be both convenient and reasonable. The definition is analogous to the classification of polygons that goes back to Steinitz [17], and is similar to those in [12] and [7]; it is illustrated in Figs. 7–9.

Two polyhedra P_0 and P_1 are of the *same geometric type* provided there is a continuous family $P(t), 0 \le t \le 1$, of polyhedra, where $P(0) = P_0$ and $P(1)$ is P_1 or a mirror image of P_1, and such that:

(i) All members of the family have the same underlying abstract polyhedron P.
(ii) All members of the family have same group of isometric symmetries.
(iii) For any two distinct faces F_1 and F_2 of the underlying abstract polyhedron P, the affine hull of the union of the images of F_1 and F_2 has the same dimension for every member $P(t)$ of the family.

In particular, two aploic isogonal prismatoids are of the *same geometric type* provided they have the same underlying abstract polyhedron and the same symmetries, and there is a continuous family of polyhedra connecting them in such a way that each polyhedron

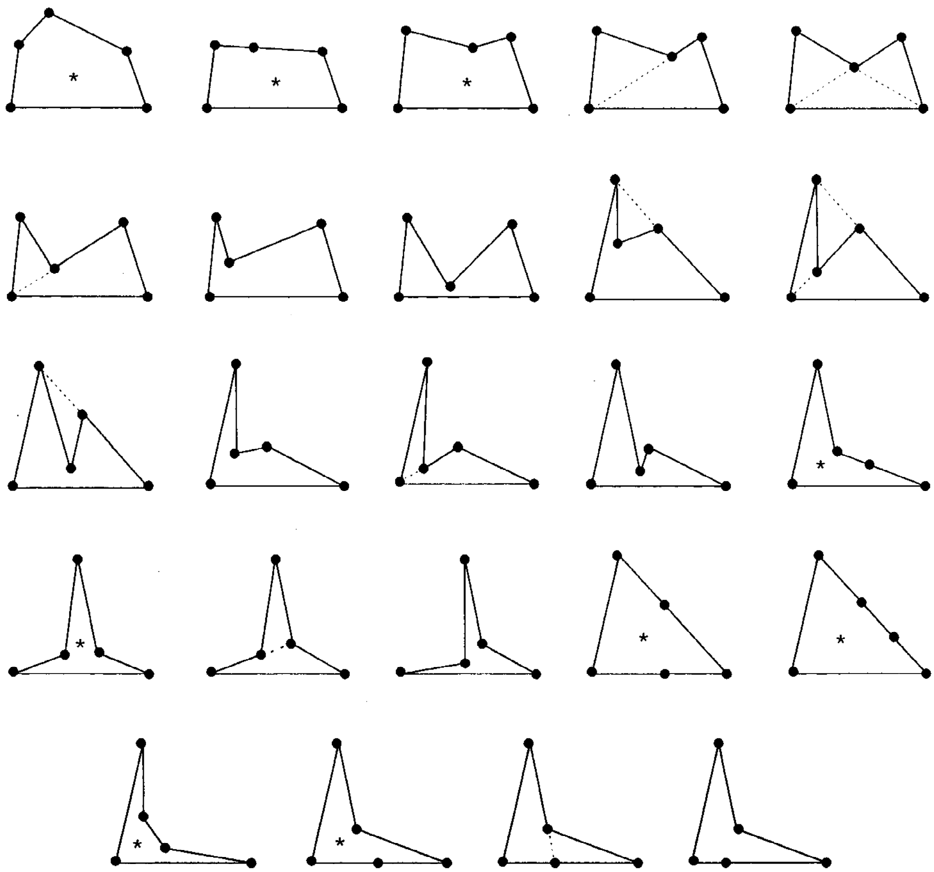


Fig. 8. Representatives of all the types of acoptic pentagons in the finer classification mentioned in the text, in which all edges and vertices are considered. One representative of each geometric type is indicated by an asterisk. The dotted lines indicate collinearity.

in the family is an aploic isogonal prismatoid with the same properties. Since here we are interested only in the geometric types of the prismatoids under discussion, we usually simplify the language and speak of their *type*.

A finer classification would result if condition (iii) were expanded to include any two faces in the wider sense (that is, the set that includes the faces, edges, and vertices) of the polyhedron P.

The definition of geometric type can obviously be applied to polygons as well, the only change being the replacement of "faces" in condition (iii) by "edges." In order to illustrate the concept of geometric type, and of the finer classification mentioned above, we show in Figs. 7 and 8 the different geometric types of quadrangles and of acoptic pentagons, and in Fig. 9 the different types of acoptic polyhedra with five vertices.

In the next section we apply the definitions given here to the simplest prismatoids—the prisms and antiprisms.

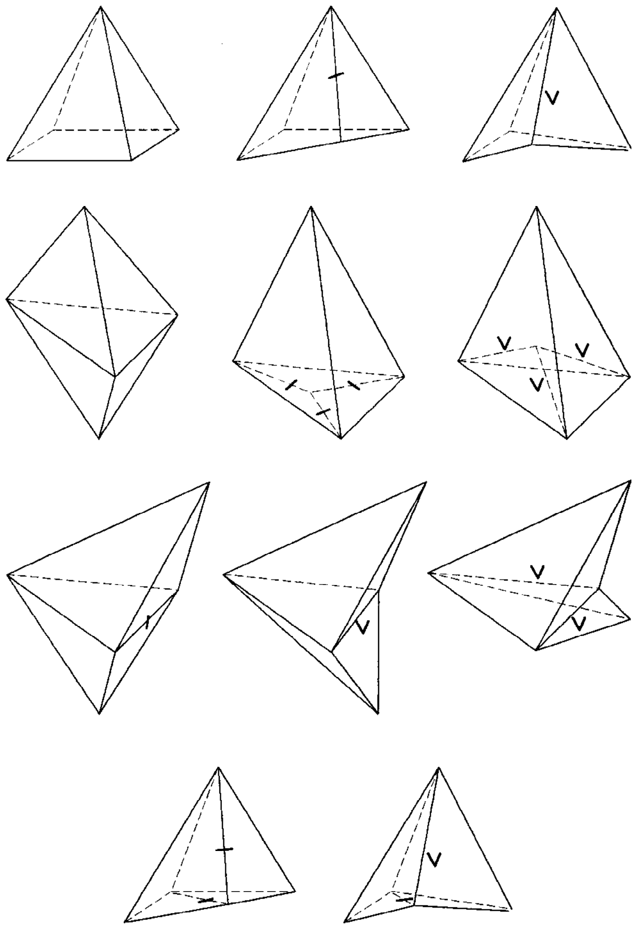


Fig. 9. Representatives of all the eleven geometric types of acoptic polyhedra with five vertices. A short dash across an edge indicates that the two faces incident with the edge are coplanar; a $\vee$ indicates that the dihedral angle at the edge is concave. The underlying abstract polyhedron of the first three is a four-sided pyramid, for the other eight it is a three-sided bipyramid.

3. The Classification of Prisms and Antiprisms

We begin by considering antiprisms, that is, polyhedra with vertex-symbol $(3.3.3.n)$, where $n \geq 3$. We note that, for a given n, all abstract antiprisms with vertex-symbol $(3.3.3.n)$ are isomorphic. In any realization of an abstract antiprism by an isogonal polyhedron P, the bases of P have to be congruent regular polygons. For each regular polygon $\{n/d\}$ there exists a continuous family of antiprisms with bases congruent to $\{n/d\}$; this family can be parametrized by a real-valued parameter. The parameter can be chosen to measure the twist of one of the bases with respect to the other; the two final antiprisms in each family (and only they) have reflective symmetry. In Figs. 10–12 the families of antiprisms with $n = 3, 4$, and 5 are illustrated. Either one or both extreme

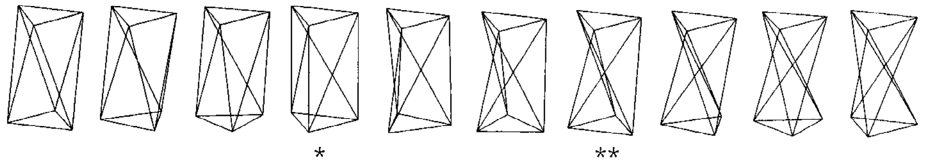


Fig. 10. Representatives of the continuous family of three-sided antiprisms. The two antiprisms marked by single or double asterisks are not aploic. The antiprism marked by two asterisks was mentioned by Schönhardt [14] in a different context; the antiprisms to the left of it are acoptic.

members of each such family has a representative with regular mantle faces, that is, is a uniform polyhedron and appears in the enumeration of Coxeter *et al.* [3]. Brückner [2] mentions such antiprisms in parts 3 and 4 of Section 140, while the existence of the other polyhedra in each family is mentioned briefly (and in very unclear and confused terms, without any details) in parts 7 and 8 of Brückner's Section 140. An antiprism with basis $\{n/d\}$ can be aploic only if n and d are relatively prime; this is satisfied for the antiprisms in Figs. 10–12. Moreover, even if n and d are relatively prime, for certain values of the twist parameter the antiprism fails to be aploic due to coplanarity of distinct mantle faces. These cases are marked by asterisks in Figs. 10–12; the nonaploic polyhedra in each family that are marked by two asterisks seem to be particularly interesting.

According to the above definition, we can say that each of the continuous families of antiprisms contains five different geometric types of aploic polyhedra: the two extreme polyhedra have more symmetries than the other members of the family, and the two nonaploic polyhedra partition the intermediate members of the family into three types. If aploic, both extreme polyhedra can be represented by uniform polyhedra if and only if $d > n/3$.

Turning to prisms, with vertex-symbols $(4.4.n)$, $n \geq 3$, we note that, again, for each n there is a single combinatorial type of abstract prisms. Concerning geometric realizations by isogonal polyhedra, we find that for each isogonal n-gon with *odd n* there exist two

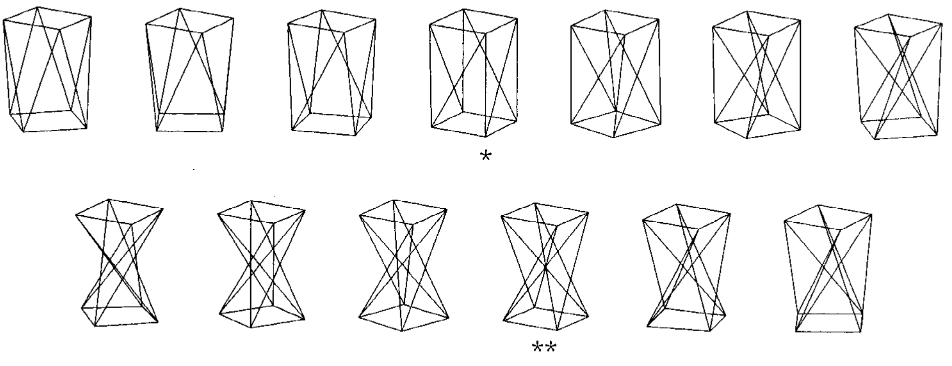


Fig. 11. Representatives of the continuous family of four-sided antiprisms with basis $\{4/1\}$. The "twist parameter" increases from left to right in the first row, from right to left in the second. The two antiprisms marked by asterisks are not aploic.

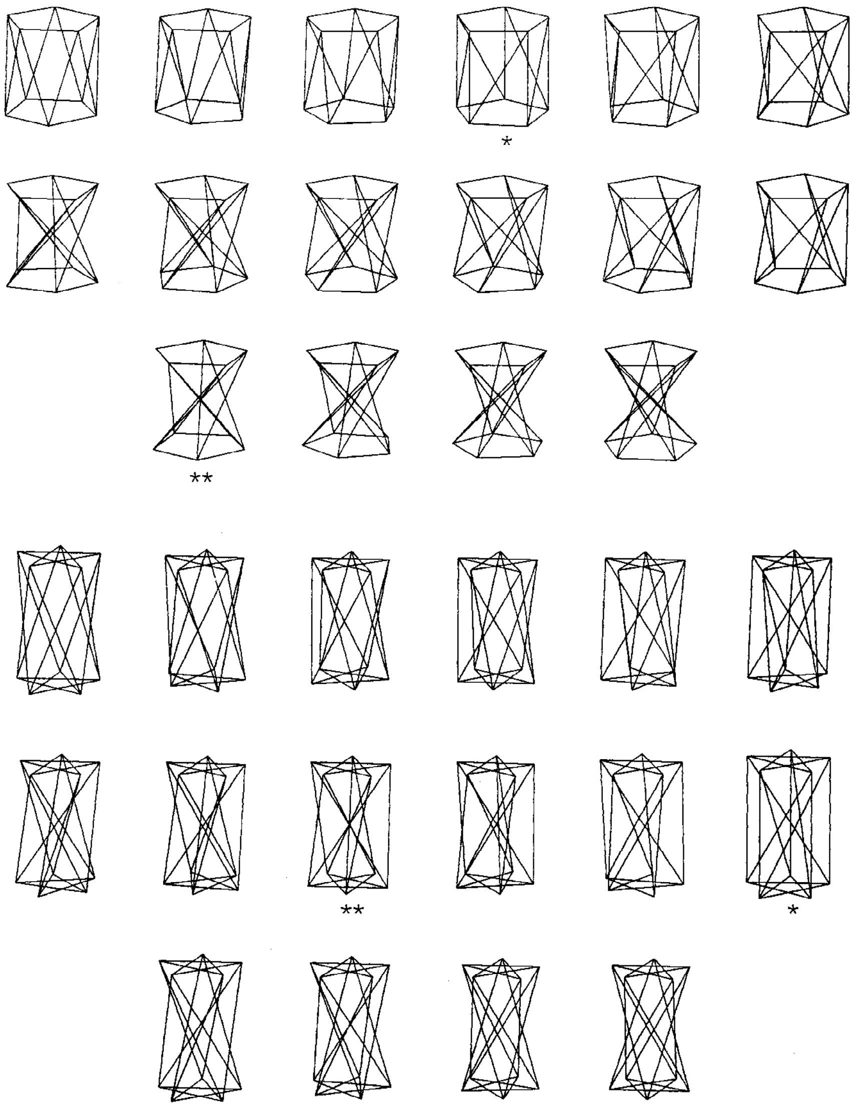


Fig. 12. Representatives of the two continuous families of five-sided antiprisms, based on $\{5/1\}$ and $\{5/2\}$, respectively. The parameter increases from left to right in the first and third rows of each half, and from right to left in the second rows. The antiprisms marked by asterisks are not aploic.

(and only two) different prisms with bases congruent to this n-gon; as mentioned above, this has to be a regular n-gon $\{n/d\}$. For reasons which should be evident from the examples in Fig. 13, we say that one of them has *parallel bases*, the other *antiparallel bases*. The former (which may be denoted by a "p" appended to the symbol of their bases) have rectangular mantle faces, the later (denoted similarly by an "a") have self-intersecting isogonal quadrangles as mantle faces. These prisms are aploic if and only

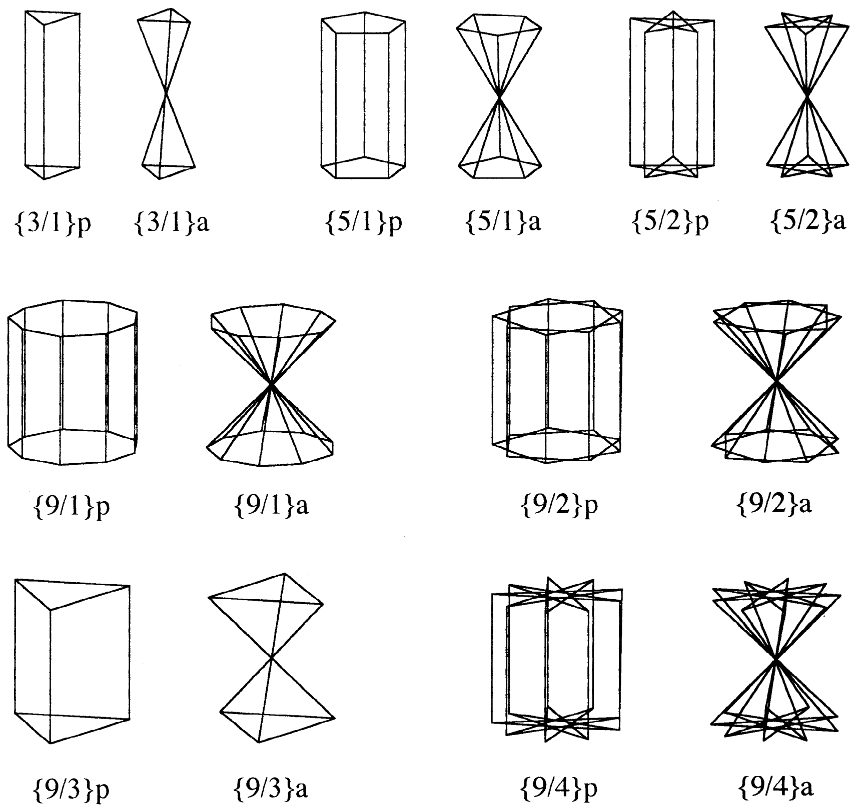


Fig. 13. The different types of isogonal n-prisms for $n = 3, 5,$ and 9. The parallel or antiparallel position of the bases is indicated by the suffix "p" or "a" attached to the symbol that characterizes the n-gonal basis.

if n and d are relatively prime; if $d = 1$, then $\{n/d\}$p is acoptic. Obviously, all prisms with parallel bases can be represented by regular-faced polyhedra.

For *even n* the situation is more interesting, due to the greater variety of isogonal polygons in this case, as well as to the greater flexibility possible for the choice of polygons that form the mantle. We consider here only the case in which n and d are relatively prime. Then the prism is aploic unless the basis is a polygon with coinciding vertices; that is, the prism is aploic if its basis is aploic. (If n and d have a common divisor $k > 1$, the appearance of the prisms is the same as for $n' = n/k$ and $d' = d/k$, with k-tuples of vertices situated at each vertex of the prism corresponding to n' and d'; all such prisms are nonaploic.) For even $n > 4$, for every nonregular polygon of each family n/d there are four different prisms with this polygon as basis; each regular polygon $\{n/d\}$ is the basis for two distinct prisms. The most interesting aspect of the situation is that, for each pair (n, d) with $d < n/4$, all the prisms whose bases are in the family n/d form a single continuum of prisms. The "space" of these prisms, which can be parametrized by a real-valued parameter, is in fact homeomorphic to a circle. This is illustrated for $n = 6$, $d = 1$ in Fig. 14, and for $n = 8$, $d = 1$ in Fig. 15; similar diagrams result for other

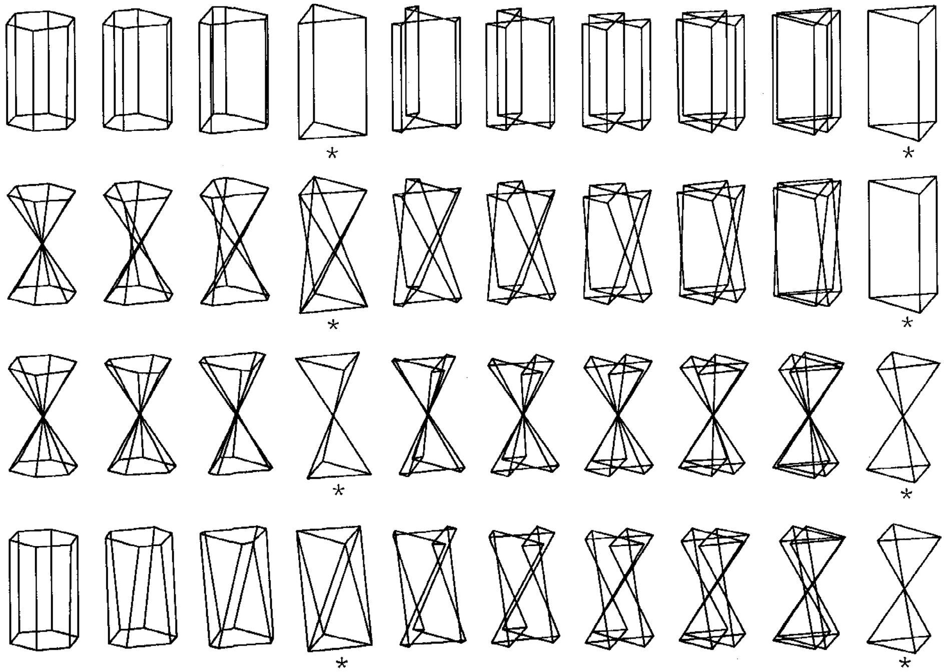


Fig. 14. The hexagonal prisms. The diagram illustrates the continuous family in which to each regular polygon as basis correspond two prisms, and to each nonregular isogonal polygon as basis correspond four prisms. The family can be followed continuously from left to right in the first and third rows, from right to left in the second and fourth; the final polyhedron is identical with the starting one. The prisms that correspond to the same base are aligned vertically; those having regular bases are repeated in two rows. The prisms marked by asterisks are not aploic.

pairs (n, d). In each case other than $n = 4d$, there are eight prisms which are the single members of their geometric type (only two of them are aploic), and eight geometric types that consist of a continuum of distinct aploic polyhedra; when $d = 1$ two of the latter families consist of acoptic prisms, which were considered also by Robertson and Carter [13] and Robertson [12]. Brückner [2] discusses prisms in Section 114 and in parts 1, 2, 5, and 6 of Section 140; the sketchy presentation completely misses antiparallel prisms, as well as those prisms which are represented in the third row of each of our Figs. 14 and 15.

The case $n = 4$, $d = 1$ is somewhat special; there is a single continuous family of prisms with quadrangular basis, as illustrated in Fig. 16.

4. Other Aploic Prismatoids with Bases

Despite the great variety of forms possible for prisms and antiprisms, their abundance is negligible compared with other isogonal prismatoids—even if only aploic ones are considered. We start by discussing the prismatoids with a basis, restricting attention to

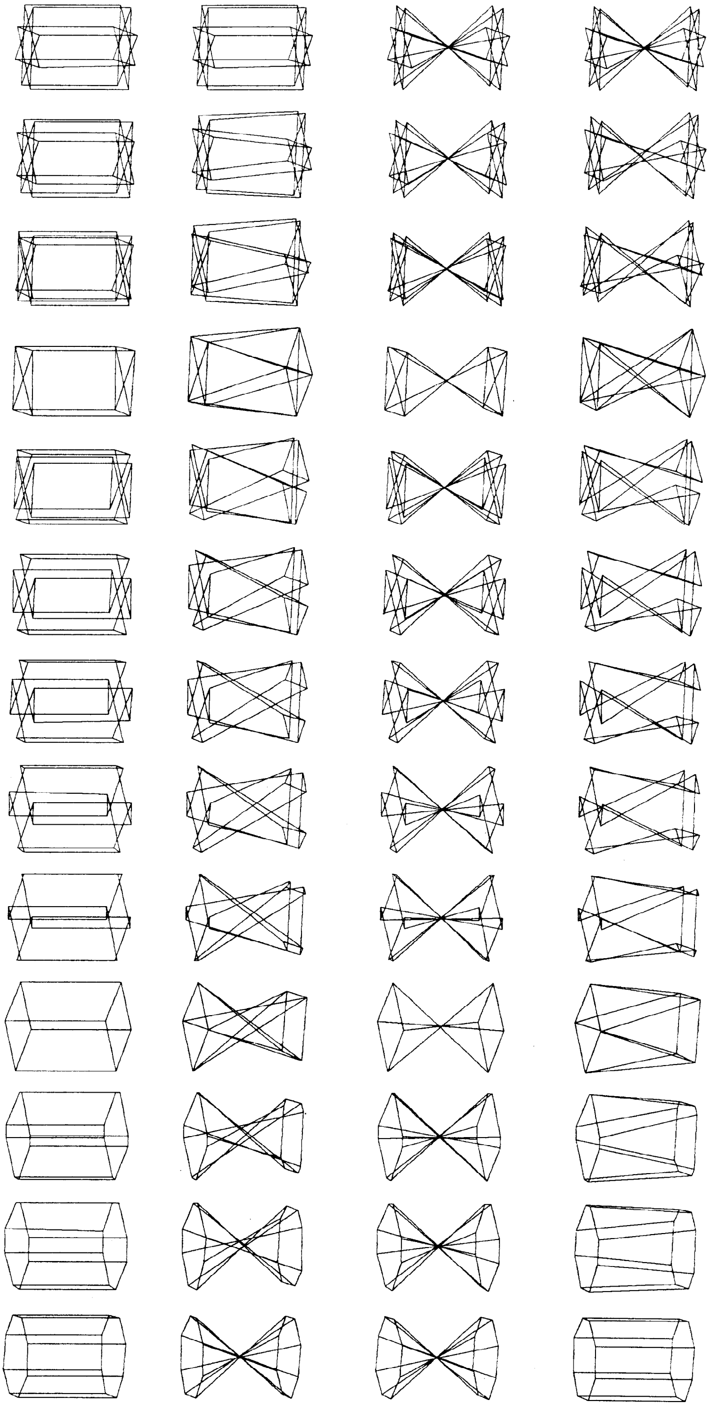


Fig. 15. The octagonal prisms, presented in a manner analogous to the one in Fig. 14. The prisms in which the bases have coinciding vertices are not aploic.

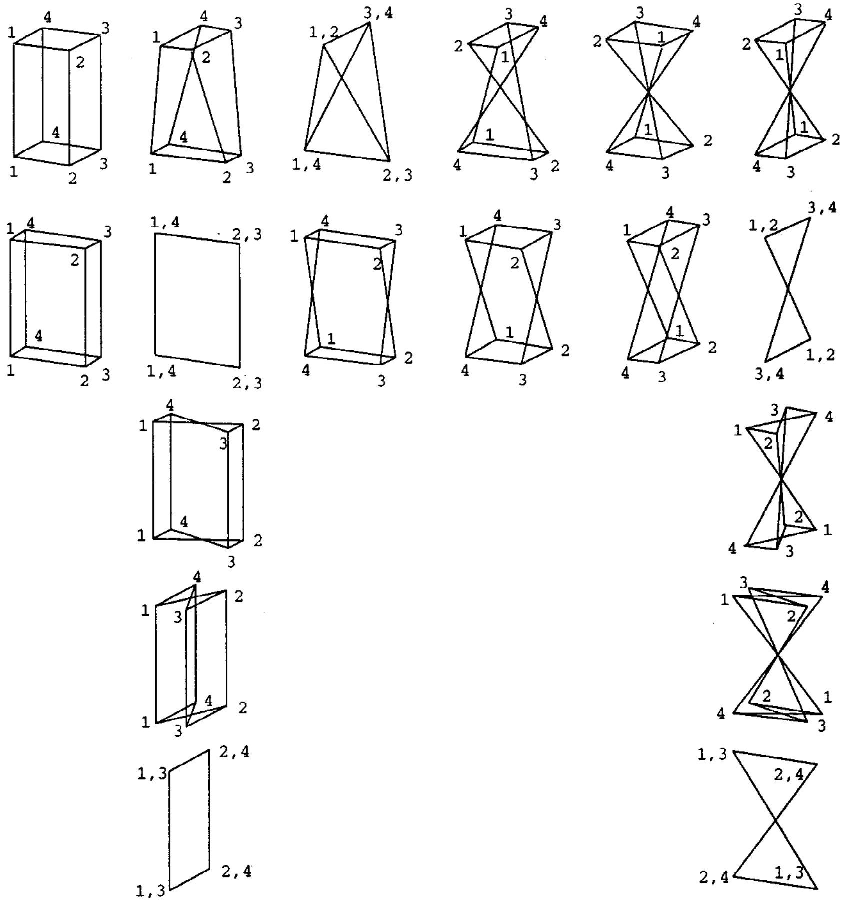


Fig. 16. The family of prisms with quadrangular basis. The family consists of one cycle, illustrated by the first two rows, together with two segments attached at different points of the cycle. All are aploic except the ones with some coinciding vertices.

those in which all mantle faces are quadrangles. Since we are interested in polyhedra other than prisms, the number of such quadrangular faces incident with each vertex must be at least three. It is most astonishing that **none** of these seems to have been mentioned anywhere in the literature; hence it is clear that they have no accepted names.

For aploic isogonal prismatoids that can be described collectively by the vertex-symbol $(4.4.4.n)$, an investigation of the possible structure of the underlying abstract polyhedra leads to precisely three distinct combinatorial types; instead of a listing of faces in the general case, we describe and illustrate the three kinds in Fig. 17 for $n = 8$. However, before continuing, it seems appropriate to acknowledge that the illustrations are hard to interpret, and to describe schematic diagrams useful both for understanding the structure of the polyhedra under discussion, and in establishing facts about them.

The diagrams in question, which are illustrated in Fig. 18, take advantage of the fact

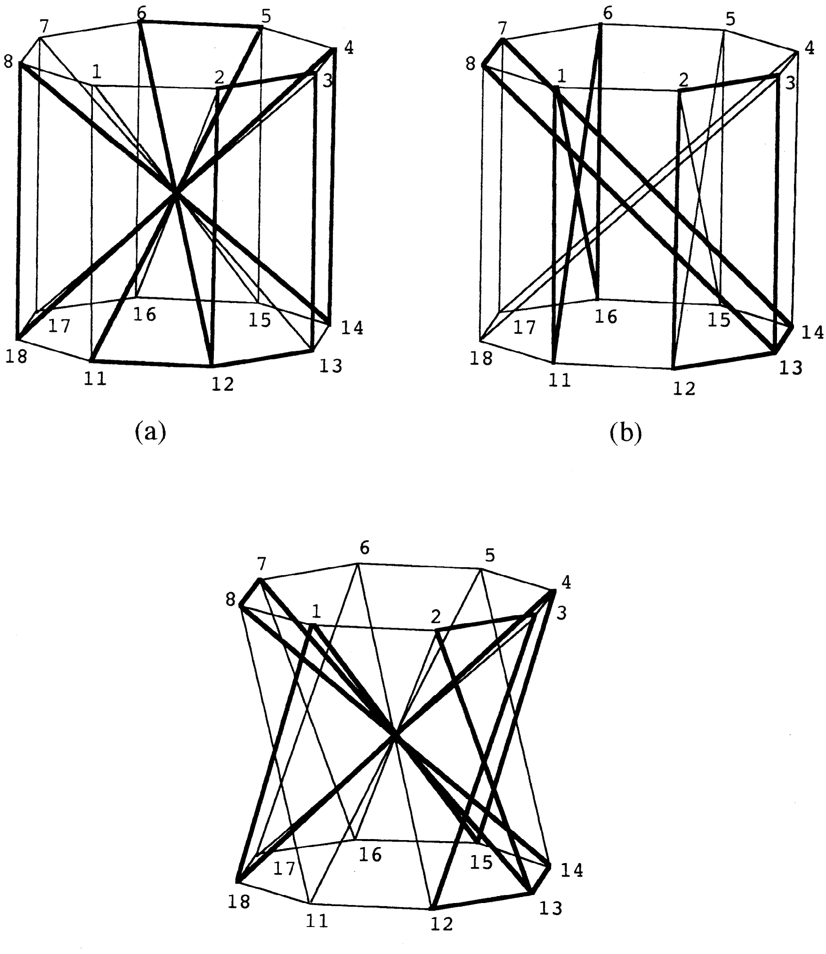
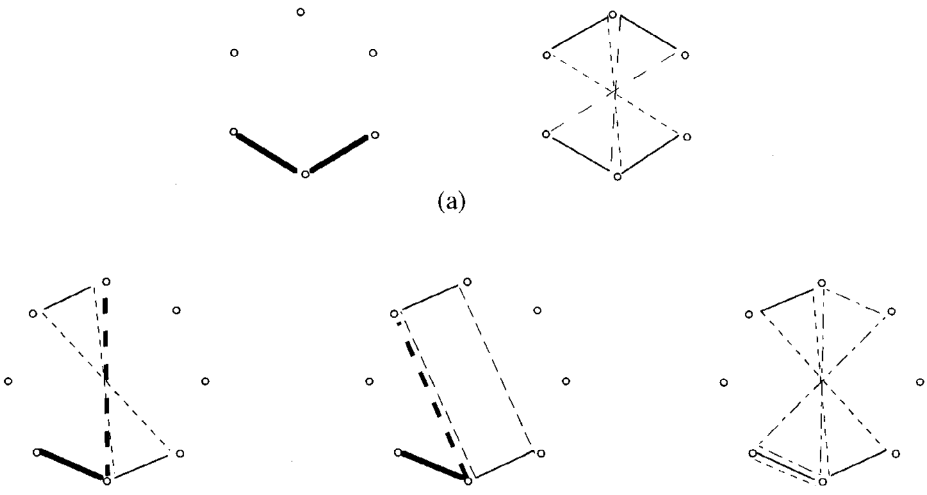


Fig. 17. Representatives of three different isomorphism types of aploic isogonal prismatoids, each with the vertex-symbol $(4.4.4.n)$, where $n = 4k$. The cases with $n = 8$ are shown. The polyhedron in (a) has as faces: 1,8,7,6,5,4,3,2,1; 11,12,13,14,15,16,17,18,11; 2,3,13,12,2; 4,5,15,14,4; 6,7,17,16,6; 8,1,11,18,8; 1,15,5,11,1; 2,12,6,16,2; 3,17,7,13,3; 4,14,8,18,4; 1,2,16,15,1; 3,4,18,17,3; 5,6,12,11,5; 7,8,14,13,7. It is orientable, of genus $\gamma = n/4 = k$. The faces of the polyhedron in (b) are: 1,8,7,6,5,4,3,2,1; 11,12,13,14,15,16,17,18,11; 2,3,13,12,2; 4,5,15,14,4; 6,7,17,16,6; 8,1,11,18,8; 1,16,6,11,1; 2,15,5,12,2; 3,18,8,13,3; 4,17,7,14,4; 1,2,15,16,1; 3,4,17,18,3; 5,6,11,12,5; 7,8,13,14,7. The faces of (c) are: 1,8,7,6,5,4,3,2,1; 11,12,13,14,15,16,17,18,11; 2,3,12,13,2; 4,5,14,15,4; 6,7,16,17,6; 8,1,18,11,8; 1,15,4,18,1; 3,17,6,12,3; 5,11,8,14,4; 7,13,2,16,7; 1,2,16,15,1; 3,4,18,17,3; 5,6,12,11,5; 7,8,14,13,7. The mantle of each of the three polyhedra consists of three kinds of quadrangles; one quadrangle of each kind is emphasized in the diagrams. The polyhedra in (b) and (c) are nonorientable, with Euler characteristic $\chi = 2 - n/2$.



(a)

(b)

Fig. 18. Diagrams used to represent and discuss isogonal prismatoids, as explained in the text. The diagrams in (a) represent the six-sided prisms shown in the leftmost columns of Fig. 14; the diagrams in (b) correspond to the prismatoids in Fig. 17.

that for isogonal prismatoids it is enough to specify which are the faces incident with one vertex. For ease of description we restrict the discussion to the case in which the bases are regular polygons. (In fact, there is no generality of structure lost by this simplification.) We start by choosing the n points that represent the vertices of one of the bases. Then for one of the vertices we indicate all the faces incident with it, using the following conventions (loosely derived from the idea that we are looking at the prismatoid from far above the center):

(i) We indicate only the faces of the mantle.
(ii) A horizontal edge is indicated by a solid line, while an edge connecting vertices of the different bases is indicated by a dashed line; vertical edges are not shown.
(iii) A quadrangle with a pair of vertical edges is indicated by drawing a single bold line (either solid or dashed).
(iv) A vertical self-intersecting quadrangle with a pair of horizontal edges is indicated by a pair of parallel thin lines, one solid and one dashed.
(v) For nonvertical faces all four edges are indicated with thin lines, solid or dashed as appropriate.

With a little patience and some practice, these diagrams can be used with advantage for discussing the polyhedra they represent. For example, the two prisms in the leftmost column of Fig. 14 are represented by the diagrams in Fig. 18(a), while the prismatoids (4.4.4.8) of Fig. 17 are represented by the three diagrams in Fig. 18(b).

It is easily verified that these three kinds of polyhedra with vertex- symbols $(4.4.4.n)$ exist whenever $n > 4$ is a multiple of 4. Polyhedra of the kind shown in Fig. 17(a) are orientable, the other two kinds are nonorientable.

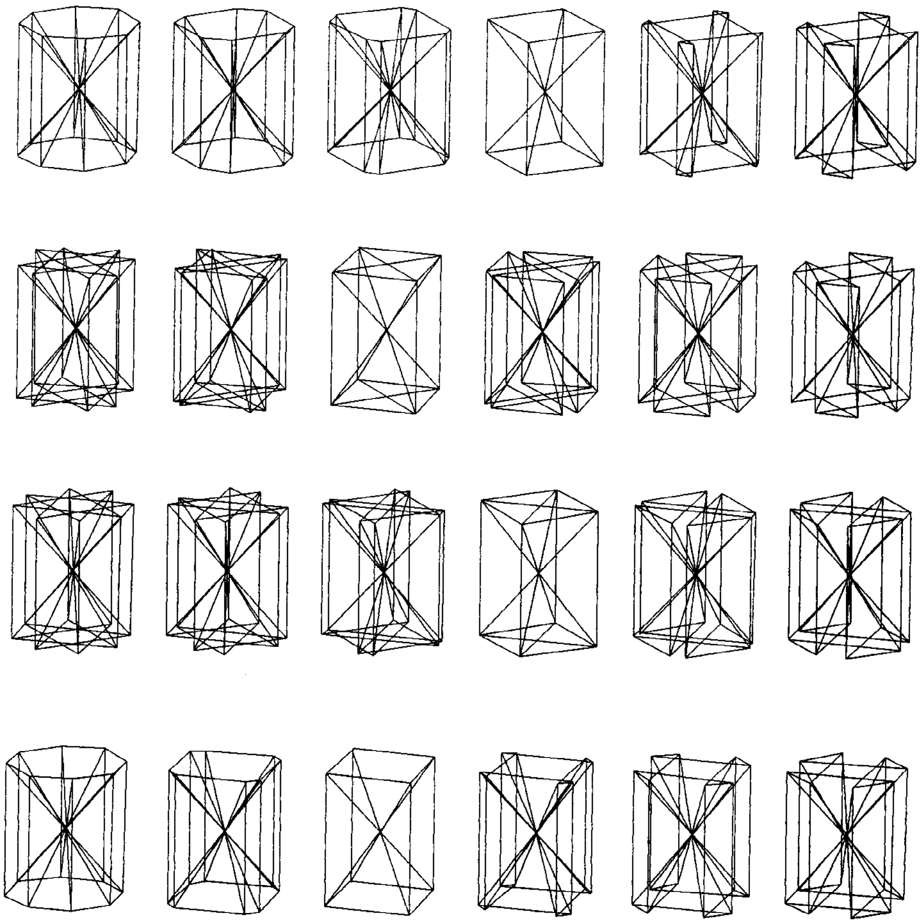


Fig. 19. The continuous family of isogonal prismatoids with vertex-symbol (4.4.4.8), that includes the polyhedron in Fig. 17(a). It starts at upper left, and snakes back and forth in alternate rows. They all are aploic, except for those with some coinciding vertices.

The examples in Figs. 17 and 18 have as bases regular octagons, but this is just for ease of visualization: any isogonal n-gon with $n = 4k \geq 8$ can serve as the basis for each of these three kinds of polyhedra, and again the polyhedra form continuous families. For the particular polyhedron of Fig. 17(a) this family is illustrated by the diagrams in Fig. 19. Details about the exact nature of these families have not been investigated so far.

After the relatively simple case of prismatoids with vertex-symbols $(4.4.4.n)$ we turn to the more complicated polyhedra that have vertex-symbols $(4.4.4.4.n)$. To begin with, there are polyhedra of this kind that are analogous to some extent to the ones of type $(4.4.4.n)$—namely, involving a variety of shapes of quadrangles, including self-intersecting ones. Three examples of such polyhedra are shown in Fig. 20 and described in its caption; the corresponding diagrams are shown in Fig. 21; in this case, however,

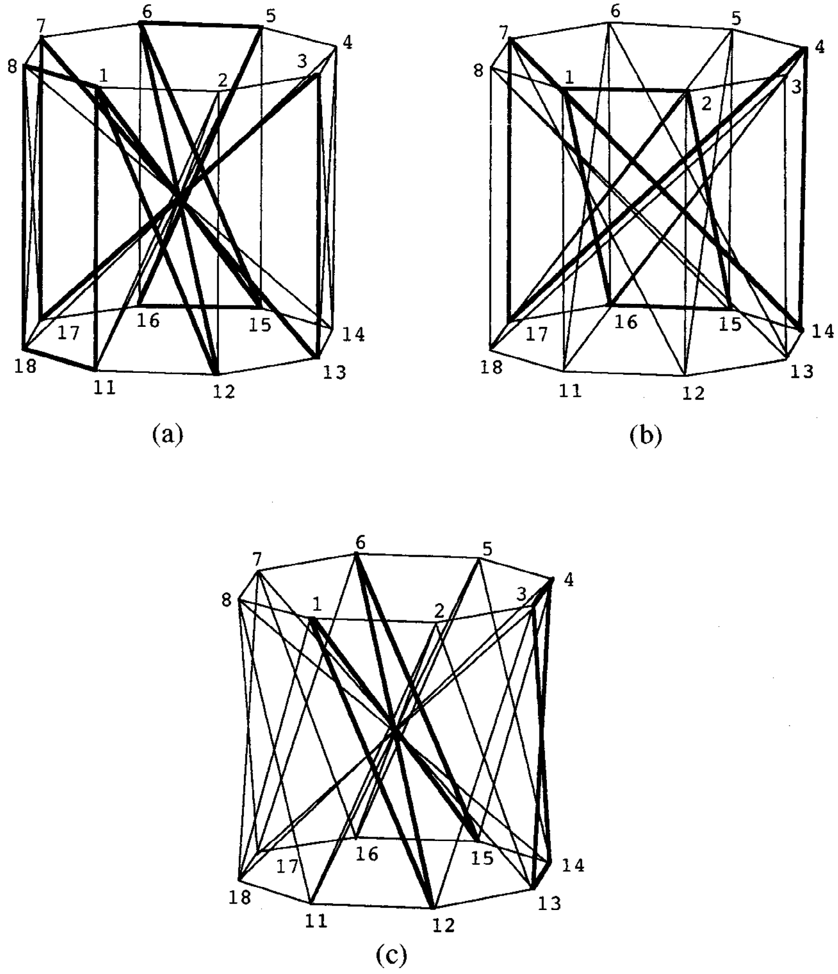


Fig. 20. Aploic isogonal prismatoids of three isomorphism types; all have vertex-symbol $(4.4.4.4.n)$, with $n = 4k$ in (a) and $n = 2k$ in (b) and (c). The cases $n = 8$ are shown. All polyhedra of these types are nonorientable, with $\chi = 2 - n$. The faces of (a) are: 1,8,7,6,5,4,3,2,1; 11,12,13,14,15,16,17,18,11; 2,3,13,12,2; 4,5,15,14,4; 6,7,17,16,6; 8,1,11,18,8; 1,15,5,11,1; 2,12,6,16,2; 3,17,7,13,3; 4,14,8,18,4; 1,2,11,12,1; 3,4,13,14,3; 5,6,15,16,5; 7,8,17,18,7; 1,12,6,15,1; 2,16,5,11,2; 3,14,8,17,3; 4,18,7,13,4. The faces of (b) are: 1,8,7,6,5,4,3,2,1; 11,18,17,16,15,14,13,12,11; 1,2,15,16,1; 2,3,16,17,2; 3,4,17,18,3; 4,5,18,11,4; 5,6,11,12,5; 6,7,12,13,6; 7,8,13,14,7; 8,1,14,15,8; 1,16,6,11,1; 2,15,5,12,2; 3,18,8,13,3; 4,17,7,14,4; 5,18,8,15,5; 6,11,1,16,6; 7,12,2,17,7; 8,13,3,18,8. The faces of (c) are: 1,8,7,6,5,4,3,2,1; 11,18,17,16,15,14,13,12,11; 1,2,11,12,1; 2,3,12,13,2; 3,4,13,14,3; 4,5,14,15,4; 5,6,15,16,5; 6,7,16,17,6; 7,8,17,18,7; 8,1,18,11,8; 1,12,6,15,1; 2,13,7,16,2; 3,14,8,17,3; 4,15,1,18,4; 5,16,2,11,5; 6,17,3,12,6; 7,18,4,13,7; 8,11,5,14,8. The mantle of each polyhedron is formed by quadrangles of several shapes; one quadrangle of each kind is emphasized.

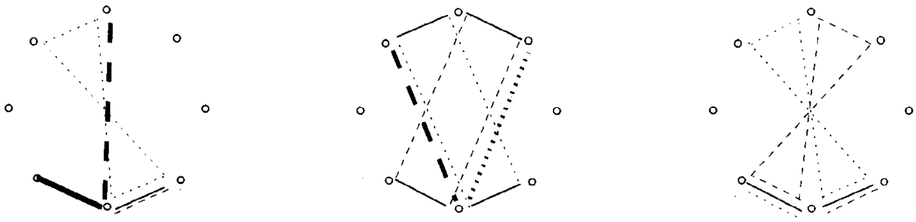


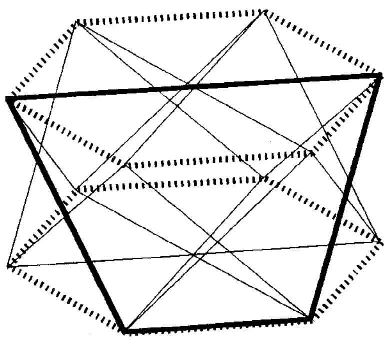
Fig. 21. Diagrams corresponding to the prismatoids with vertex-symbol (4.4.4.4.8) shown in Fig. 20.

these examples are only some of the possibilities. Again, the use of regular octagons as bases is just for ease of visualization; any isogonal polygon could be used, and the polyhedra form continuous families. These have not been investigated in any detail, but it should be noted that if the basis is a nonregular isogonal polygon, the number of distinct quadrangles in the mantle may be larger than in Fig. 20.

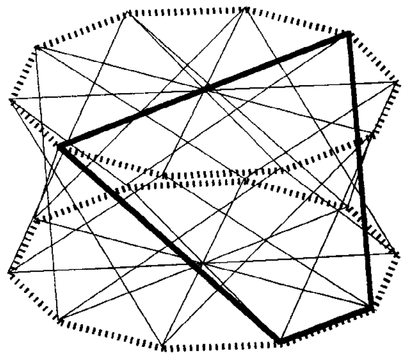
Besides these "strange" polyhedra, there are two families of isogonal prismatoids with vertex-symbol $(4.4.4.4.n)$ that have mantles consisting entirely of simple quadrangles which, in suitable representatives, can all be made congruent. Thus they are, in a sense, closest to acoptic polyhedra and in particular to prisms and antiprisms. Prismatoids in the two families are denoted by symbols of the form $P(t_0, t_1; n)$ and $A(t_0, t_1; n)$.

In Fig. 22 we show some of the simplest representatives of the first family; the corresponding diagrams are shown in Fig. 23. Since these polyhedra are just the smallest instances of prismatoids with more than three quadrangles incident with each vertex, we describe them by symbols that are easily adaptable to more general situations. Moreover, to simplify the exposition, for the time being we restrict attention to the case in which the basis is a regular polygon. Other possibilities are considered later. We start by observing that each mantle face is a trapezoid T_1; the parallel edges of T_1 belong to the different basis planes of the prismatoid—but only one of these edges is an edge of the basis polygon. All these trapezoids are congruent. In the symbol $P(t_0, t_1; n)$ of a prismatoid of this kind, n is the number of vertices of each base, t_0 is the *span* of that edge of T_1 which is also an edge of the basis, and t_1 is the span of each side-edge of T_1. Here (and in what follows) by "span" we mean across how many steps along the vertices that are in the basis does the edge in question reach; note that this refers to the vertices as they are encountered along the circle on which they lie, and not necessarily along the edges of the basis polygon. The two bases are aligned, and the side-edges of T_1, besides having span t_1, also reach from one basis to the other. The fourth edge of T_1 has span $t_0 + 2t_1$, and is a diameter of the n-gonal basis. Therefore $n = 2t_0 + 4t_1$. Since polyhedra are (by definition) connected, the positive integers t_0 and t_1 must be relatively prime, and t_0 must be odd.

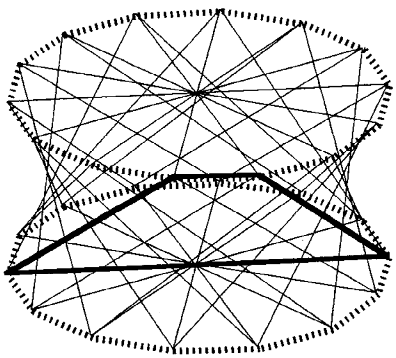
The construction of polyhedra in the second family, which are denoted by the symbol $A(t_0, t_1; n)$, can be described as follows; again we consider here only the case in which each basis is a regular n-gon. We take these two n-gons in an antiprismatic position, and number all $2n$ vertices consecutively as they appear on an orthogonal projection. The mantle faces T_1 are, as in the first family, trapezoids which share one of their parallel edges with a basis. That edge has span t_0, which therefore must be an even positive integer,



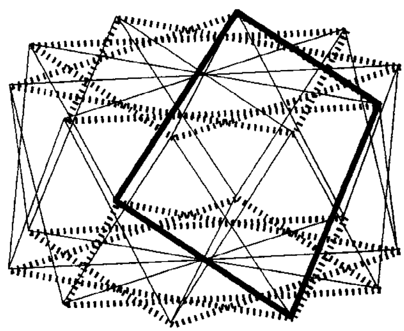
P(1, 1; 6)



P(1, 2; 10)



P(1, 3; 14)



P(3, 1; 10)

Fig. 22. Examples of isogonal prismatoids of the combinatorial type $(4.4.4.4.n)$, belonging to the family of polyhedra denoted by the symbol $P(t_0, t_1; n)$, which is explained in the text. One of the congruent quadrangles that form the mantle is shown by heavy solid lines, while the two bases are represented by the heavy dashed lines.

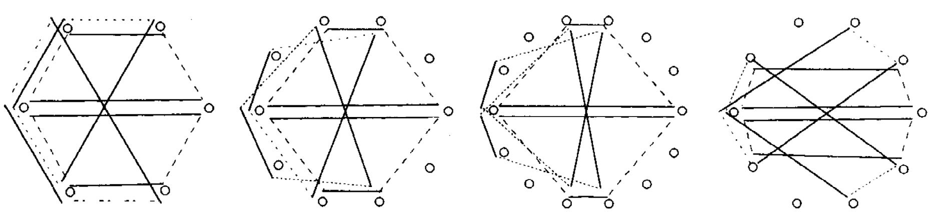


Fig. 23. Diagrams corresponding to the prismatoids with vertex-symbol (4.4.4.4.8) shown in Fig. 20. All mantle faces incident with the leftmost vertex are indicated in each diagram.

while each of the side-edges of the trapezoid has span t_1, which is an odd positive integer. Here the relation between the parameters is $n = t_0 + 2t_1$. Figure 24 shows some examples of these prismatoids; the diagrams of three of them are contained in Fig. 25.

The two families of polyhedra with vertex-symbols $(4.4.4.4.n)$ are the starting members (for $k = 1$) of two “superfamilies” of isogonal prismatoids $P(t_0, t_1, \ldots, t_k; n)$ and $A(t_0, t_1, \ldots, t_k; n)$, with vertex-symbols $(4^{4k}.n)$, where k is any positive integer, and n is determined in a suitable manner. (The “exponential” notation in the vertex-symbol is just an abridgement of the “product” notation.) The definition of these families is analogous to the one we have seen in case $k = 1$, except that now the “free” edge (of span $t_0 + 2t_1$) of the trapezoid T_1, instead of being a diameter of the basis polygon, is one of the parallel edges of a second trapezoid T_2, whose side-edges have span t_2. If $k > 2$, the fourth edge (of span $t_0 + 2t_1 + 2t_2$) of T_2 is one of the parallel edges of a trapezoid T_3, the side-edges of which have span t_3, and so on in a self-explanatory manner. Hence the mantle consists of k kinds of trapezoids, each represented by $2n$ congruent copies. For prismatoids from the family $P(t_0, t1, \ldots, t_k; n)$ the value of n is determined by $n = 2t_0 + 4(t_1 + \cdots + t_k)$, while for prismatoids from $A(t_0, t_1, \ldots, t_k; n)$ we have $n = t_0 + 2(t_1 + \cdots + t_k)$; the labeling of the vertices is in both cases the same as for $k = 1$. As in the case $k = 1$, we need t_0 to be positive, and odd in the P-families, while even in the A-families. In both cases the parameters $t_0, t_1, \ldots, t_k$ cannot all have a common factor, and some additional conditions need to be satisfied in order to avoid unwanted coincidences. The precise conditions have not been determined, but it appears that if all parameters are positive it is sufficient to require that none equals the sum of any collection of the others. Examples of such polyhedra in the case $k = 2$ are shown in Figs. 26 and 27.

In addition to the two families of isogonal prismatoids with vertex-symbols $(4^h.n)$, where $h \equiv 0 (\mathrm{mod}\, 4)$, there are two analogous families for which $h \equiv 2 (\mathrm{mod}\, 4)$. The polyhedra in these families are denoted by symbols $P(t_0, t_1, \ldots, t_k, t^*; n)$ and $A(t_0, t_1, \ldots, t_k, t^*; n)$; they have vertex-symbols $(4^{4k+2}.n)$. The polyhedra of these types are constructed in exactly the same way as indicated by those parts of their symbol which coincide with the symbols of the polyhedra discussed above. The only additional part, t^*, indicates that the fourth edge of the trapezoid T_k is also an edge of a self-intersecting quadrangle T^* which has a pair of parallel edges, the distance between the parallel edges being t^*, and the nonparallel edges having span $n/2$ in the first family and n in the second. Here $n = 2t_0 + 4(t_1 + \cdots + t_k) + 2t^*$ for polyhedra with the symbol $P(t_0, t_1, \ldots, t_k, t^*; n)$, and $n = t_0 + 2(t_1 + \cdots + t_k) + t^*$ for polyhedra with the symbol $A(t_0, t_1, \ldots, t_k, t^*; n)$. Again various conditions, which have not been completely determined so far, have to be satisfied by the parameters in order to avoid unwanted coincidences or disconnected “polyhedra.” Examples of polyhedra of these types for $k = 1$ are shown in Figs. 28 and 29.

As stated above, the discussion of the P- and A-families was conducted assuming the basis polygons to be regular. However, the symmetry of these polyhedra allows various modifications of the bases and of the polyhedra themselves, without loss of their character as isogonal prismatoids. As is easily verified, in the case of polyhedra $P(t_0, t_1, \ldots, t_k; n)$ or $P(t_0, t_1, \ldots, t_k, t^*; n)$, any isogonal polygon can serve as the basis; in fact, these polyhedra give rise to continuous families such as the ones in Figs. 14–16 and 19. This is illustrated in Fig. 30 for the polyhedron $P(1, 1; 6)$. Analogously, the bases of polyhedra $A(t_0, t_1, \ldots, t_k; n)$ can be twisted with respect to each other, and then each

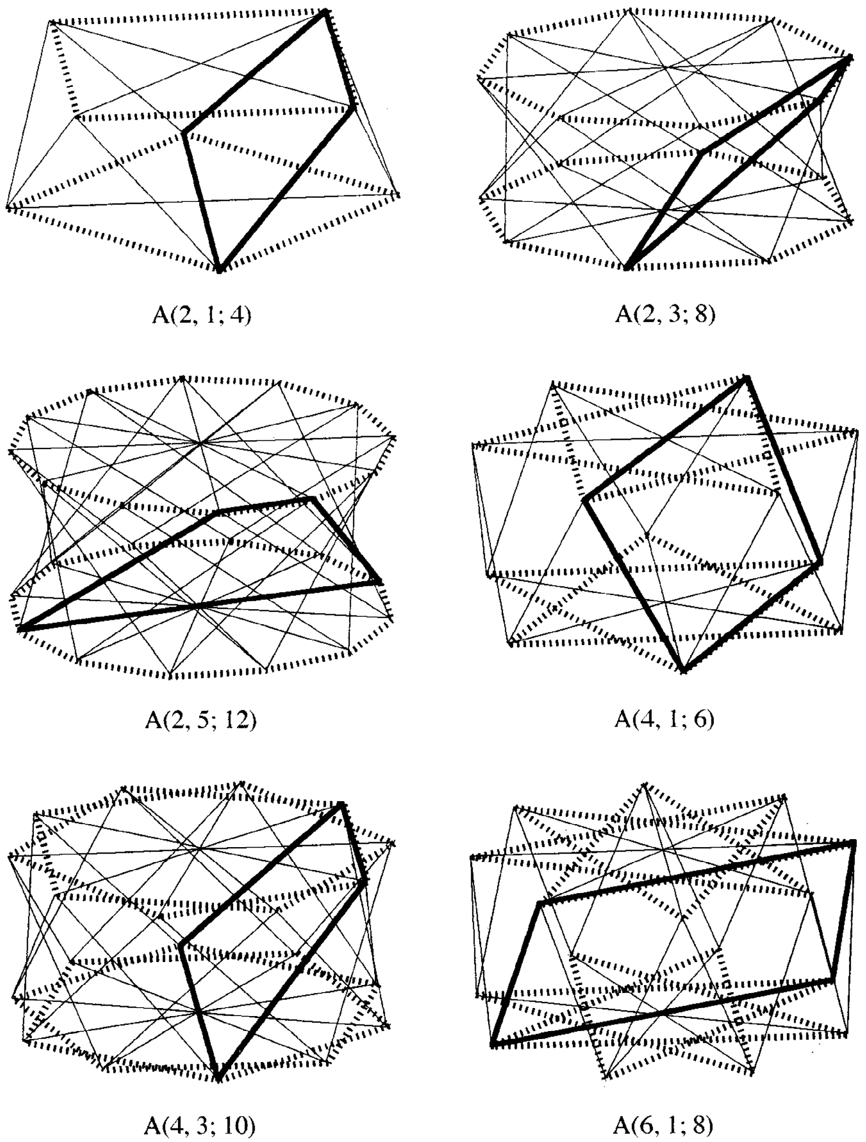


Fig. 24. Examples of isogonal prismatoids denoted by $P(t_0, t_1; n)$, with vertex-symbol $(4.4.4.4.n)$. The symbol $P(t_0, t_1; n)$ is explained in the text. One of the congruent quadrangles that form the mantle is shown by heavy solid lines, while the two bases are represented by the heavy dashed lines.

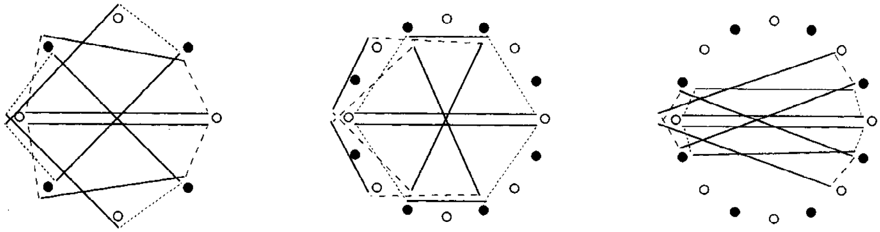
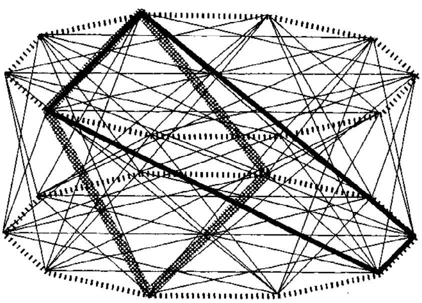
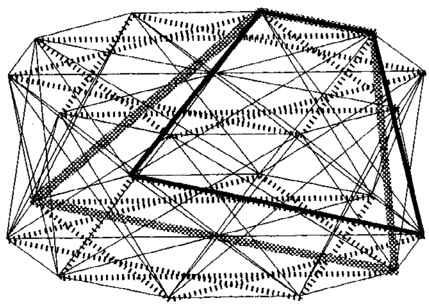


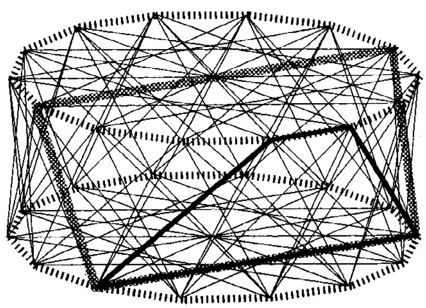
Fig. 25. Diagrams corresponding to the first two prismatoids with vertex-symbol (4.4.4.4.8), and the last one, shown in Fig. 24. The vertices of one of the basis polygons are indicated by hollow dots, those of the other by solid dots.



P(1, 3, −1; 10)

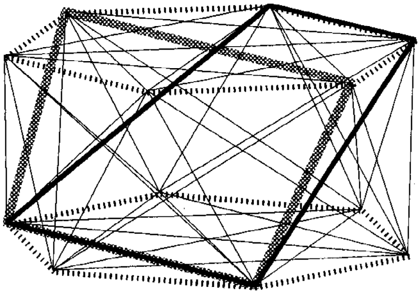


P(3, −1, 2; 10)

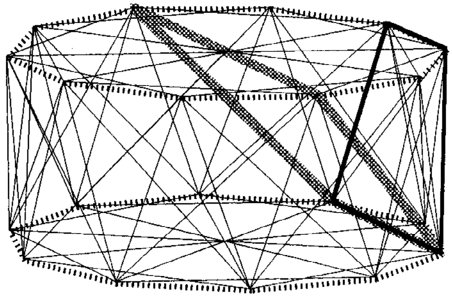


P(1, 2, 1; 14)

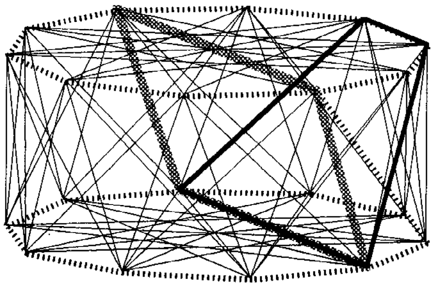
Fig. 26. Examples of isogonal prismatoids $P(t_0, t_1, t_2; n)$ with vertex-symbol $(4^8.n)$. The meaning of the symbol $P(t_0, t_1, t_2; n)$ is explained in the text. One of the trapezoids T_1 is shown in heavy solid lines, while one T_2 is shown shaded. The two basis polygons are shown in heavy dashed lines.



A(2, 3, −1; 6)



A(2, 1, 3; 10)



A(2, 3, 1; 10)

Fig. 27. Examples of isogonal prismatoids $A(t_0, t_1, t_2; n)$ with vertex-symbol $(4^8.n)$. The meaning of the symbol $A(t_0, t_1, t_2; n)$ is explained in the text. One of the trapezoids T_1 is shown in heavy solid lines, while one T_2 is shown shaded. The two basis polygons are shown in heavy dashed lines.

trapezoid can be replaced by two triangles; if the replacement is done systematically, isogonal prismatoids with vertex-symbol $(3^{6k}.n)$, where $k \geq 1$, are obtained. These can be considered as natural relatives of the traditional antiprisms. As an example, one isogonal prismatoid with vertex-symbol $(3^6.4)$, obtained from the polyhedron $A(2, 1; 4)$ by a slight twist of one basis polygon and replacement of each trapezoid by two triangles, is illustrated in Fig. 31. Clearly, the same method can also be applied to all isogonal prismatoids $P(t_0, t_1, \ldots, t_k; n)$ with regular polygons as the basis.

5. Basis-free Aploic Prismatoids

There are several interesting families of basis-free aploic polyhedra, some of which we describe next. However, these are only a small part of the possible polyhedra of this kind. A systematic investigation would seem to be a challenging but rewarding task.

(i) The simplest (and most widely known) are the *sphenoids,* that is, isogonal tetrahedra, examples of which are shown in Fig. 32. If we had included "digons" among polygons, sphenoids would be antiprisms with digonal bases. All sphenoids are isohedral as well (that is, their symmetry group acts transitively on the faces); in the terminology

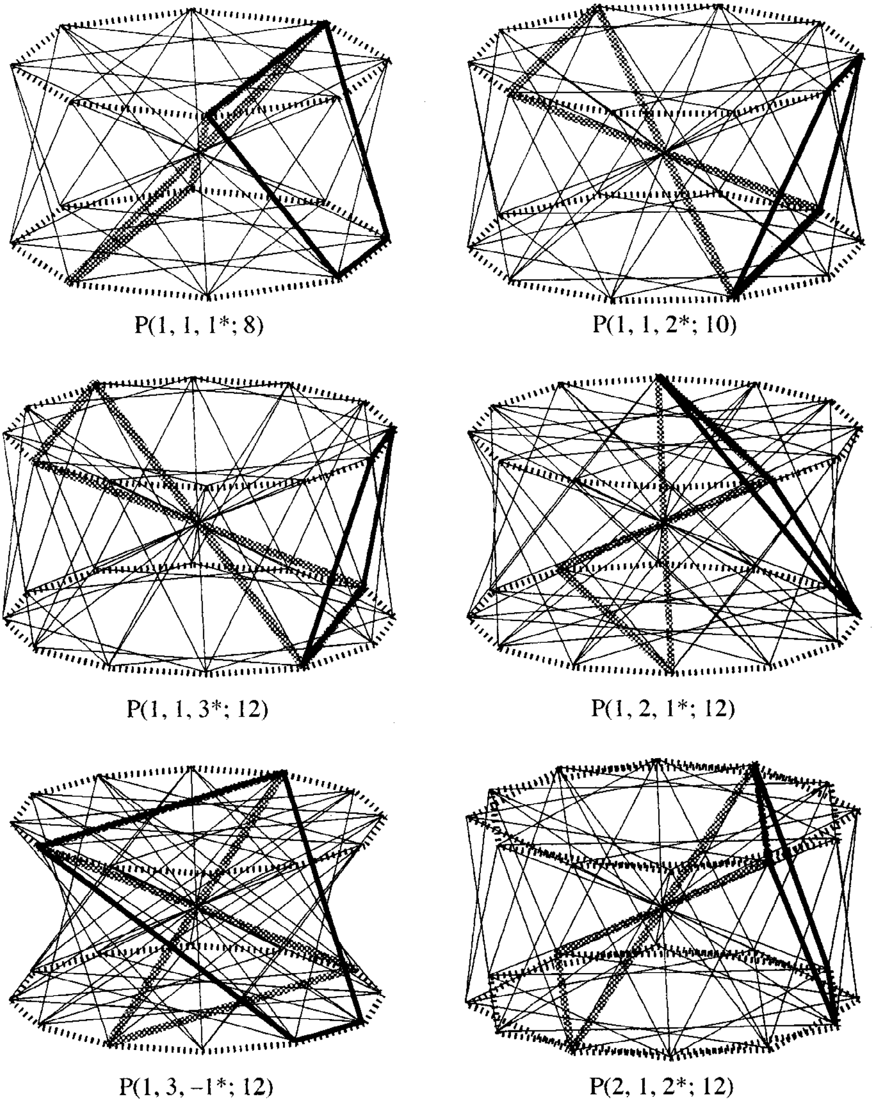


Fig. 28. Examples of isogonal prismatoids of type $P(t_0, t_1, t^*; n)$, with vertex-symbol $(4^6.n)$. The meaning of the symbol $P(t_0, t_1, t^*; n)$ is explained in the text. One of the trapezoids T_1 is shown in heavy solid lines, while one T^* is shown shaded. The two basis polygons are shown in heavy dashed lines.

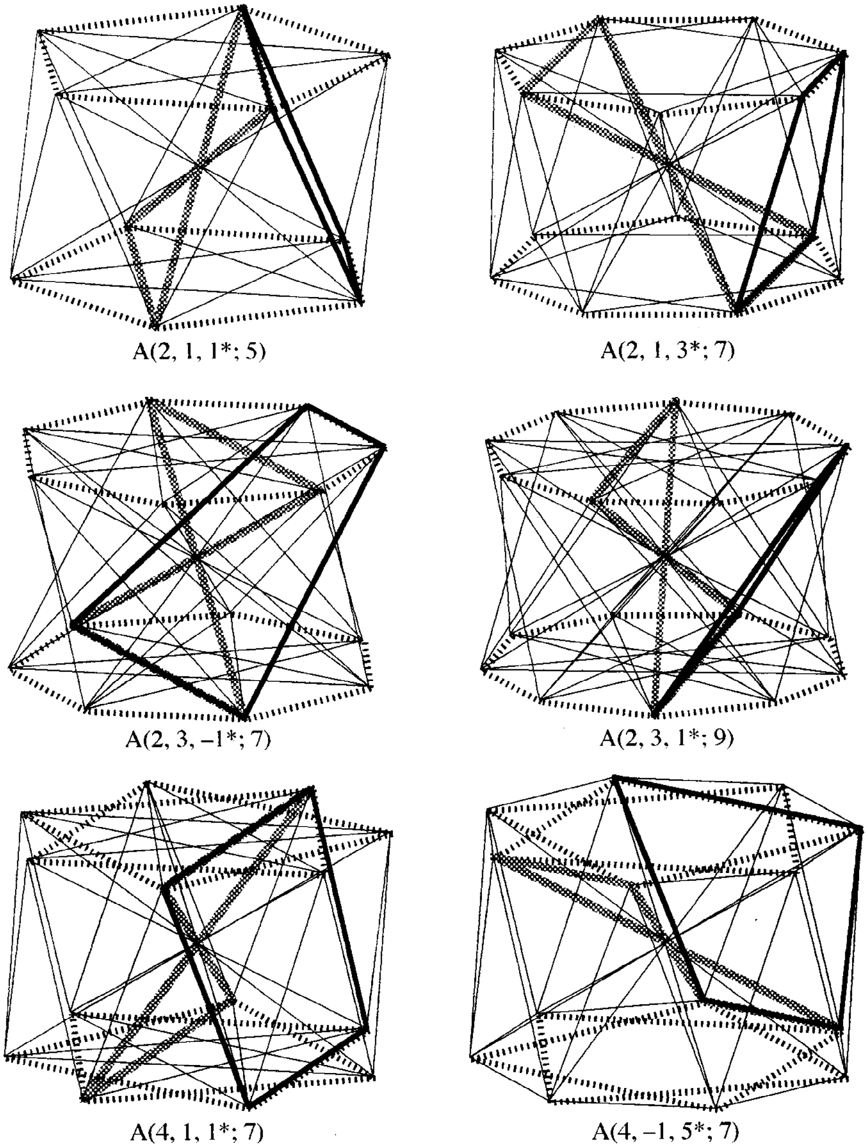


Fig. 29. Examples of isogonal prismatoids of type $A(t_0, t_1, t^*; n)$, with vertex-symbol $(4^6.n)$. The meaning of the symbol $A(t_0, t_1, t^*; n)$ is explained in the text. One of the trapezoids T_1 is shown in heavy solid lines, while one T^* is shown shaded. The two basis polygons are shown in heavy dashed lines.

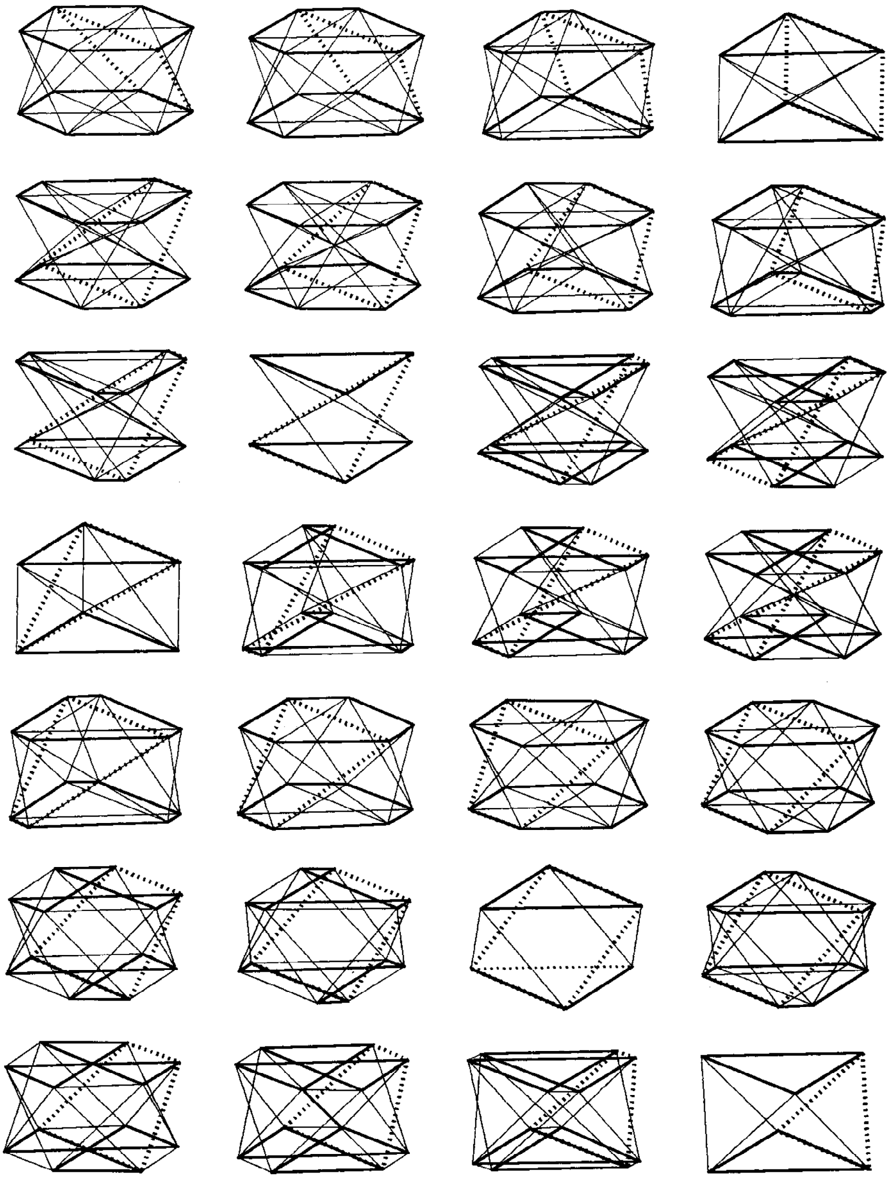


Fig. 30. The family of polyhedra obtained from $P(1, 1; 6)$ by continuously changing the bases from the regular hexagon to other isogonal hexagons. All these isogonal prismatoids have the same underlying abstract polyhedron with vertex-symbol (4.4.4.4.6).

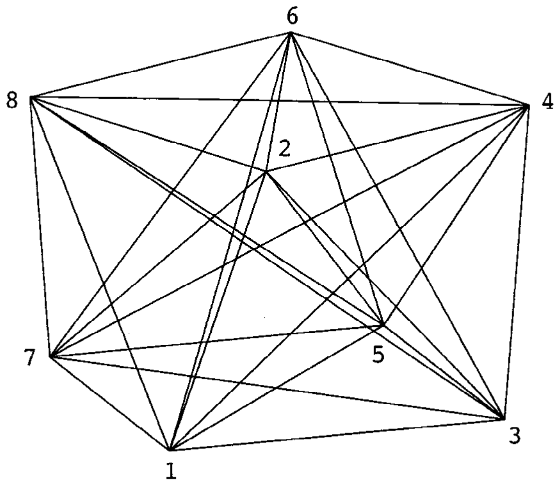


Fig. 31. A member of the continuous family isogonal prismatoids obtained from the polyhedron $A(2, 1; 4)$ in Fig. 24 by a slight twist of one basis polygon with respect to the other, with appropriate replacement of the skew quadrangle by two triangles. The polyhedra in this family are orientable and have vertex-symbols $(3^6.n)$. For the prismatoid shown, $n = 4$ and the faces are: 1,7,5,3,1; 2,4,6,8,2; 1,3,4,1; 3,5,6,3; 5,7,8,5; 7,1,2,7; 2,8,3,2; 4,2,5,4; 6,4,7,6; 8,6,1,8; 1,4,8,1; 3,6,2,3; 5,8,4,5; 7,2,6,7; 2,1,5,2; 4,3,7,4; 6,5,1,6; 8,7,3,8.

of Grünbaum [7] such polyhedra are called "noble." The sphenoids and the Platonic (regular) polyhedra are the only noble polyhedra that are acoptic; below we describe additional noble polyhedra which are not acoptic. There are two geometric types of aploic sphenoids, distinguished by their symmetry group. The less symmetric sphenoids form a continuous family that depends on one real parameter.

(ii) A general method for the generation of basis-free isogonal prismatoids uses the *Boolean sum* of two or more suitable isogonal prismatoids with bases. The prismatoids have to be chosen so that their bases coincide in pairs, and those pairs are deleted from the new polyhedron. With appropriate choices, the resulting isogonal prismatoids are aploic; in fact, in some special cases they are even acoptic. The method of Boolean sums is an open-ended one, since the number of prismatoids with bases that can be involved in the formation of a basis-free prismatoid can be arbitrarily large; hence there is no

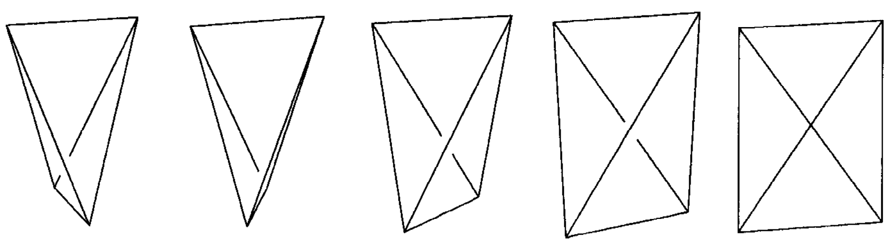
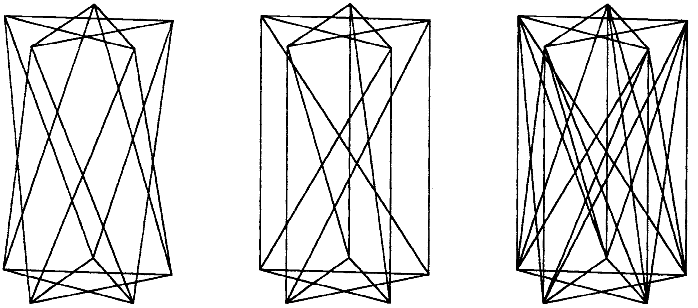
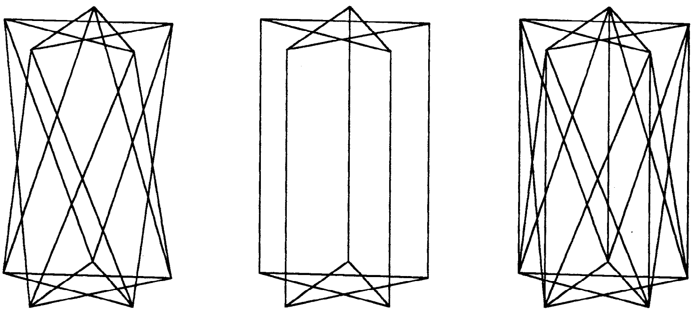


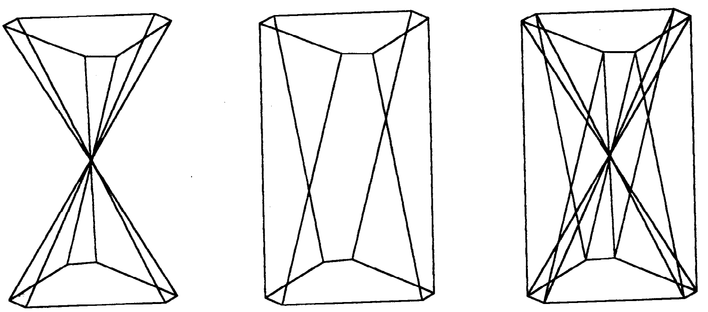
Fig. 32. The continuous family of sphenoids. The extreme polyhedra have greater symmetry than those of the intermediate type. All except the last are acoptic; the last one is not aploic, since all four faces are in the same plane.



(a)



(b)



(c)

Fig. 33. Examples of basis-free isogonal prismatoids obtained from (a) two antiprisms; (b) an antiprism and a prism; (c) two prisms. In each case the two starting polyhedra (shown at the left and in the center) must have coinciding bases; the resulting polyhedron is shown at the right. The basis-free prismatoids in (b) and (c) are aploic, the one in (a) is not.

hope of giving a complete enumeration. Some of the possibilities are discussed in the following paragraphs.

The simplest case is the Boolean sum of two antiprisms, or two prisms, or one prisms and one antiprism, which share both bases; eliminating these bases gives a basis-free isogonal prismatoid. This is illustrated in Fig. 33. More complicated Boolean sums, each involving four prisms, are shown by the examples in Fig. 34. The basis-free prismatoids

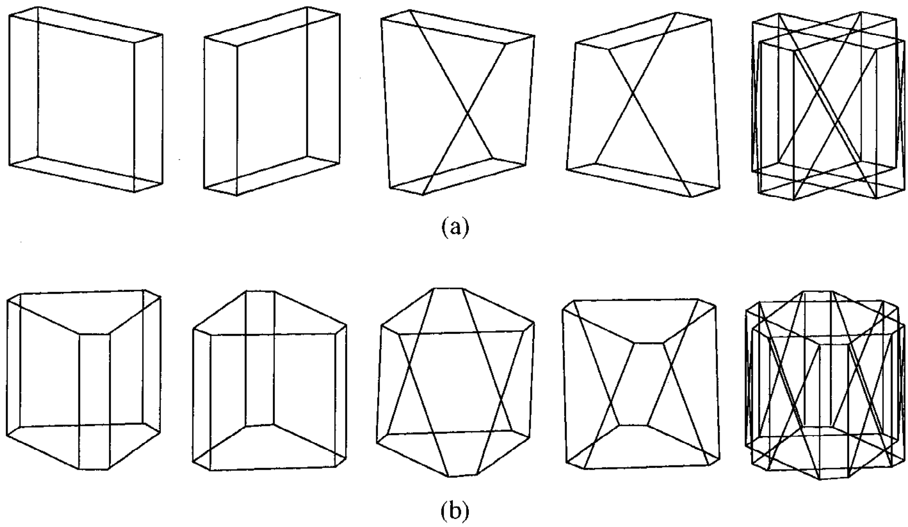


Fig. 34. Examples of basis-free aploic isogonal prismatoids (shown at the right) with vertex-symbol (4.4.4.4), obtained from four prisms each (shown to the left of the basis-free polyhedron). The basis-free prismatoid in (a) is a realization of the regular toroidal map with symbol $\{4, 4\}_{4,0}$ in the notation of Coxeter and Moser [4].

in Figs. 33(b, c) and 34 are aploic, those in Fig. 34 are orientable. In fact, it is easy to verify that, understanding the faces of the prismatoid in Fig. 34(a) as determining quadrangular regions, a realization of the regular toroidal map $\{4, 4\}_{4,0}$ is obtained. The results of analogous Boolean sums involving four antiprisms are shown in Fig. 35. The polyhedron in Fig. 35(c) is not aploic, but it is noble; this type of noble polyhedra was described in [7] under the name "wreath polyhedra."

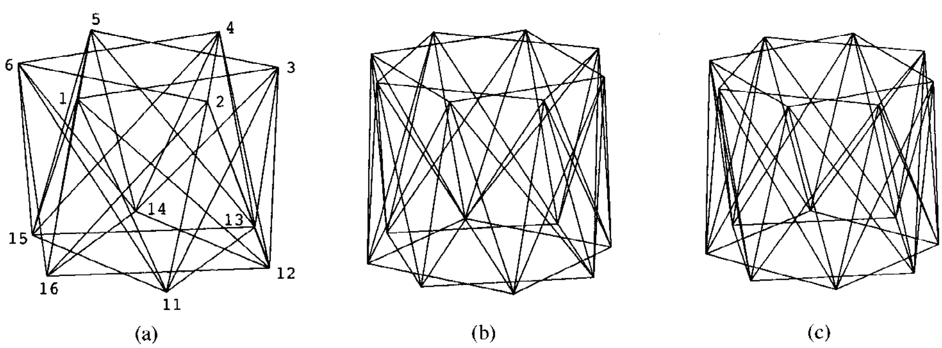


Fig. 35. The polyhedra in (a) and (b) are examples of basis-free aploic isogonal prismatoids with vertex-symbol (3.3.3.3.3.3). Each is obtained from four antiprisms with bases coinciding in suitable pairs. The underlying abstract polyhedron is topologically a torus. The faces of the prismatoid in (a) are: 1,11,3,1; 3,11,13,3; 3,13,5,3; 5,13,15,5; 5,15,1,5; 1,15,11,1; 1,3,12,1; 3,14,12,3; 3,5,14,3; 5,16,14,5; 5,1,16,5; 1,12,16,1; 2,13,11,2; 2,4,13,2; 4,15,13,4; 4,6,15,4; 6,11,15,6; 6,2,11,6; 2;16,12,2; 2,12,4,2; 4,12,14,4; 4,14,6,4; 6,14,16,6; 6,16,2,6. The faces in (b) are determined analogously. Suitable rotations of the upper bases yield families of polyhedra of the same type. Transitions between types lead to nonaploic polyhedra, some of which are isohedral as well; an example of such a polyhedron is shown in (c).

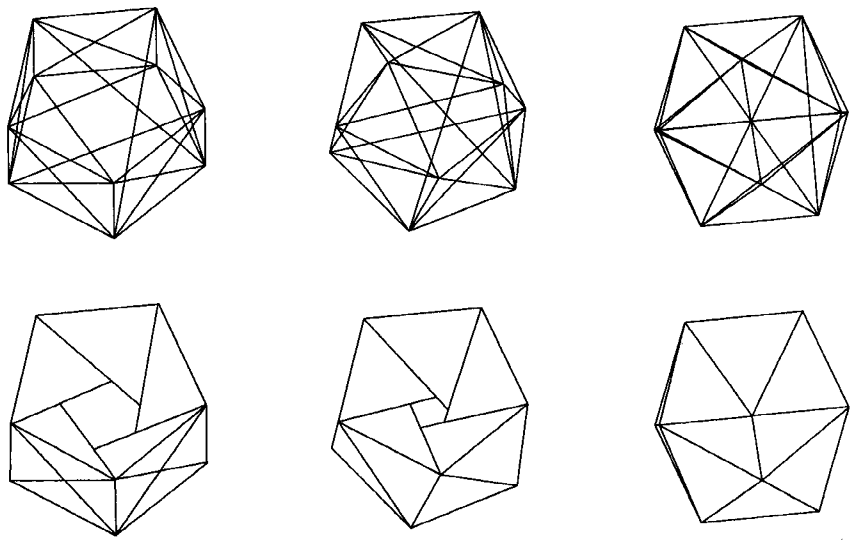


Fig. 36. Examples of basis-free isogonal prismatoids obtained from two pentagonal antiprisms with common bases. Each prismatoid is shown in skeletal as well as in surficial form—the latter being appropriate since the faces are simple polygons (triangles) that do not cross each other. The middle one is acoptic; the other two, which represent the extremes of a continuous family of acoptic isogonal prismatoids, fail to be acoptic since some of their edges are in the planes of other faces.

As a particular cases of Boolean sums, if two suitable acoptic antiprisms are used, or an antiprism and a prism, acoptic isogonal tori can be obtained. Examples of such toroidal polyhedra, with vertex-symbol (3.3.3.3.3.3) are shown in Figs. 36 and 37. Like the antiprisms or prisms used in their construction, these polyhedra depend on various parameters; beyond certain parameter values they cease to be acoptic. Polyhedra of this kind were described (along with isogonal acoptic polyhedra of higher genera) in [8]. According to a private communication from Prof. J. M. Wills, some of these toroidal prismatoids were described earlier, by U. Brehm at a meeting in Oberwolfach in 1977; however, the only published account of this presentation [1, p. 438] contains no specifics, and does not mention isogonality.

Boolean sums of suitable prisms with prismatoids of type $P(t_0, t_1, \ldots, t_k; n)$ or $P(t_0, t_1, \ldots t_k, t^*; n)$ can yield aploic basis-free isogonal prismatoids with vertex-symbol (4^{2h}), where $h \geq 6$. Other combinations using the same technique are possible as well; for example, from antiprisms and prismatoids of type $A(t_0, t_1, \ldots, t_k; n)$ or $A(t_0, t_1, \ldots, t_k, t^*; n)$, aploic basis-free isogonal prismatoids with vertex-symbols $(3.3.3.4^{2h})$, where $h \geq 4$, and many other polyhedra can be obtained.

(iii) Another remarkable family of basis-free isogonal prismatoids are the *crown polyhedra*; they were first described by Edmund Hess (see references in [7]) under the name "stephanoids" (from the Greek word for "crown"). Like the sphenoids, all crown polyhedra are isohedral as well (thus they are noble). The crown polyhedra are of two kinds, which can be called the prismatic and the antiprismatic, each kind depending on three positive integers as parameters; for appropriate values of these parameters,

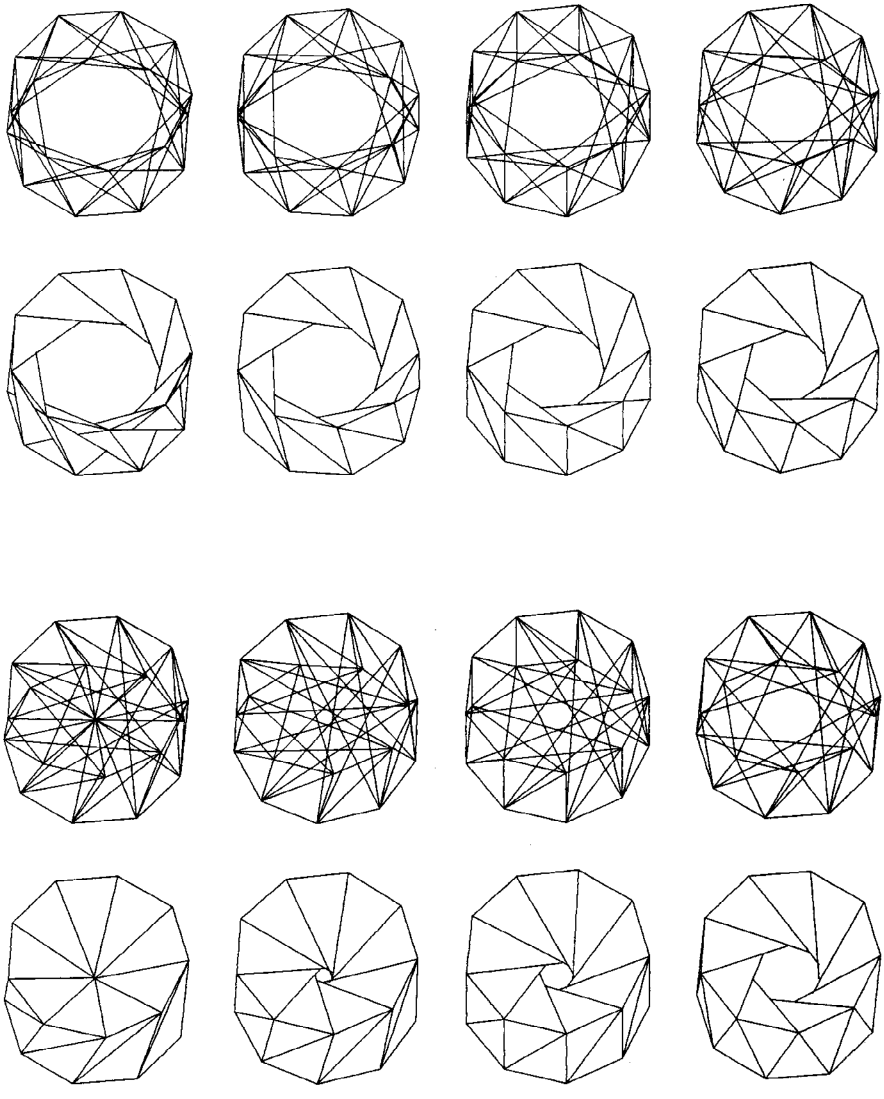
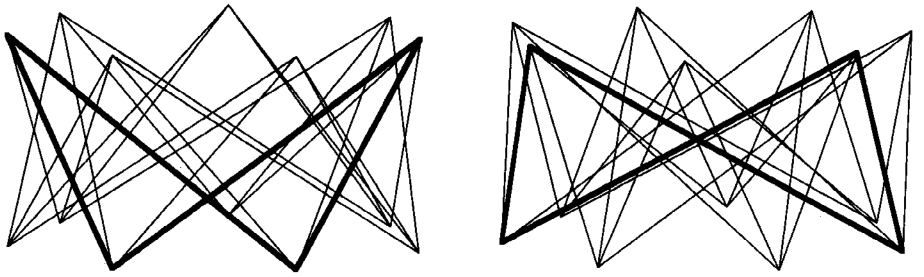


Fig. 37. A continuous family (going from left to right in the upper two rows, and from right to left in the two bottom rows) of toroidal isogonal prismatoids, obtained from pairs of nine-sided antiprisms. Both skeletal and surficial forms of each polyhedron are shown. All these polyhedra have combinatorial type (3.3.3.3.3.3). Except for the starting and the ending members of the family, all polyhedra are acoptic. At two stages the polyhedra contain pairs of coplanar triangular faces, which can be replaced by rectangles; the resulting polyhedra have combinatorial type (3.3.3.4.4) and can also be obtained directly from suitable prisms and antiprisms.



(a) (b)

Fig. 38. Two examples of crown polyhedra. The one in (a) is prismatic, while the one in (b) is antiprismatic. Crown polyhedra are not only isogonal but isohedral as well. One polygon in each polyhedron is emphasized.

the crown polyhedra are aploic. In all cases, the faces are congruent self-intersecting quadrangles. The two kinds are illustrated in Fig. 38 by one aploic representative each. For more details see Section 6 of [7].

6. Comments and Open Questions

(i) This paper developed from the observation that the presentation of the classification of isogonal prisms and antiprisms in [2] is confused and incomplete. Since there appears to be no other treatment of this topic, the author decided to write up his observations in what became Section 3 of this note. However, further investigation showed that Brückner was not only deficient in the treatment of isogonal prisms and antiprisms, but that he missed completely the huge collection of isogonal prismatoids discussed in Sections 4 and 5 above. (The only exception to this is his mention of crown polyhedra ("stephanoids") which, curiously enough, he did not discuss in the presentation of isogonal polyhedra.) In the beginning we believed that Brückner (unconsciously?) wished to avoid including in the discussion of isogonal polyhedra those that have self-intersecting quadrangles as faces, or polyhedra that do not have isohedral polar polyhedra; but the existence of polyhedra of type $P(t_0, t_1, \ldots, t_k; n)$, which he also failed to mention, disproves this idea. Now we are inclined to think that he simply did not look for any isohedra except those that are isomorphic to uniform polyhedra. Why he would have thought this appropriate (in particular, without mentioning it), and especially in view of the fact that he was aware of the existence of various noble polyhedra discovered by Hess, is really mystifying. However, on the other hand, it is equally hard to understand that during the almost full century since the publication of Brückner's book no attempt was made to rectify his omission—in fact, to the best of the author's knowledge, no mention of the shortcomings of his enumeration of isogonal polyhedra made its way into print! Naturally, since his enumeration of the isogonal prismatoids was so faulty, one has to wonder whether his enumeration of the other isogonal polyhedra is complete. Since Brückner in this context again deferred the discussion of some noble polyhedra to a later section, and since his list of noble polyhedra is incomplete, a negative answer is obvious. A thorough investigation of these questions would appear to be long overdue.

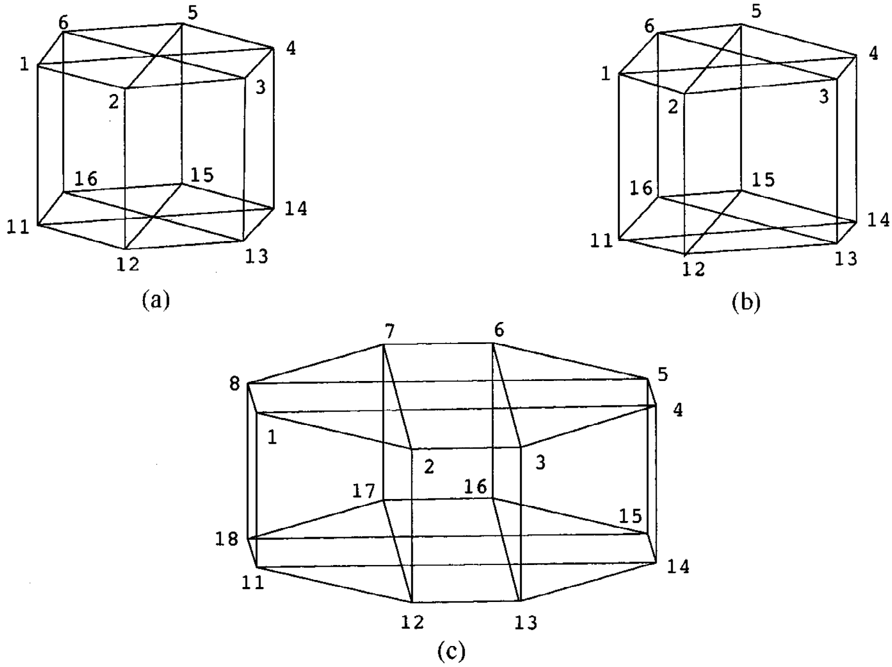


Fig. 39. Examples of nonaploic prismatoids obtained by replacements of the bases of hexagonal or octagonal prisms. In the illustrations the mantle faces are the rectangles 1,2,12,11,1, etc. The upper basis in (a) is formed by the quadrangles 1,2,5,4,1, 3,4,1,6,3, and 5,6,3,2,5; by the hexagons 1,4,3,6,5,2,1 and 2,5,4,1,6,3,2 in (b), and by the quadrangles 1,4,5,8,1 and 3,6,7,2,3 and the octagon 1,2,7,8,5,6,3,4,1 in (c). The prismatoid (a) is not orientable, while those in (b) and (c) are orientable; the polygons in the bases in (a) are congruent, while in (b) the two polygons in each basis have distinct shapes and in (c) they are even of different numbers of sides.

(ii) Nonaploic isogonal prismatoids arise not only as limiting cases of aploic families, but in many other ways as well. One general method, which is essentially again Boolean addition, is to replace one polygon P by an edge-sharing family of polygons that have free edges coinciding with those of P, and that have, as a family, the same symmetry as P. This is illustrated by the examples in Fig. 39. Another application of the same idea is the observation that given a rectangle and the two self-intersecting quadrangles which have the same vertices, any one of these three polygons can be replaced by the family consisting of the other two, or can be used to replace that family. For example, if the polyhedron in Fig. 17(c) is combined in the manner discussed in (ii) of Section 5 with an octagonal prism (with the deletion of the bases of both), the resulting polyhedron is not aploic since four faces of the original polyhedron have the same vertices as four of the eight mantle faces of the prism. However, if each such pair is replaced by the other self-intersecting quadrangle with the same vertices, the aploic isogonal prismatoid shown in Fig. 40 is obtained; this polyhedron has vertex-symbol (4.4.4.4), and it is orientable with genus 0. Many other such examples can be formed, resulting in either aploic or nonaploic polyhedra.

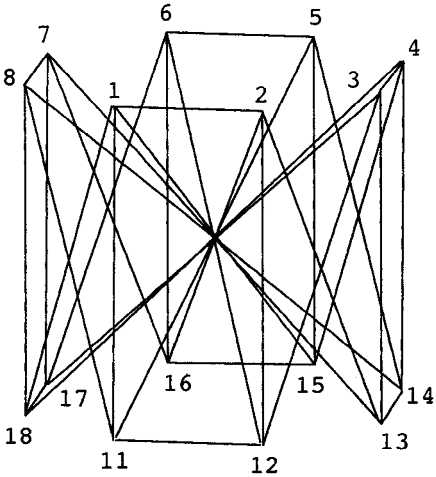


Fig. 40. An aploic isogonal prismatoid with vertex-symbol (4.4.4.4), derived from the polyhedron in Fig. 17(c) by a combination of the procedures described in (ii) of Sections 5 and 6. It is orientable, and of genus 0. The faces of the polyhedron are: 1,2,16,15,1; 3,17,18,4,3; 5,6,12,11,5; 7,13,14,8,7; 1,11,12,2,1; 3,4,14,13,3; 5,15,16,6,5; 7,8,18,17,7; 1,15,4,18,1; 3,12,6,17,3; 5,11,8,14,5; 7,16,2,13,7; 1,18,8,11,1; 3,13,2,12,3; 5,14,4,15,4; 7,17,6,16,7.

(iii) A complete determination of isogonal prismatoids with vertex-symbols $(4.4.4.4.n)$, $n \geq 5$, would be desirable, although probably quite hard. On the other hand, even for small values of n there are other interesting questions that may be pursued. For example, it is easy to verify that the prismatoids $A(2, 3; 8)$ and $A(6, 1; 8)$ shown in Fig. 24 are isomorphic (have the same underlying abstract polyhedron). The same relation exists between $A(2, 3, -1^*; 7)$ and $A(4, -1, 5^*; 7)$, as well as between $P(1, 2; 10)$ and $P(3, 1; 10)$. This is clearly a consequence of some relations in the modular arithmetic; however, the general behavior of the various prismatoids with respect to isomorphism has not been clarified so far. For example, there is a bijection between the vertices of $A(2, 1, 3^*; 7)$ and $A(4, 1, 1^*; 7)$ which maps faces to faces, but is not an isomorphism. Also open are questions regarding the character of the continuous families that can be derived from the various P- and A-prismatoids.

References

1. U. Brehm and W. Kühnel, Smooth approximation of polyhedral surfaces regarding curvatures. *Geom. Dedicata* **12** (1982), 435–461.
2. M. Brückner, *Vielecke und Vielflache*. Teubner, Leipzig, 1900.
3. H. S. M. Coxeter, M. S. Longuet-Higgins, and J. C. P. Miller, Uniform polyhedra. *Philos. Trans. Roy. Soc. London Ser. A* **246** (1953/54), 401–450.
4. H. S. M. Coxeter and W. O. J. Moser, *Generators and Relations for Discrete Groups*, 4th edn. Springer-Verlag, Berlin, 1980.
5. B. Grünbaum, Regular polyhedra. *Companion Encyclopedia of the History and Philosophy of the Mathematical Sciences*, Vol. 2, I. Grattan-Guinness, ed. Routledge, London, 1994, pp. 866–876.
6. B. Grünbaum, Metamorphoses of polygons. *Proc. Strens Memorial Conference on Recreational Mathemat-*

ics and Its History, R. K. Guy and R. E. Woodrow, eds. Mathematical Association of America, Washington, DC, 1994, pp. 35–48.
7. B. Grünbaum, Polyhedra with hollow faces. *Proc. NATO–ASI Conference on Polytopes: Abstract, Convex, and Computational*, Toronto, 1993, T. Bisztriczky, P. McMullen, R. Schneider, and A. Ivic' Weiss, eds. Kluwer, Dordrecht, 1994, pp. 43–70.
8. B. Grünbaum and G. C. Shephard, Polyhedra with transitivity properties. *Math. Rep. Acad. Sci. Canada* **6** (1984), 61–66.
9. Z. Har'El, Uniform solutions for uniform polyhedra. *Geom. Dedicata* **47** (1993), 57–110.
10. E. Hess, Ueber gleicheckige und gleichkantige Polygone. *Schriften Ges. Beförd. gesammten Naturwiss. Marburg*, **10** (1874), 611–743 + plates.
11. A. L. F. Meister, Generalia de genesi figurarum planarum et inde pendentibus earum affectionibus. *Novi Comm. Soc. Reg. Scient. Gotting.* **1** (1769/70), 144–180 + plates.
12. S. A. Robertson, *Polytopes and Symmetry*. London Mathematical Society Lecture Notes Series, Vol. 90. Cambridge University Press, Cambridge, 1984.
13. S. A. Robertson and S. Carter, On the Platonic and Archimedean solids. *J. London Math. Soc.* (2) **2** (1970), 125–132.
14. E. Schönhardt, Über die Zerlegung von Dreieckspolyedern in Tetraeder. *Math. Ann.* **98** (1927), 309–312.
15. J. Skilling, The complete set of uniform polyhedra. *Philos. Trans. Roy. Soc. London Ser. A* **278** (1975), 111–135.
16. S. P. Sopov, Proof of the completeness of the enumeration of uniform polyhedra. *Ukrain. Geom. Sb.* **8** (1970), 139–156.
17. E. Steinitz, Polyeder und Raumeinteilungen. *Encykl. math. Wiss.* (*Geom*) **3** (3AB12), (1922), pp. 1–139.
18. *Webster's Third New International Dictionary of the English Language*. Encyclopædia Britannica, Chicago, 1966.
19. C. Wiener, *Über Vielecke und Vielflache*. Teubner, Leipzig, 1864.

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