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Ceva, Menelaus, and Selftransversality

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Abstract. The purpose of this paper is to state and prove a theorem (the CMS Theorem) which generalizes the familiar Ceva's Theorem and Menelaus' Theorem of elementary Euclidean geometry. The theorem concerns n-acrons (generalizations of n-gons) in affine space of any number of dimensions and makes assertions about circular products of ratios of lengths, areas, volumes, etc. In particular it contains, as special cases, many results in this area proved by earlier authors.

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1. Introduction

The theorems of Ceva and Menelaus are well-known results in the elementary geometry of triangles. Since the early 19th century they have been generalized to polygons with more sides and in various other directions (for example, by Carnot [4] and Poncelet [20]; see [12] for more details and the history of these results). Variants of these theorems for pentagons are illustrated in Fig. 1, together with a related result which we call selftransversality. This seems to have been first formulated in [12]. Each asserts that, in a suitable n-gon, the circular product of n ratios of the oriented lengths of certain line segments has a fixed value $+1$ or -1; the segments in question are determined either by a given line (for Menelaus), by a given point (for Ceva), or by the polygon itself (for the selftransversality result). For the formulation of these and related theorems see, for example, Carnot [4, p. 295], Poncelet [20 p. xix], Gonzalez [11], Shklyarskii *et al.* [22, pp. 48, 318–319], Schröder [21, p. 113], Eves [7, pp. 63, 64], Grünbaum and Shephard [12]; the proofs of the theorems of Ceva and Menelaus in the case of triangles can also be found in many other books. The precise statements of the cases illustrated in Figure 1 are given in the caption to the figure. As noted by Carnot, Poncelet and others, Menelaus' theorem can be extended to n-gons in Euclidean d-space. Other authors discussed various generalizations of the theorems of Ceva and Menelaus to $(d+1)$-gons in Euclidean d-space, or in spherical or hyperbolic d-spaces; see, for example, Nádeník [19], Molnár [17], [18], Gluskov [9], [10], Boldescu [1],

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Budinsky' and Nádeník [3], Budinsky' [2], Klein [13], Fearnley-Sander [8], Mao [15], Landy [14], Erdniev and Mancaev [6], Masal'cev [16].

The purpose of this paper is to formulate and prove a theorem which we call the CMS theorem, where CMS stands for Ceva–Menelaus-Selftransversality. The CMS theorem contains as special cases many of the results of the papers mentioned above. Our theorem makes assertions not only about the ratios of lengths of line segments, but also about ratios of areas and volumes of triangles and simplices defined by the vertices of the n-gon and by transversals of appropriate dimension. The method of proof is an extension of the 'area principle' used in [12]. We begin, in Section 2, with the necessary definitions and notation. Section 3 is devoted to a statement the theorem, and to a diagrammatic method for the explicit determination of the cases to which the theorem is applicable. In Section 4 we give a proof of the theorem, and conclude, in Section 5, with a number of examples and general comments.

2. Definitions and Notation

Throughout this note, we work in affine space $\mathbb{A}^d$ of $d \geq 2$ dimensions. Let $V_1, \ldots, V_n$ be a set of $n \geq 3$ points in general position, that is, such that for $1 \leq m \leq \min\{d, n-1\}$ every subset consisting of $m+1$ of the points is affinely independent. By the polyacron (or n-acron) $P = [V_1, \ldots, V_n]$ we mean the points $V_1, \ldots, V_n$ (the vertices of P) considered as a cyclic sequence, together with the sides and diagonals of P of various dimensions. For an integer f such that $1 \leq f \leq d$, and $f+1$ integers $i_0, i_1, \ldots, i_f$, (all different modulo n) an f-diagonal of P of type $(i_0, i_1, \ldots, i_f)$ is the affine f-flat $\mathrm{aff}(V_{i_0}, \ldots, V_{i_f})$ spanned by the set $V_{i_0}, \ldots, V_{i_f}$ of vertices of P. In particular, if the given vertices are consecutive, then the f-diagonal is also called an f-side of P. In the case $f = 1$ we shall often write 'side' or 'diagonal' instead of '1-side' or '1-diagonal'. Here, and throughout, the subscript i runs from 1 to n, and all subscripts j are reduced modulo n so that they satisfy $1 \leq j \leq n$. A polyacron is regarded as unchanged by any cyclic permutation of the vertices and so is oriented in the sense that the orientation is changed by reversing the order in which its vertices are listed. Polyacrons are special kinds of families of affine flats; in the context of the results discussed here, they seem to be a natural generalization of polygons to $d > 2$ dimensions.

The term 'polyacron' is not new; it was coined by the nineteenth century mathematician T. P. Kirkman (famous for his 'school girls problem' and deserving to be better known for his many other contributions to mathematics) and it appears in the *Oxford English Dictionary*.

Given two r-simplices $[U_0, \ldots, U_r]$ and $[V_0, \ldots, V_r]$ contained in the same r-flat, we use the symbol $[U_0 \ldots U_r / V_0 \ldots V_r]$ to denote the quotient of the absolute values of the r-contents of the two simplices, prefixed by a $+$ or a $-$ sign according to whether the simplices are oriented in the same way or oppositely. This symbol is a very useful affine invariant. In particular, $[AB/CD]$ is the ratio of the lengths of

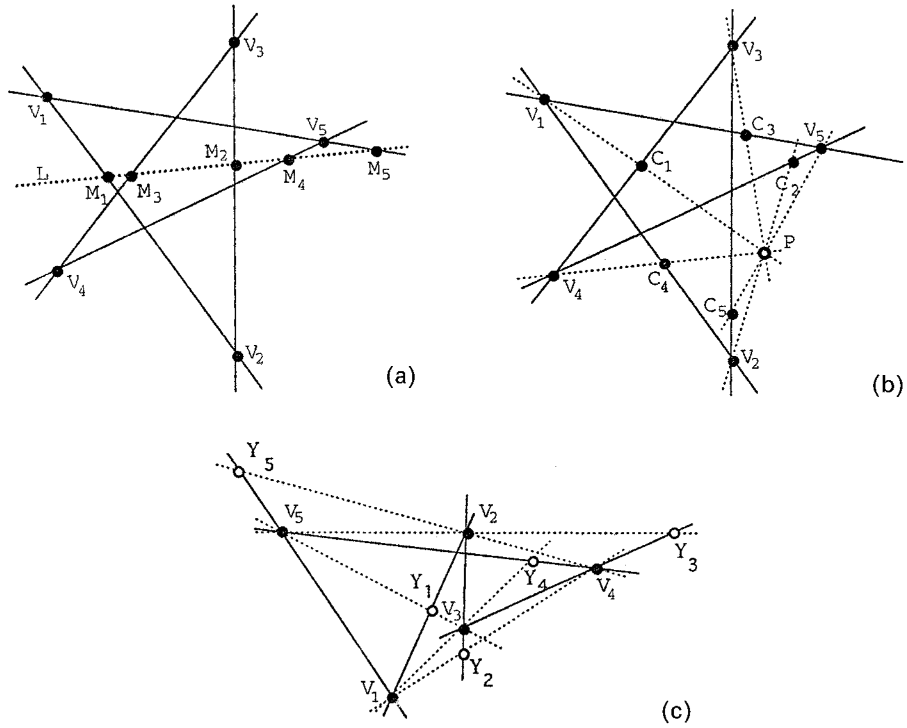

Figure 1. Illustrations, for a pentagon P (more precisely, as explained in the text, for a 5-acron) with vertices V_1, V_2, V_3, V_4, V_5 of one variant each of the classical theorems of Menelaus and Ceva, and of the recent selftransversality result. All points and lines are assumed to be in suitably general positions so that the ratios involved are defined; $\ell(B, C)$ denotes the oriented length of the segment $[B, C]$.

(a) Menelaus' theorem: If any line L intersects the side A_iA_{i+1} of P in the point M_i, then

$$\prod_{i=1}^{5} \frac{\ell(V_iM_i)}{\ell(V_{i+1}M_i)} = 1.$$

(b) Ceva's theorem: If P is any point and C_i is the intersection point of the line PV_i with the line $V_{i+2}V_{i+3}$, then

$$\prod_{i=1}^{5} \frac{\ell(V_{i+2}C_i)}{\ell(V_{i+3}C_i)} = -1.$$

(c) Selftransversality: If Y_i is the intersection point of the lines V_iV_{i+1} and $V_{i+2}V_{i+4}$, then

$$\prod_{i=1}^{5} \frac{\ell(V_iY_i)}{\ell(V_{i+1}Y_i)} = 1.$$

In terms of Table 1 (the meaning of which is explained below), (a) corresponds to entry (5), (b) to entry (3), and (c) to entry (1). Moreover, (c) can be interpreted also as illustrating entry (2), according to which

$$\prod_{i=1}^{5} \frac{\ell(V_iY_{i+3})}{\ell(V_{i+2}Y_{i+3})} = 1.$$

the line segments $[A, B]$ and $[C, D]$, with a $+$ or $-$ sign to indicate whether these line segments (or vectors) have the same, or directly opposite directions.

The idea of our theorem is simple. We start from an n-acron P and a (fixed) q-flat Q in $\mathbb{A}^d$. Assume these and all other objects under discussion are in general position. For each r-diagonal R_i of P of a chosen type we specify, in a prescribed manner, an s-diagonal S_i where $s = d-q-r-1$. Then the *transversal* aff$(Q \cup R_i)$, specified by the *anchor* Q and the diagonal R_i, meets S_i in a single point. This point, together with vertices of P, is used to specify two s-dimensional simplices. The theorem asserts that (under certain specified conditions), the circular product of the ratios of the s-dimensional contents of these simplices, has a constant value $+1$ or -1. The precise statement of the theorem is given in the next section.

The traditional version of Ceva's theorem for a triangle (actually a 3-acron) corresponds to the case $d = 2$, $n = 3$, $q = 0$, $r = 0$, $s = 1$. Here the anchor Q is a fixed point, R_i runs through the vertices of P, and for each R_i, S_i is the 1-side of P which does not contain R_i. The theorem is concerned with the ratios of the lengths of line segments defined by vertices of P, and the intersection points of the transversals defined by Q and R_i with the sides S_i. Menelaus' theorem corresponds to $d = 2$, $n = 3$, $q = 1$, $r = -1$, $s = 1$ so the anchor Q is a fixed line and each R_i is the empty set. The theorem concerns the ratios of the lengths of line segments defined by the intersection points of Q with the sides S_i of P. For the pentagons in Figure 1(a) and (b), the parameters take the same values except that $n = 5$. Figure 1(c) shows the case $d = 2$, $n = 5$, $q = -1$, $r = s = 1$, so the anchor Q is the empty set. The theorem makes an assertion about the ratios of the lengths of the line segments defined by vertices of P and the intersection points Y_i of diagonals $R_i =$ aff(V_{i+2}, V_{i+4}) with sides $S_i =$ aff(V_i, V_{i+1}) of P.

3. Formulation of the Result

Let $P = [V_1, \dots, V_n]$ be a polyacron in $\mathbb{A}^d$, where $1 \leq d \leq n-1$. Let q, r, s be integers such that $-1 \leq q \leq d-1, -1 \leq r \leq d-1, 1 \leq s \leq \min\{d, n-r-2\}$ and $q + r + s + 1 = d$. Further let $A = (a_0, \dots, a_s)$ and $B = (b_0, \dots, b_r)$ be sequences of integers such that all the elements of $A \cup B$ are distinct modulo n. Let S_i denote the s-diagonal aff$(V_{i+a_0}, V_{i+a_1}, \dots, V_{i+a_s})$ and R_i the r-diagonal aff$(V_{i+b_0}, V_{i+b_1}, \dots, V_{i+b_r})$. Let Q be a q-flat such that, for each $i = 1, \dots, n$, the $(q+r+1)$-flat aff$(Q \cup R_i)$ spanned by Q and R_i meets the s-diagonal S_i in a single point Y_i which (by the assumed general position of the vertices) must be distinct from $V_{i+a_0}, V_{i+a_1}, \dots, V_{i+a_s}$. (If $q = -1$ or $r = -1$ then the corresponding flat Q or R is interpreted as being empty. Note that $q = r = -1$ is excluded by the assumptions on the parameters.) Now define

$$\rho(P; A, B, Q) = \prod_{i=1}^{n} \left[\frac{V_{i+a_0} V_{i+a_1} \dots V_{i+a_{s-1}} Y_i}{V_{i+a_1} V_{i+a_2} \dots V_{i+a_s} Y_i} \right]. \tag{1}$$

This is the circular product of ratios mentioned at the end of the previous section. Conditions under which the product (1) takes a fixed value $+1$ or -1 are given by the following result:

THE CMS THEOREM. *Given an n-acron P in $\mathbb{A}^d$, sequences of integers A, B, and a flat Q as specified above, then $\rho(P; A, B, Q)$ is a constant independent of P if and only if there exists an integer k such that, modulo n, the sequence*

$$(a_0 + k, a_1 + k, \ldots, a_{s-1} + k, b_0 + k, b_1 + k, \ldots, b_r + k)$$

is a permutation π of the sequence

$$(a_1, a_2, \ldots, a_s, b_0, b_1, \ldots, b_r).$$

The value of the constant is given by $\rho(P; A, B, Q) = (e(\pi))^n$, where $e(\pi) = 1$ if π is an even permutation and $e(\pi) = -1$ if π is an odd permutation.

If $s = 1$, the right side of (1) becomes the product of n quotients of lengths of line segments; hence our theorem is seen to include, as special cases, both Ceva's and Menelaus' theorems for n-gons, as well as the selftransversality result from [12].

Before giving a proof of the theorem we shall give a geometric interpretation of the condition stated in the theorem. To do this we use what we shall call CMS-diagrams. Start with a regular plane n-gon N, number the vertices of N consecutively in a positive direction, and mark those that correspond to the integers a_i and b_i. The vertices corresponding to a_0 and a_s are marked in such a way that they can be distinguished from those corresponding to $a_1, \ldots, a_{s-1}$ and $b_0, \ldots, b_r$. See Figure 2(a) for an example with $n = 9$, $A = (0, 1, 2, 3)$ and $B = (4, 6, 8)$; the markings are explained in the caption to the figure. The condition of the theorem holds if and only if the two *sets* of vertices, corresponding to the integers $a_0, \ldots, a_{s-1}, b_0, \ldots, b_r$ and to $a_1, \ldots, a_s, b_0, \ldots, b_r$ are *directly congruent* (as unmarked sets); that is, one can be made to coincide with the other by a suitable rotation about the centre of the polygon N. For the example of Figure 2, part (b) shows the two sets of vertices, and it will be observed that the second set can be obtained from the first by a rotation through angle $4\pi/9$. Hence, in this case, the condition of the theorem holds.

Although the theorem holds for all values of the parameters specified at the beginning of this section (with appropriate choices of A and B), if $s = d$ the assertion becomes trivial in the following sense: the terms in the numerator and denominator of (1) are identical (apart from a possible permutation of the vertices defining the simplices), so complete cancellation can be carried out, yielding 1 or -1.

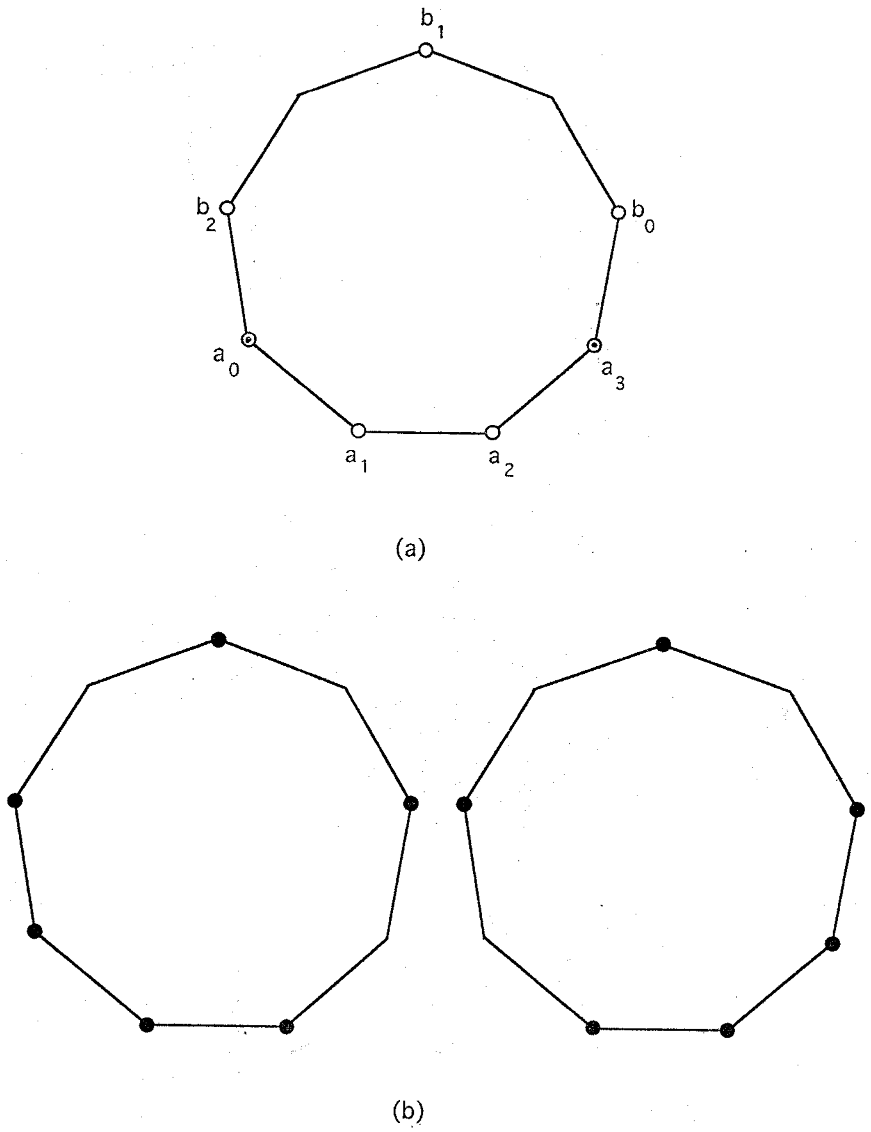

Figure 2. (a) A CMS-diagram for $n = 9$, with $A = (0, 1, 2, 3)$ and $B = (4, 6, 8)$. Points corresponding to a_0, a_3 are marked $\odot$, those corresponding to a_1, a_2 and to b_0, b_1, b_2 are marked $\bigcirc$. Here $s = 3$ and $r = 2$. Because the two sets of points marked in (b) are directly congruent, the sets A and B satisfy the condition of the CMS theorem.

This is illustrated in Figure 4. In (a) we have $n = 5$, $d = 2$, $q = -1$, $r = 0$, $s = 2$, $A = (1, 2, 4)$ and $B = (3)$. Then the product (1) involves quotients of the areas of triangles. Explicitly this is

$$\left[\frac{V_1V_2V_3}{V_2V_4V_3}\right] \cdot \left[\frac{V_2V_3V_4}{V_3V_5V_4}\right] \cdot \left[\frac{V_3V_4V_5}{V_4V_1V_5}\right] \cdot \left[\frac{V_4V_5V_1}{V_5V_2V_1}\right] \cdot \left[\frac{V_5V_1V_2}{V_1V_3V_2}\right]$$

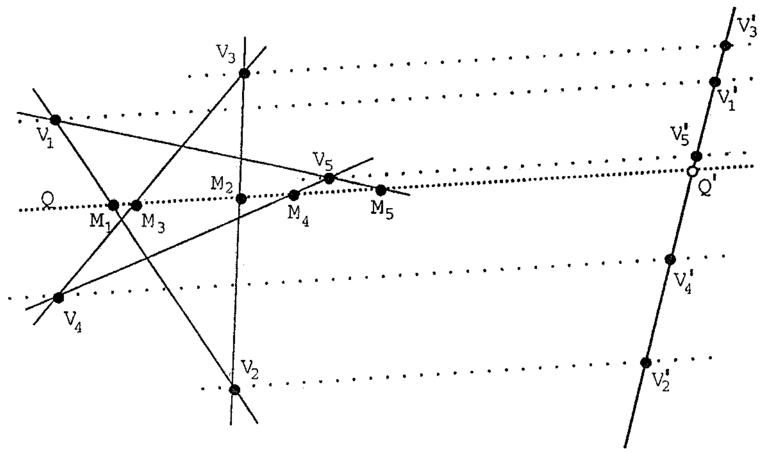

Figure 3. By projection along Q, the theorem of Menelaus for dimension $d = 2$, $n = 5$ reduces to the theorem of Ceva for dimension $d' = 1$:

$$\prod_{i=1}^{5}\left[\frac{V_i M_i}{V_{i+1} M_i}\right] = \prod_{i=1}^{5}\left[\frac{V_i' Q'}{V_{i+1}' Q'}\right] = 1,$$

and the latter is trivial in the sense described in Section 3.

and the cancellations become evident. On the other hand, if all the parameters (as well as A and B) take the same values, except that $d = 3$ and $q = 0$, we arrive at the situation shown in Figure 4(b). Here the anchor Q is a fixed point and the line $\mathrm{aff}(Q, V_3)$ meets the plane $\mathrm{aff}(V_1, V_2, V_4)$ in Y_5. The other points Y_i are determined by cyclically changing all the subscripts (mod 5). The theorem makes an assertion about the product $\rho(P; A, B, Q)$ of 5 terms of the form

$$[(V_{i+1}V_{i+2}Y_i + 3)/(V_{i+2}V_{i+4}Y_{i+3})] \quad (i = 0, 1, 2, 3, 4)$$

and clearly this result is far from trivial. In a similar way we obtain a non-trivial result (not illustrated) with the same values of the parameters (as well as A and B) except that $d = 4, q = 1$. Another trivial example, corresponding to $d = 2, q = 0$, $r = -1$, $s = 2$, $A = (1, 2, 3)$ and $B = \emptyset$, is illustrated in Figure 4(c).

In Table I we list all the non-trivial assertions of the theorem in the case $n = 5$. For each of the possible values of the parameters and of the dimension (in this case $d = 2, 3$ or 4) we can readily determine the permissible sets A and B using the CMS-diagrams. It will be seen that, in the case of pentacrons, the CMS theorem makes 34 non-trivial assertions, many of which appear to be new.

Table I. The essentially different possibilities of the parameters d, q, r, s and sets A and B for which the CMS theorem is valid and non-trivial when $n = 5$. Since n is odd we have $e(\pi) = \rho(P; A, B, Q)$.

List number	d	q	r	s	A	B	$e(\pi) = \rho(P; A, B, Q)$	Remarks
1	2	−1	1	1	(1, 2)	(3, 5)	1	Figue 1(c)
2	2	−1	1	1	(1, 3)	(4, 5)	1	Figure 1(c)
3	2	0	0	1	(1, 2)	(4)	−1	Ceva's theorem Figure 1(b)
4	2	0	0	1	(1, 3)	(2)	−1	
5	2	1	−1	1	(1, 2)	Ø	1	Menelaus' theorem Figures 1(a), 3
6	2	1	−1	1	(1, 3)	Ø	1	
7	3	−1	1	2	(1, 2, 3)	(4, 5)	1	
8	3	−1	1	2	(1, 3, 2)	(4, 5)	1	
9	3	−1	1	2	(1, 2, 4)	(3, 5)	1	
10	3	−1	1	2	(1, 4, 2)	(3, 5)	1	
11	3	−1	2	1	(1, 2)	(3, 4, 5)	−1	
12	3	−1	2	1	(1, 3)	(2, 4, 5)	−1	
13	3	0	0	2	(1, 2, 4)	(3)	−1	Figure 4(b)
14	3	0	0	2	(1, 3, 2)	(5)	−1	
15	3	0	1	1	(1, 2)	(3, 5)	1	
16	3	0	1	1	(1, 3)	(4, 5)	1	
17	3	1	−1	2	(1, 2, 3)	Ø	1	
18	3	1	−1	2	(1, 4, 2)	Ø	1	
19	3	1	0	1	(1, 2)	(4)	−1	
20	3	1	0	1	(1, 3)	(2)	−1	
21	3	2	−1	1	(1, 2)	Ø	1	
22	3	2	−1	1	(1, 3)	Ø	1	
23	4	0	0	3	(1, 2, 3, 4)	(5)	−1	
24	4	0	0	3	(1, 2, 4, 3)	(5)	−1	
25	4	0	0	3	(1, 2, 4, 5)	(3)	−1	
26	4	0	1	2	(1, 2, 3)	(4, 5)	1	
27	4	0	1	2	(1, 3, 2)	(4, 5)	1	
28	4	0	1	2	(1, 2, 4)	(4, 5)	1	
29	4	0	1	2	(1, 4, 2)	(3, 5)	1	
30	4	0	2	1	(1, 2)	(4)	−1	
31	4	0	2	1	(1, 3)	(2)	1	
32	4	1	−1	3	(1, 2, 3, 4)	Ø	1	
33	4	1	−1	3	(1, 3, 5, 2)	Ø	1	
34	4	1	0	2	(1, 2, 4)	(3)	−1	
35	4	1	0	2	(1, 3, 2)	(5)	−1	
36	4	1	1	1	(1, 2)	(3, 5)	1	
37	4	1	1	1	(1, 3)	(4, 5)	1	
38	4	2	−1	2	(1, 2, 3)	Ø	1	
39	4	2	−1	2	(1, 4, 2)	Ø	1	
40	4	2	0	1	(1, 2)	(4)	−1	
41	4	2	0	1	(1, 3)	(2)	−1	
42	4	3	−1	1	(1, 2)	Ø	1	
43	4	3	−1	1	(1, 3)	Ø	1	

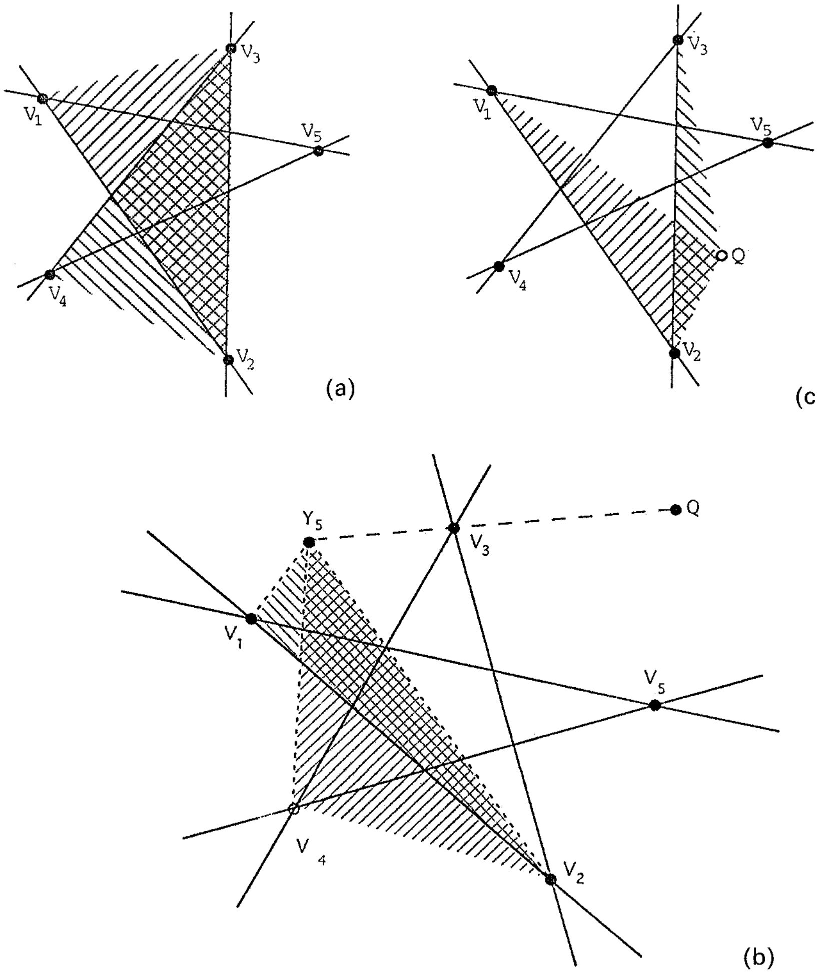

Figure 4. (a) The case $n = 5, d = 2, q = -1, r = 0, s = 2$, $A = (1, 2, 4)$ and $B = (3)$. One of the ratios used in the compilation of $\rho(P; A, B, Q)$ is $[V_1V_2V_3/V_2V_4V_3]$ which is the ratio of the areas of the two shaded triangles. The other four ratios are obtained by cyclic changes of the subscripts (mod 5). Since the areas of the triangles cancel, were refer to this case as trivial. (b) A non-trivial result in three dimensions (entry (13) in Table 1) in which the parameters (as well as A and B) take the same values as in (a) except that $q = 1, d = 3$. The line aff(Q, V_3) meets the plane aff(V_1, V_2, V_4) in the point Y_5. One of the ratios used in the compilation of $\rho(P; A, B, Q)$ is $[V_1V_2Y_5/V_2V_4Y_5]$ which is the ratio of the areas of the two shaded triangles. The other four factors are obtained by cyclic changes of the subscripts (mod 5). In this case there is no trivial cancellation. (c) If this figure is interpreted in $d = 2$ dimensions, with Q as a fixed point, the result is trivial.

4. Proof of the Theorem

Choose $q + 1$ points $X_0, \ldots, X_q$ in Q in such a way that the $n + q + 1$ points $V_1, \ldots, V_n, X_0, \ldots, X_q$ are in general position. Let the position vectors of points in $\mathbb{A}^d$ be represented by the corresponding lower case letters, so that a point U_i has position vector u_i with components $(u_{i1}, u_{i2}, \ldots, u_{id})$. In terms of determinants, the condition for $d+1$ points $U_0, U_1, \ldots, U_d$ to lie in a hyperplane is

$$D(u_0, \ldots, u_d) = \det \begin{bmatrix} u_{01} & u_{11} & \cdot & \cdot & u_{d,1} \\ u_{01} & u_{12} & \cdot & \cdot & u_{d,2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ u_{0d} & u_{1d} & \cdot & \cdot & u_{d,d} \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} = 0.$$

For $i = 1, \ldots, n$ let H_i be the hyperplane spanned by the $(s-1)+(r+1)+(q+1) = d$ points $V_{i+a_1}, \ldots, V_{i+a_{s-1}}, V_{i+b_0}, \ldots, V_{i+b_r}, X_0, \ldots, X_q$, and suppose H_i meets the 1-diagonal $\mathrm{aff}(V_{i+a_0}V_{i+a_s})$ in the point W_i. Because of the assumed generality of position of the points, W_i will be uniquely determined and distinct from V_{i+a_0} and V_{i+a_s}. Then $w_i = (1-\lambda_i)v_{i+a_0} + \lambda_i v_{i+a_s}$ for some value of λ_i, and

$$\begin{aligned} \frac{\lambda_i}{\lambda_i - 1} &= \left[\frac{V_{i+a_0}W_i}{V_{i+a_s}W_i}\right] \\ &= \left[\frac{V_{i+a_0}V_{i+a_1}\cdots V_{i+a_{s-1}}Y_i}{V_{i+a_s}V_{i+a_1}\cdots V_{i+a_{s-1}}Y_i}\right] \\ &= (-1)^{s-1}\left[\frac{V_{i+a_0}\cdots V_{i+a_{s-1}}Y_i}{V_{i+a_1}\cdots V_{i+a_s}Y_i}\right]. \end{aligned} \tag{2}$$

The second equality holds since the simplices in the numerator and denominator of the last expression have the same base $[V_{i+a_1}, \ldots, V_{i+a_{s-1}}, Y_i]$ and so their signed volumes are proportional to their heights, namely the signed lengths of the line segments $[V_{i+a_0}, W_i]$ and $[V_{i+a_s}, W_i]$. This is the extension of the 'area principle' from [12] mentioned in the Introduction; it could be called the 'volume principle'. (In the case $s = 1$, the second equality of (2) is an identity since $Y_i = W_i$.)

Now, as W_i lies in the hyperplane H_i,

$$\begin{aligned} 0 &= D(w_i, v_{i+a_1}, \ldots, v_{i+a_{s-1}}, v_{i+b_0}, \ldots, v_{i+b_r}, x_0, \ldots, x_q) \\ &= D((1-\lambda_i)v_{i+a_0} + \lambda_i v_{i+a_s}, v_{i+a_1}, \ldots, v_{i+a_{s-1}}, v_{i+b_0}, \ldots, v_{i+b_r}, x_0, \ldots, x_q) \\ &= (1-\lambda_i)D(v_{i+a_0} + v_{i+a_1}, \ldots, v_{i+a_{s-1}}, v_{i+b_0}, \ldots, v_{i+b_r}, x_0, \ldots, x_q) \\ &\quad +\lambda_i D(v_{i+a_s}, v_{i+a_1}, \ldots, v_{i+a_{s-1}}, v_{i+b_0}, \ldots, v_{i+b_r}, x_0, \ldots, x_q) \end{aligned}$$

and solving for $\lambda_i/(\lambda_i - 1)$ we obtain

$$\frac{\lambda_i}{\lambda_i - 1} = \frac{D(v_{i+a_0}, v_{i+a_1}, \ldots, v_{i+a_{s-1}}, v_{i+b_0}, \ldots, v_{i+b_r}, x_0, \ldots, x_q)}{D(v_{i+a_s}, v_{i+a_1}, \ldots, v_{i+a_{s-1}}, v_{i+b_0}, \ldots, v_{i+b_r}, x_0, \ldots, x_q)}$$

$$= (-1)^{s-1}\frac{D(v_{i+a_0}, v_{i+a_1}, \ldots, v_{i+a_{s-1}}, v_{i+b_0}, \ldots, v_{i+b_r}, x_0, \ldots, x_q)}{D(v_{i+a_1}, \ldots, v_{i+a_{s-1}}, v_{i+a_s}, v_{i+b_0}, \ldots, v_{i+b_r}, x_0, \ldots, x_q)}. \quad (3)$$

Taking the product from $i = 1$ to $i = n$, we see that the determinants in the numerator are, up to a permutation of columns, exactly those in the denominator if and only if condition given in the statement of the theorem holds. Moreover, this permutation of columns introduces n times the factor e(π) into the value of the determinant. In all, we have, from (2) and (3),

$$\begin{aligned}\rho(P; A, B, Q) &= \prod_{i=1}^{n}\left[\frac{v_{i+a_0}\cdots v_{i+a_{s-1}}Y_i}{v_{i+a_1}\cdots v_{i+a_s}Y_i}\right] \\ &= \prod_{i=1}^{n}\frac{(-1)^{s-1}\lambda_i}{\lambda_i - 1} = (e(\pi))^n\end{aligned}$$

as claimed.

5. Remarks and Examples

(i) We note that the proof of the CMS theorem in Section 4 could be amplified by showing that the condition of the theorem is not only necessary for the algebraic cancellation of the ratios involved in the product, but that even numerically such cancellation can occur for all polyacrons only if the condition is fulfilled. This is an easy consequence of the fact that if a polynomial (in any given number of variables) has the value zero for all choices of real values of its variables, then it must be identically zero (that is, all its coefficients must equal 0). Of course, it is possible for the product in the theorem to have value 1 (or -1) for a particular choice of the parameters and points even if the condition is not satisfied.

(ii) The parity of the permutation π, which appears in the formulation of our theorem, obviously does not depend on the number of elements in B as long as $B \neq \emptyset$. As a consequence, the cases $q > 0$ of the theorem (which can be interpreted as generalizations of the theorem of Menelaus) are simple consequences of the cases in which $q = 0$ (which correspond to Ceva's theorem). More geometrically, the assertion of the theorem for a given polyacron P with any given d and $q > 0$ can be reduced to the one for a polyacron P' with $d' = d - q$ and $q' = 0$ by projecting the polyacron along Q onto a d'-dimensional flat complementary to Q. An example is shown in Figure 3. Clearly the statement of Menelaus' theorem for the pentacron P follows immediately from that for the projected pentacron P', and as the latter is trivial, we arrive at a simple and elementary proof of Menelaus' theorem when $n = 5$ (and, by obvious extension, to all $n \geq 3$).

(iii) Since it is difficult to illustrate the theorem by intelligible diagrams if $d \geq 3$, we shall describe some of the cases where $d = 3$ and $n = 5$. Entry (12) of Table I shows that if W_i is the intersection of the line aff(V_{i+1}, V_{i+3}) with the plane aff$(V_{i+2}V_{i+4}, V_i)$, then the product (for $1 \leq i \leq 5$) of the ratios of directed

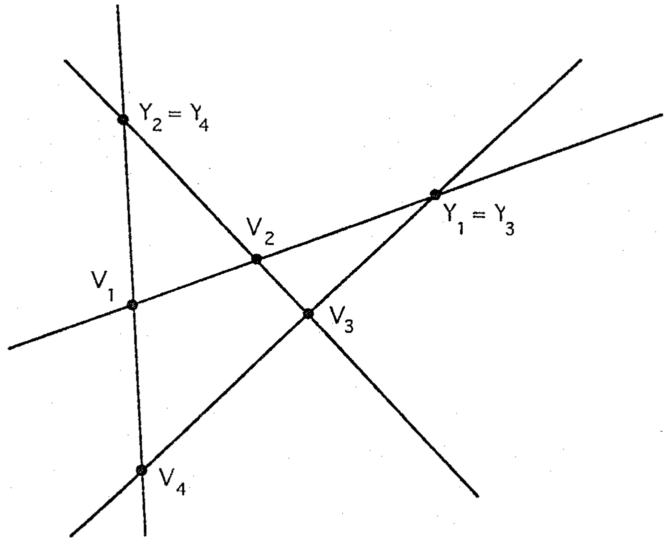

Figure 5. The simplest selftransversality result. Here $d = 2$, $n = 4$, $q = -1$, $r = 1$, $s = 1$, $A = (1, 2)$ and $B = (3, 4)$. In the notation of the diagram,

$$\left[\frac{V_1Y_1}{V_2Y_1}\right] \cdot \left[\frac{V_2Y_2}{V_3Y_2}\right] \cdot \left[\frac{V_3Y_3}{V_4Y_3}\right] \cdot \left[\frac{V_4Y_4}{V_1Y_4}\right] = 1.$$

It is remarkable that such a simple result has not previously appeared in the literature, except, so far as we are aware, in Carnot's book [4] published in 1803.

lengths of $[V_{i+1}, W_i]$ and $[V_{i+3}, W_i]$ equals -1. On the other hand, entry (7) shows that if $W_i = \text{aff}(V_{i+1}, V_{i+2}, V_{i+3}) \cap \text{aff}(V_{i+4}, V_i)$ then the product of the five ratios of oriented areas of the triangles $V_{i+1}V_{i+2}W_i$ and $V_{i+2}V_{i+3}W_i$ is 1. Entry (13) corresponds to the case in which a fixed point Q is given, and the intersection of the line aff$(Q\,V_{i+3})$ with the 2-diagonal aff$(V_{i+1}, V_{i+2}, V_{i+4})$ determines a point W_i. Then $\rho(P; A, B, Q)$ is the product of the ratios of the areas of the oriented triangles $V_{i+1}V_{i+2}W_i$ and $V_{i+2}V_{i+4}W_i$, and is equal to -1.

(iv) In some contexts it is not necessary to assume, in the definition of a polyacron, that its vertices are in general position. Instead, it may be sufficient to require the affine independence only of sets of points (of certain specified sizes) that occur in the statement of a theorem or its proof. It seemed unnecessary to burden the formulation of our theorem with details of these possibilities.

(v) It is rather remarkable that the 'selftransversality' results, corresponding to $Q = \emptyset$ in the CMS theorem, seem not to have been mentioned in the literature – even in the first non-trivial case $d = 2$, $n = 4$, $A = (1, 2)$, $B = (3, 4)$, see Figure 5, or in the next two cases (for $d = 2, n = 5$) illustrated in Figure 1(c). The only exception to this of which we are aware is that the first-mentioned result appears (in different notation) in [4, p. 279, Théorème VI].

(vi) The selftransversality theorem of this paper was originally suggested by empirical results for polyacrons (for small n and d) obtained using a Macintosh IIcx with Mathematica© software. The proofs of these results suggested the general form of the theorem stated above. It seems likely that other theorems about plane n-gons admit analogous generalizations to polyacrons.

(vii) Even if the sets A and B fail to satisfy the condition of the theorem, it may be possible to determine the value of the product $\rho(P; A, B, Q)$ in terms of other circular products. For example, using the 'volume principle' it is possible to prove results of the following kind: Given any n-acron P, with $n \geq 5$ and five integers $\alpha, \beta, \gamma, \delta, \epsilon$ distinct modulo n, then, with $A = (\alpha, \beta, \gamma, \delta)$, $B = (\epsilon)$, $A^* = (\alpha, \beta, \gamma)$, $B^* = (\delta, \epsilon)$, $A^{**} = (\alpha, \beta)$ and $B^{**} = (\gamma, \delta, \epsilon)$, we have

$$\rho(P; A, B, \emptyset) = (-1)^n \rho(P; A^*, B^*, \emptyset) = \rho(P; A^{**}, B^{**}, \emptyset).$$

(viii) The 'volume principle' is not new. Without a special name, it has probably been used by many people, and in many contexts. For example, in [5, p. 131], it is used to prove a theorem on the sum of the ratios of lengths in which a point partitions transversals from the vertices of a tetrahedron to the opposite faces. However, it seems that the method has not been previously applied to the topic of this note, nor has its wider utility been noted.

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