COMMON TRANSVERSALS FOR FAMILIES OF SETS†

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1. Let $\mathscr{K}$ denote a family of subsets of a Euclidean or projective space. $\mathscr{K}$ will be said to have property $\mathscr{T}(k)$, where k is a natural number, [resp. property $\mathscr{T}$] if every k [resp. all] members of $\mathscr{K}$ have a common transversal (*i.e.*, may be intersected by a suitable straight line).

In [**26**] and [**23**] the following theorem was proved (among other related results):

For a family of parallel segments in the plane $\mathscr{T}(3)$ *implies* $\mathscr{T}$.

The determination of other sets of conditions under which $\mathscr{T}(k)$, for some fixed k, implies $\mathscr{T}$, has been the object of numerous investigations. (The following list is believed to be complete: [**3, 6-16, 19-21, 23-29**].)

In the present note we shall prove additional theorems of this nature, generalizing some of the previously established results. Our main tool will be Helly's [**18**] general theorem on intersections of "cells". Some of the results of this note were announced in [**9**].

In §2 we discuss some sets of conditions, sufficient for the existence of common transversals for families of sets in n-dimensional Euclidean space E^n. As an application, we obtain a sharper version of the result in [**13**] dealing with common transversals of "thin families" of spheres in E^n. The case $n = 3$ is investigated in more detail in §3. The two-dimensional case is treated in §4; the main result obtained generalizes some of the theorems of [**21**] and [**24**]. As corollaries, results similar to some of [**16**] are obtained.

2. A compact subset C of E^n will be called a *cell* (in E^n) if C is homotopic (in itself) to a point. A formulation of Helly's theorem [**18**] on intersections of cells, suitable for the present purposes, is:

If $\mathscr{C}$ *is a family of cells in* E^n *such that the intersection of any* $2, 3, \ldots, n$ *members of* $\mathscr{C}$ *is a cell and such that the intersection of any* $n+1$ *members of* $\mathscr{C}$ *is not empty, then the intersection of all the members of* $\mathscr{C}$ *is not empty.*

(The above definition of "cell" is more restrictive than Helly's original one [**18**]; a slightly weaker version of Helly's theorem is given in [**1**].)

In order to apply Helly's theorem to problems dealing with common transversals, we need to introduce some concepts and notations.

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A family $\mathscr{K} = \{K_i,\ 1 \leqslant i \leqslant m\}$ of subsets of E^n shall be called *separated by parallel hyperplanes* (in short, *separated*) if there exists a family $\mathscr{H} = \{H_i,\ 0 \leqslant i \leqslant m\}$ of parallel hyperplanes H_i such that K_i is contained in the open part of E^n bounded by H_{i-1} and H_i, for $1 \leqslant i \leqslant m$, while H_i is between H_{i-1} and H_{i+1}, for $1 \leqslant i \leqslant m-1$.

If $K_i \subset E^n$ belongs to a family $\mathscr{K}$ of separated sets, a subset C_i of a $2n-2$ dimensional Euclidean space E^{2n-2} may be used to represent the set of straight lines intersecting K_i, H_0 and H_m, in the following way. Each such intersecting line L is completely determined by the two points $x_0 = L \cap H_0$ and $x_m = L \cap H_m$, and may therefore be uniquely represented by the point (x_0, x_m) of E^{2n-2}. C_i is the set of all the points corresponding in this way to such intersecting lines.

With this terminology we have

THEOREM 1. *If $\mathscr{K}$ is a separated family of compact, convex subsets of E^n such that the intersection of any 3, 4, ..., $2n-2$ of the sets $C_i \subset E^{2n-2}$ is a cell, then $\mathscr{T}(2n-1)$ implies $\mathscr{T}$.*

Proof. The assertion of the theorem would follow at once from Helly's theorem applied to the sets $C_i \subset E^{2n-2}$ if the sets C_i were cells in E^{2n-2} (which they are not), and if we knew that $C_i \cap C_j$, for $i \neq j$, are cells. Now, this property of $C_i \cap C_j$ follows immediately from a theorem of Brunn [2] according to which the set $\{x_0;\ (x_0, x_m) \in C_i \cap C_j \subset E^{2n-2}\} \subset H_0$ is convex. On the other hand, without changing any of the other arguments, instead of the sets C_i we may consider the cells $C_i^* \subset C_i$, obtained by taking only those points of C_i for which the corresponding lines $L \subset E^n$ form an angle $\geqslant \alpha$ with H_0, for some α small enough to insure that $C_i^* \cap C_j^* = C_i \cap C_j$ for all $i \neq j$. This ends the proof of Theorem 1.

If H is a hyperplane parallel to H_0 and H_m and situated between them, and if $K \subset H$ is a convex set, then the corresponding set C is convex. Therefore we have

COROLLARY 1. *If $\mathscr{K}$ is a family of compact, convex sets in E^n, whose members are contained in distinct parallel hyperplanes, then $\mathscr{T}(2n-1)$ implies $\mathscr{T}$.*

Remark 1. Using standard arguments it is easily seen that in Theorem 1 (as well as in other results of this paper) either the assumption that $\mathscr{K}$ is finite, or the assumption that each K_i is compact, but not both, may be omitted. Obviously, the convexity of the sets K_i is necessary (for $n > 2$).

Remark 2. Corollary 1, which generalizes the theorem of Santaló [**26**] cited in §1, may be proved directly from Helly's theorem on intersections of convex sets [**17**] (in analogy to the proof of Santaló's theorem in **23**]).

Remark 3. Corollary 1 may be proved also under the weaker assumption that not all the parallel hyperplanes, containing the members of $\mathscr{K}$, coincide. Well-known examples ([15], [19], [24]) of families in E^2 for which $\mathscr{T}(k)$ does not imply $\mathscr{T}(k+1)$ show that this weaker condition may not be dropped.

Remark 4. The necessity of assuming (in Corollary 1 as well as in Theorem 1) $\mathscr{T}(2n-1)$ instead of, *e.g.*, $\mathscr{T}(n+1)$ follows at once from the following example in E^3. Let K_i, $1 \leqslant i \leqslant 5$, be the segments with end-points A_i and B_i, where $A_1 = (0, 0, 0)$, $A_2 = (3, -3, 1)$, $A_3 = (2, 0, 2)$, $A_4 = (1, 3, 3)$, $A_5 = (0, 0, 4)$, $B_1 = (6, 0, 0)$, $B_2 = (3, 1, 1)$, $B_3 = (0, 2, 2)$, $B_4 = (-3, 3, 3)$, $B_5 = (0, 6, 4)$. As easily verified, this family has property $\mathscr{T}(4)$ but not $\mathscr{T}(5)$.

(The necessity of the assumption (in Theorem 1) that $C_i \cap C_j$, etc., be cells will be discussed later (Remarks 6 and 7, §3).)

For another corollary of Theorem 1 we need the concept of a "thin" family of sets. A family $\mathscr{K} = \{x_i + \lambda_i K\}$ of sets similar to a compact, convex set $K \subset E^n$ which has the origin as centre of symmetry, is called ρ-thin, for a real $\rho \geqslant 1$, if $(x_i + \rho\lambda_i K) \cap (x_j + \rho\lambda_j K) = \emptyset$ for $i \neq j$.

Corollary 2. *For 2-thin families of (closed, solid) spheres in E^n, $\mathscr{T}(2n-1)$ implies $\mathscr{T}$.*

Proof. It is not difficult to see that a 2-thin family $\mathscr{S} = \{S_i\}$ of spheres which has property $\mathscr{T}(3)$ if $n = 2$ or $\mathscr{T}(4)$ if $n \geqslant 3$, is separated by parallel hyperplanes. Indeed, the separating hyperplanes may be taken orthogonal to the line determined by the centres of any two members of $\mathscr{S}$ whose convex hull does not meet any other member of $\mathscr{S}$. [The existence of such a pair follows easily from 2-thinness and $\mathscr{T}(3)$.] In order to apply Theorem 1 it is therefore necessary only to show that the intersection of any 3, 4, ..., $2n-2$ of the sets C_i, corresponding to transversals of S_i, is a cell. But this follows immediately from the reasoning given in the "Nachtrag" to [14]. This ends the proof of Corollary 2.

Remark 5. With $\mathscr{T}(n^2)$ assumed instead of $\mathscr{T}(2n-1)$, Corollary 2 is due to Hadwiger [13], [14]. The case $n = 2$ of Corollary 2 appears in [16].

3. A family $\mathscr{K}$ of convex sets in E^n shall be called k-simple (k a natural number) if, whenever L_0 and L_1 are straight lines both of which intersect any k members $K_1, \ldots, K_k$ of $\mathscr{K}$, there exists a continuous family $L(t)$, $0 \leqslant t \leqslant 1$, of straight lines such that $L(t)$ intersects K_i for all t, $0 \leqslant t \leqslant 1$, and all i, $1 \leqslant i \leqslant k$, while $L(0) = L_0$ and $L(1) = L_1$. (For families $\mathscr{K}$ separated by parallel hyperplanes, k-simplicity is equivalent to the condition that the intersection of any k of the sets $C_i \subset E^{2n-2}$ be connected.)

By using the arguments in [2] it is easily seen that any family of compact, convex sets separated by parallel hyperplanes is 3-simple. On the other hand, such families do not have to be 4-simple.

In the case $n = 3$, Theorem 1 may be given the simpler form

THEOREM 2. *If $\mathscr{K}$ is a 4-simple separated family of compact, convex subsets of E^3, then $\mathscr{T}(5)$ implies $\mathscr{T}$.*

Proof. In view of Theorem 1, we have only to show that for any four members K_1, K_2, K_3, K_4 of $\mathscr{K}$ the intersections $C_1 \cap C_2 \cap C_3$ and $C_1 \cap C_2 \cap C_3 \cap C_4$ are cells. But this, by the above remark on 3-simplicity, is an immediate consequence of the following lemma.

LEMMA 1. *Let $\mathscr{K} = \{K_i, 1 \leqslant i \leqslant m\}$ be a family of compact, convex subsets of E^3, separated by parallel planes H_i, $0 \leqslant i \leqslant m$. Let A be the subset of H_0 consisting of points through which pass straight lines intersecting all the members of $\mathscr{K}$. If A is connected, it is simply connected.*

Proof. Assume that A is not simply connected, and let x be a point of a bounded component A^* of the complement of A in H_0. Let B_i denote the cone with vertex x generated by K_i, and let $D_i = B_i \cap H_m$. Obviously, D_i is convex, and $\bigcap_{i=1}^{m} D_i = \emptyset$ because $x \notin A$. Therefore, by Helly's theorem on convex sets [17], there exist p, q, r with $1 \leqslant p \leqslant q < r \leqslant m$, such that $D_p \cap D_q \cap D_r = \emptyset$. On the other hand, for any i, j we have $D_i \cap D_j \neq \emptyset$ since otherwise there would exist a plane passing through x and strictly separating K_i from K_j, in contradiction to our assumption that A^* is a bounded component of the complement of A. Thus, the complement of $D_p \cup D_q \cup D_r$ in H_m has a bounded component D^*; let E be the ellipse of maximal area inscribed in $\bar{D}^*$. We denote by P_q resp. P_r the plane passing through x and separating E from B_q resp. B_r, and by F the intersection of H_0 with those closed half-spaces determined by P_q and P_r which do not contain E. Then obviously $x \in F$, F is unbounded and connected, and $F \cap A = \emptyset$ in contradiction to the assumption that A^* is a bounded component of the complement of A. This ends the proof of the lemma, and therefore proves Theorem 2.

Remark 6. The condition of 4-simplicity in Theorem 2 (and therefore also some condition on the intersections of the sets C_i in Theorem 1) is necessary, as is shown by the following example. Let K_i, $i = 1, 2, 5, 6$, be segments with end-points A_i, B_i, where $A_1 = (0, 0, 0)$, $A_2 = (1, 0, 0)$, $A_5 = (4, 0, 0)$, $A_6 = (6, 0, 0)$, $B_1 = (0, 0, 6)$, $B_2 = (1, 1, 5)$, $B_5 = (5, 5, 1)$, $B_6 = (6, 6, 0)$, in a system of coordinates (x_1, x_2, x_3) of E^3. It is obvious that precisely two straight lines intersect all these four sets, one of them L_A, containing the points A_i, the other, L_B, the points B_i. Let H_3 resp. H_4 denote the plane $x_1 = 2$ resp $x_1 = 3$, and let L_i, $i = 1, 2, 5, 6$, denote a straight line intersecting, in interior points of the segments, the three members of $\{K_1, K_2, K_5, K_6\}$ different from K_i. Then $L_i \cap K_i = \emptyset$. We denote by K_3 the convex hull of the five points $H_3 \cap L_A = (2, 0, 0)$ and $H_3 \cap L_i$, $i = 1, 2, 5, 6$, and by K_4 the convex hull of $H_4 \cap L_B = (3, 3, 3)$ and $H_4 \cap L_i$, $i = 1, 2, 5, 6$. Then the family $\{K_i, 1 \leqslant i \leqslant 6\}$ is separated by parallel hyperplanes (indeed, except for K_5, the sets are contained in parallel hyperplanes) and has property $\mathscr{T}(5)$ but does not have $\mathscr{T}(6)$.

Remark 7. It is possible that Theorem 1 holds with the condition "$\mathscr{K}$ is k-simple, for $k = 4, \ldots, 2n-2$" imposed instead of the conditions on the intersections of the sets C_i.

4. In the case of the plane, more complete results on common transversals may be obtained, mainly because of the possibility of applying the projective duality and because of the following simpler form of Helly's theorem valid for $n = 2$ ([18], [22]):

If $\mathscr{K}$ is any family of cells (i.e. compact, simply connected sets) in E^2 such that the intersection of any two members of $\mathscr{K}$ is connected, and the intersection of any three members is non-empty, then the intersection of all the members of $\mathscr{K}$ is not empty.

Properties $\mathscr{P}$ like boundedness, connectedness, etc., of compact subsets of the projective plane π will in the sequel be understood in the following sense:

The set $A \subset \pi$ has property $\mathscr{P}$ if there exists a homeomorphism ϕ of π onto another projective plane π^*, and a straight line $L \subset \pi^*$, such that $L \cap \phi(A) = \emptyset$ and such that $\phi(A)$ has property $\mathscr{P}$ in the affine plane obtained from π^* by taking L as the "line at infinity".

The following properties of subsets of the projective plane are easy consequences of their well-known Euclidean counterparts:

(i) If the intersection of two connected sets [cells] is connected, their union is connected [their union and intersection are cells].

(ii) If the intersection of three connected sets [cells] is not empty, and if the intersection of any two of them is connected, then their union is connected [their union and intersection are cells].

We may formulate Helly's theorem for the projective plane as follows:

LEMMA 2†. *If $\mathscr{C}$ is a family of cells in the projective plane π, such that the intersection of any two members of $\mathscr{C}$ is connected and the intersection of any three members is non-empty, then the intersection of all the members of $\mathscr{C}$ is not empty.*

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†*Added in proof.* Dr. J. Molnár informed the author that Lemma 2 of the present paper is a corollary of his paper "A kétdimenziós topológikus Helly-tételről" ("On Helly's two-dimensional topological theorem"), *Matematikai Lapok*, 8 (1957), 108–114.

Proof. We prove the Lemma first in the case $\mathscr{C} = \{C_0, C_1, C_2, C_3\}$. Since $C = C_1 \cup C_2 \cup C_3$ is, by (ii), a cell and since all the properties considered are invariant with respect to homeomorphisms of π, we may assume that C is contained (and therefore bounded) in an affine plane of π. Now, $\{C^* = C \cap C_0, C_1, C_2, C_3\}$ is a family satisfying the conditions of Helly's theorem for the affine (Euclidean) plane. Indeed, each of the four sets is a cell (for C^* this follows from (i), since C and C_0 are cells and C^* is easily seen to be connected) and, as $C^* \cap C_i = C_0 \cap C_i$ and $C^* \cap C_i \cap C_j = C_0 \cap C_i \cap C_j$, the conditions on the intersections of two or three sets are also satisfied. Therefore

$$C^* \cap C_1 \cap C_2 \cap C_3 = C_0 \cap C_1 \cap C_2 \cap C_3 \neq \emptyset.$$

For arbitrary finite families $\mathscr{C} = \{C_i, 0 \leqslant i \leqslant n\}$ the Lemma may now easily be established by induction, using the above special case and the inductive assumption on the family $\{C_i^* = C_0 \cap C_i;\ 1 \leqslant i \leqslant n\}$. The general (infinite) case then follows by compactness. This ends the proof of Lemma 2.

Following the accepted terminology, we shall call a compact subset of a projective space π_n *convex* if its intersection with any straight line is either empty, or a point, or a segment (but not the whole line) [4], [5]. We shall say that a compact subset of π_n is *strongly bounded* if there exists a hyperplane π_{n-1} disjoint from it. It is well known that compact, convex sets are strongly bounded.

Remark 8. By defining appropriately the notion of cell in π_n, Helly's theorem may be established also for families of cells in π_n, $n \geqslant 2$. On the other hand, Helly's theorem on convex sets may be formulated as follows: Any family $\mathscr{K}$ of compact convex sets in π_n, such that the intersection of any *two* members of $\mathscr{K}$ is convex and the intersection of any $n+1$ of them non-empty, has a non-empty intersection.

If A is a compact, strongly bounded subset of the projective plane π, a set K will be called a *convex hull* of A if K is convex, contains A and is minimal with respect to these properties. A set A may have, in general, more than one convex hull; but it is easily seen that if A is connected then the convex hull of A is unique.

In this terminology we have

THEOREM 3. *Let $\mathscr{A} = \{A_i\}$ be a family of compact, connected, strongly-bounded subsets of the projective plane π, such that for some $A_0 \in \mathscr{A}$ the following conditions are satisfied:*

(*a*) *The convex hull of A_0 is disjoint from the convex hull of any other member of $\mathscr{A}$;*

(*b*) *For any $A_i, A_j \in \mathscr{A}$, $i \neq j$, a connected set of points in the projective plane dual to π corresponds to the set of all lines intersecting simultaneously A_0, A_i and A_j;*

(c) *For any* $A_i, A_j, A_k \varepsilon \mathscr{A}$ *there exists a straight line intersecting* A_0, A_i, A_j *and* A_k.

Then the family $\mathscr{A}$ *has property* $\mathscr{T}$.

Proof. In the plane dual to π, let C_i, for $i \neq 0$, denote the set of all points corresponding to straight lines in π which intersect A_0 and A_i Then, as a consequence of (*a*), it follows that C_i is a cell (see [**21**], Lemma 2.2). Since (*b*) insures that $C_i \cap C_j$ is connected, and (*c*) that $C_i \cap C_j \cap C_k \neq \emptyset$, it results from Lemma 2 that $\bigcap_i C_i \neq \emptyset$, *i.e.* all the members of $\mathscr{A}$ have a common transversal. This ends the proof of Theorem 3.

Remark 9. Theorem 3 obviously applies also to families of subsets of E^2. It generalizes Theorem 2 of [**21**] and the corresponding result of [**24**], the main difference being that in [**21**] and [**24**] conditions (*a*), (*b*), (*c*) (or conditions equivalent to them) are imposed on any pair, triple, resp. quadruple of members of $\mathscr{A}$.

Remark 10. Using the case $n = 4$ of Theorem 1 of [**6**] (or the corresponding result of §4 of [**21**], or Formula (7) of [**24**]), a somewhat generalized formulation of Theorem 1 of [**21**] (and of the related result of [**24**]) follows easily from the above Theorem 3.

Remark 11. It is easy to see that none of the conditions (*a*), (*b*), (*c*) of Theorem 3 may be dropped.

Applying Theorem 3 we shall now prove

COROLLARY 3. *For any* $\sqrt{2}$*-thin family* $\mathscr{C}$ *of congruent circles in the plane,* $\mathscr{T}(3)$ *implies* $\mathscr{T}$.

Proof. The condition of $\sqrt{2}$-thinness implies, because of $\mathscr{T}(3)$, that any triangle whose vertices are centres of circles in $\mathscr{C}$, has an obtuse angle. By considering the (essentially only 3) different possible configurations of four-membered subfamilies of $\mathscr{C}$, it is easily seen that $\mathscr{C}$ has property $\mathscr{T}(4)$. Since the condition (*b*) of Theorem 3, is obviously satisfied (with any member of $\mathscr{C}$ as A_0), this establishes Corollary 3.

Remark 12. As is shown by the example of four circles, centred at the vertices of a unit square and having radius $\frac{1}{4}\sqrt{2}$, it is impossible to replace in Corollary 3 the condition of $\sqrt{2}$-thinness by that of ρ-thinness, for any $\rho < \sqrt{2}$. On the other hand, Corollary 3 fails for $\sqrt{2}$-thin families of incongruent circles. This is shown by the four circles with centres (9, 0), (0, 2), (0, 8), (−9, 0) and radii 5, 1, 3, 5, respectively.

If K is a compact, convex set, a family $\{x_i + K\}$ of translates of K shall be called *dispersed* if $(x_i + K) \cap \left(\frac{1}{2}(x_i + x_j) + K\right) = \emptyset$ for $i \neq j$. Obviously, if K has the origin as centre of symmetry, the notion of a dispersed family of translates of K coincides with that of a 2-thin family (see §2).

We have the following

COROLLARY 4. *If* $\mathscr{K} = \{x_i + K\}$ *is a dispersed family of translates of a compact, convex subset* K *of* E^2, *then* $\mathscr{T}(3)$ *implies* $\mathscr{T}$.

Proof. Let $K^* = \frac{1}{2}\big(K + (-K)\big)$, and let $\mathscr{K}^* = \{x_i + K^*\}$. Then it is easily seen that $\mathscr{K}^*$ is dispersed (*i.e.* 2-thin), and that $\mathscr{T}(3)$, resp $\mathscr{T}$, hold for $\mathscr{K}$ and $\mathscr{K}^*$ simultaneously. Without loss of generality we may therefore assume that K has the origin as centre of symmetry. Let E be the (unique) ellipsoid of minimal area containing K. Obviously, we may assume that E is a circle. On the other hand it is easily seen that $E \subset \sqrt{2}K$, and therefore the family $\{x_i + E\}$ is $\sqrt{2}$-thin. As in the proof of Corollary 3 it follows, by considering four-membered subfamilies of $\mathscr{K}$, that $\mathscr{T}(4)$ holds. The final part of the proof is then completely analogous to the corresponding part of the proof of Corollary 3.

Remark 13. It is easy to find examples showing that the assumptions of Corollary 4 are necessary. Even if only centrally symmetric sets are considered, ρ-thinness, for some $\rho < 2$, is not sufficient, and the corollary fails also for $\rho = 2$ if $\mathscr{K}$ is allowed to be of the form $\{x_i + \alpha_i K\}$.

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