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A new Ceva-type theorem

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1. *Introduction*

The classical theorems of Ceva and Menelaus make assertions about the value of certain products of ratios of lengths in configurations in the affine plane. We shall use the term *Ceva-type* to describe any result of this general kind: one that specifies a configuration in affine space of n dimensions, defined only by incidences, about which one can make an assertion about a product of ratios of lengths, areas, etc. Several results of this kind are known. Apart from the classical results there are, for example, Ceva's and Menelaus' Theorems for n-gons, Hoehn's Theorem for pentagrams [1], and the Selftransitivity Theorem of [2].

The purpose of this note is to introduce a new result of this kind. Before stating it in Section 2 we shall review some earlier results and show how this Ceva-type theorem is related to them. Throughout we shall work in the affine plane.

Let $T = [V_1, V_2, V_3]$ be a triangle and O be a fixed point which is not collinear with any two of the vertices V_1, V_2, V_3 of T. If each of the lines OV_i $(i = 1, 2, 3)$ meets the opposite side of T in the point W_i, then the classical theorem of Ceva states

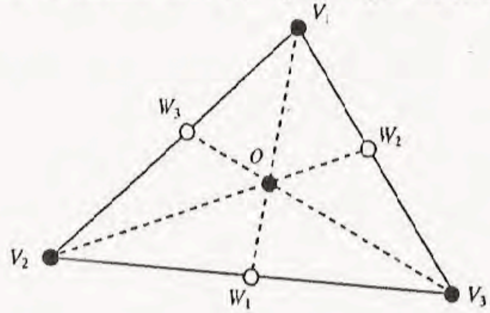
$$\frac{|V_2W_1|}{|W_1V_3|}\cdot\frac{|V_3W_2|}{|W_2V_1|}\cdot\frac{|V_1W_3|}{|W_3V_2|} = 1 \qquad (1)$$

(see Figure 1). Here the notation $|AB|$ means the ordinary (Euclidean) length of the line segment $[A, B]$. We can write (1) in the slightly stronger form

$$\prod_{i=1}^{3}\left[\left[\frac{V_iW_{i+2}}{W_{i+2}V_{i+1}}\right]\right] = 1$$

where each subscript j is reduced modulo 3 so that it lies in the range $1 \leq j \leq 3$, and the shadowed brackets mean that *signed* lengths are to be taken into account. This means that if A, B, C are distinct collinear points, then $[[AB/BC]] = \pm|AB|/|BC|$ where the sign is positive if B lies between A and C, and is negative otherwise. It should be observed that $[[AB/BC]]$ is an affine invariant. (For further explanation of this notation, see [2].)

FIGURE 1

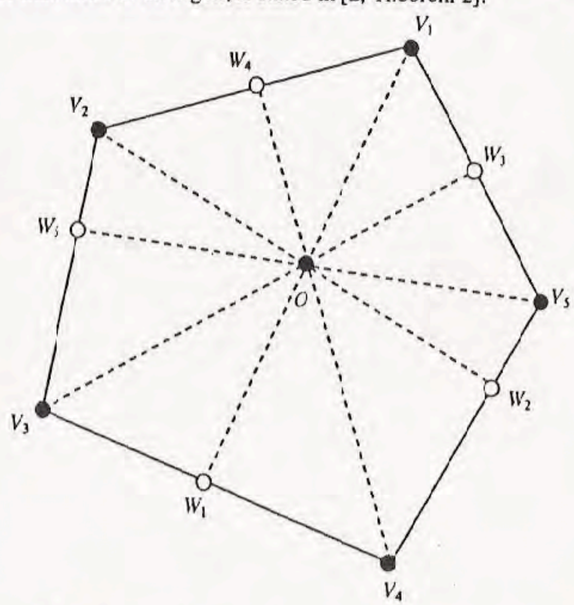


The extension of Ceva's Theorem to n-gons (n odd, $n > 3$) concerns a fixed point O in the plane of a general n-gon $P = [V_1, V_2, \ldots, V_n]$. If the lines OV_i (joining O to vertex V_i) meets the opposite side V_rV_{r+1} of P $(r = i + \frac{1}{2}(n-1))$ in the point W_i, then

$$\prod_{i=1}^{n}\left[\left[\frac{V_rW_i}{W_iV_{r+1}}\right]\right] = 1. \qquad (2)$$

(See Figure 2 for the case $n = 5$.) We tacitly assume here and throughout the rest of this note that everything is well defined, the points of intersection exist (lines defining them are not parallel), and all the points are distinct so that the quotients in (2) and similar expressions have non-zero denominators. We emphasise that this result, like all others in this note, applies to *general* n-gons; non-adjacent vertices may coincide and edges may intersect or overlap in any way, subject only to the restrictions stated in the previous sentence. Another variant of Ceva's Theorem, relating to diagonals rather than sides of an n-gon, is stated in [2, Theorem 2].

FIGURE 2



In 1993, L. Hoehn stated two theorems about pentagrams. Generalisations of these were stated and proved in [2]. We quote these results here since they are required later in the proof of the main theorem.

Hoehn's First Theorem Let $P = [Z_1, \ldots, Z_n]$ be a given n-gon and j be an integer such that, for each i, the integers $i - 2j$, $i - j$, i, $i + j$, $i + 2j$ are distinct (modulo n). Define V_i ($i = 1, \ldots, n$) to be the point of intersection of $Z_{i-2j}Z_i$ and $Z_{i-j}Z_{i+j}$. Then the points V_i and V_{i+j} lie on the line $Z_{i-j}Z_{i+j}$ and

$$\prod_{i=1}^{n}\left[\left[\frac{Z_{i+j}V_{i+j}}{V_i Z_{i-j}}\right]\right] = 1.$$

This theorem is illustrated for $n = 7$ and $j = 1$ in Figure 3(a). The conditions of the theorem imply $n > 5$.

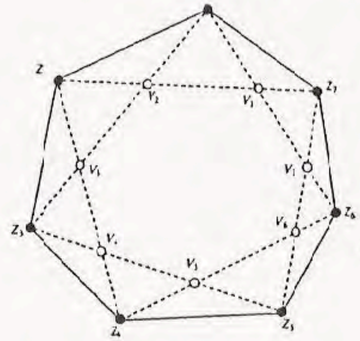


FIGURE 3(a)

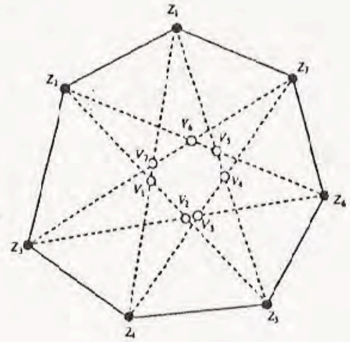


FIGURE 3(b)

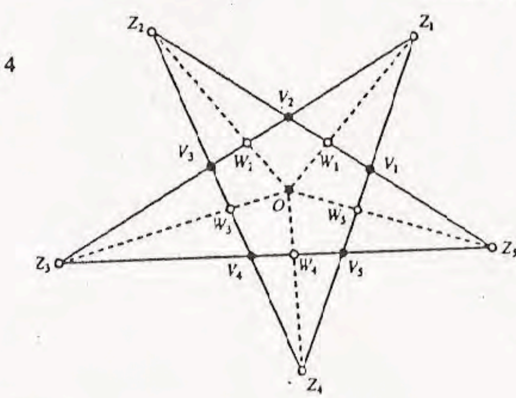
Hoehn's Second Theorem Let $P = [Z_1, \ldots, Z_n]$ be a given n-gon, and j, k be positive integers such that $j + 2k \equiv 0$ (modulo n), and for each $i = 1, \ldots, n$ the integers i, $i + k$, $i + j$, $i + j + k$ are distinct (modulo n) and i, $i + k$, $i + 2k$, $i + 3k$ are distinct (modulo n). Define V_i as the intersection of Z_iZ_{i+k} and $Z_{i+j}Z_{i+j+k}$. Then the points V_i and V_{i+2k} lie on the line Z_iZ_{i+k} and

$$\prod_{i=1}^{n}\left[\left[\frac{Z_iV_i}{V_{i+2k}Z_{i+k}}\right]\right] = 1.$$

This theorem is illustrated for $n = 7$ and $(j, k) = (1, 3)$ in Figure 3(b). The simplest case of the theorem arises when $n = 4$ and $(j, k) = (2, 1)$. Then $V_1 = V_3$, $V_2 = V_4$ and the theorem reduces to a non-trivial statement about the ratios of the lengths of sides of a complete quadrangle.

In both the classical and n-gonal form of Ceva's Theorem, the fixed point O is joined to the vertex *opposite* to a given side of the n-gon P. An alternative interpretation (which is equivalent in the case of a triangle) is that we join *O to the point of intersection of the sides of* P *adjacent to the given side*. To introduce our main theorem and proof (which will be given in the next section) we consider a very simple example, namely with $n = 5$ so all the subscripts are reduced modulo 5 (see Figure 4). For a given side V_iV_{i+1} of the pentagon $P = [V_1, V_2, V_3, V_4, V_5]$ write $Z_i = V_iV_{i-1} \cap V_{i+1}V_{i+2}$

FIGURE 4



and $W_i = OZ_i \cap V_iV_{i+1}$ for $i = 1, 2, 3, 4, 5$. Then we assert that

$$\prod_{i=1}^{5}\left[\left[\frac{V_iW_i}{W_iV_{i+1}}\right]\right] = 1. \tag{3}$$

To prove this we use a technique which we introduced in [2] and called the *Area Principle*. This extremely simple tool turns out to be very powerful in establishing results of this kind. It states that because the triangles $T_i = [O, Z_i, V_i]$ and $S_i = [O, Z_i, V_{i+1}]$ have the same base $[O, Z_i]$,

$$\left[\left[\frac{V_iW_i}{W_iV_{i+1}}\right]\right] = -\left[\left[\frac{OZ_iV_i}{OZ_iV_{i+1}}\right]\right]$$

where the term on the right is the ratio of the *signed areas* of the triangles T_i and S_i; we recall that the signed area of a triangle is positive if the vertices are listed in a counterclockwise direction, and negative if they are listed in a clockwise direction. More details concerning this notation can be found in [2]. Hence

$$\prod_{i=1}^{5}\left[\left[\frac{V_iW_i}{W_iV_{i+1}}\right]\right] = -\prod_{i=1}^{5}\left[\left[\frac{OZ_iV_i}{OZ_iV_{i+1}}\right]\right]. \tag{4}$$

However

$$\left[\left[\frac{V_{i+2}Z_{i+2}}{Z_iV_{i+1}}\right]\right] = -\left[\left[\frac{OZ_{i+2}V_{i+2}}{OZ_iV_{i+1}}\right]\right]$$

since the triangles on the right have bases $[Z_i, V_{i+1}]$ and $[Z_{i+2}, V_{i+2}]$ and the same apex O (and so the same heights). Thus

$$\prod_{i=1}^{5}\left[\left[\frac{V_{i+2}Z_{i+2}}{Z_iV_{i+1}}\right]\right] = -\prod_{i=1}^{5}\left[\left[\frac{OZ_{i+2}V_{i+2}}{OZ_iV_{i+1}}\right]\right]. \tag{5}$$

Clearly the right sides of (4) and (5) are identical, and the left side of (5) takes the value 1 by Hoehn's First Theorem applied to the pentagram $[Z_1, Z_2, Z_3, Z_4, Z_5]$ with $j = 1$. (This is the original form of the theorem as stated by Hoehn in [1].) We deduce that the left side of (4) takes the value 1, so (3) is established and the theorem is proved. (Hoehn placed restrictions on the pentagram, namely that the inner pentagram is convex, but as we have shown in [2, Theorems 4 and 5], these are unnecessary.)

2. *The Main Theorem*

Our main theorem extends the simple example in the previous section to n-gons and their diagonals.

Theorem

Let $P = [V_1, \ldots, V_n]$ be a general n-gon, O be a fixed point, and j, k be given positive integers. Define

$$Z_i = V_iV_{i-j} \cap V_{i+k}V_{i+k+j}$$

and

$$W_i = OZ_i \cap V_iV_{i+k}$$

where the subscripts are reduced modulo n (see Figure 5). Then if either of the conditions

(a) $j \equiv k$ and the integers $0, j, 2j, 3j, 4j, 5j$ are distinct (modulo n), *or*

(b) $j + 2k \equiv 0$, the integers $0, k, j, j + k$ are distinct (modulo n) and the integers $0, k, 2k, 3k$ are distinct (modulo n),

then

$$\prod_{i=1}^{n}\left[\!\left[\frac{V_iW_i}{W_iV_{i+k}}\right]\!\right] = 1. \qquad (6)$$

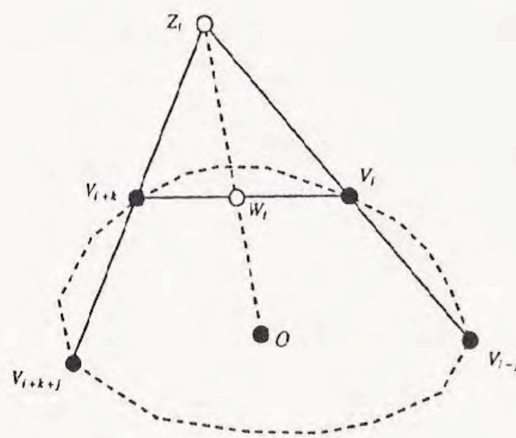


FIGURE 5

Proof

By the area principle applied to triangles with base $[O, Z_i]$ we obtain

$$\left[\!\left[\frac{V_iW_i}{W_iV_{i+k}}\right]\!\right] = -\left[\!\left[\frac{OZ_iV_i}{OZ_iV_{i+k}}\right]\!\right]$$

and so

$$\prod_{i=1}^{n}\left[\!\left[\frac{V_iW_i}{W_iV_{i+k}}\right]\!\right] = (-1)^n \prod_{i=1}^{n}\left[\!\left[\frac{OZ_iV_i}{OZ_iV_{i+k}}\right]\!\right]. \qquad (7)$$

In case (a), writing $k = j$ as in Figure 6, we see that Z_{i+j} and Z_{i-j} lie on the line V_iV_{i+j} and

$$\left[\!\left[\frac{Z_{i+j}V_{i+j}}{V_iZ_{i-j}}\right]\!\right] = -\left[\!\left[\frac{OZ_{i+j}V_{i+j}}{OZ_{i-j}V_i}\right]\!\right].$$

(The triangles on the right have the same apex O and their bases are the line segments in the fraction on the left.) Hence

$$\prod_{i=1}^{n}\left[\!\left[\frac{Z_{i+j}V_{i+j}}{V_iZ_{i-j}}\right]\!\right] = (-1)^n \prod_{i=1}^{n}\left[\!\left[\frac{OZ_{i+j}V_{i+j}}{OZ_{i-j}V_i}\right]\!\right]. \qquad (8)$$

Since $j \equiv k$, the right sides of (7) and (8) are identical, and the left side of (8) takes the value 1 by Hoehn's First Theorem, quoted above, with the same value of j. We deduce that the left side of (7) takes the value 1, so (6) is true and the theorem is proved in this case.

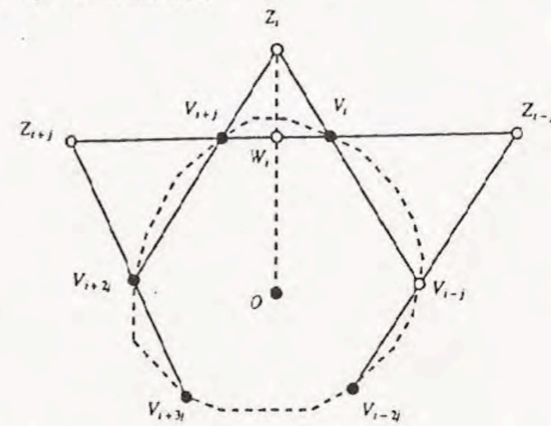


FIGURE 6

In case (b) with $j + 2k \equiv 0$ we observe that Z_i and Z_{i+k} lie on the line $V_iV_{i+2k} (= V_iV_{i-j})$ and that

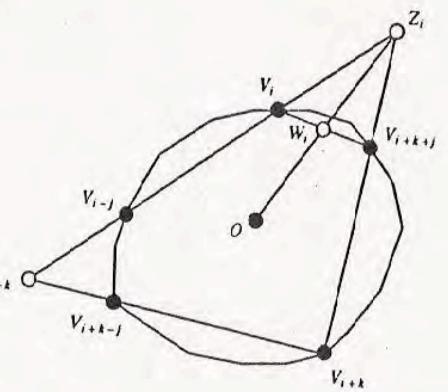
$$\left[\!\left[\frac{Z_iV_i}{V_{i+2k}Z_{i+k}}\right]\!\right] = -\left[\!\left[\frac{OZ_iV_i}{OZ_{i+k}V_{i+2k}}\right]\!\right]$$

(see Figure 7) since the triangles on the right have the same apex O and their bases are the line segments in the fraction on the left. Hence

$$\prod_{i=1}^{n}\left[\!\left[\frac{Z_iV_i}{V_{i+2k}Z_{i+k}}\right]\!\right] = (-1)^n \prod_{i=1}^{n}\left[\!\left[\frac{OZ_iV_i}{OZ_{i+k}V_{i+2k}}\right]\!\right]. \qquad (9)$$

The right sides of (7) and (9) are identical, and the left side of (9) takes the value 1 by Hoehn's Second Theorem with the same values of j and k. We deduce the left side of (7) takes the value 1, so (6) is true and the proof of the theorem is complete.

FIGURE 7



3. *Comments*

The simplest cases of the main theorem arise when $n = 4$, $j = 2$, $k = 1$ which satisfy condition (b). Here all the points Z_i coincide, and the result (see Figure 8) that

$$\prod_{i=1}^{4}\left[\left[\frac{V_iW_i}{W_iV_{i+1}}\right]\right] = 1 \qquad (10)$$

can readily be proved directly, that is, without using Hoehn's Theorem. The values $n = 4$, $j = 1$, $k = 1$ do not satisfy either condition (a) or (b), yet the theorem remains true; relation (10) holds in this case also (see Figure 9). Again this is easy to prove. It seems remarkable that such simple theorems on quadrangles do not seem to appear in the literature. However, as the

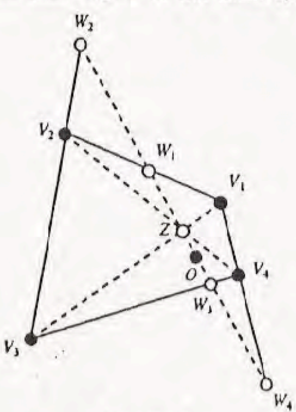


FIGURE 8

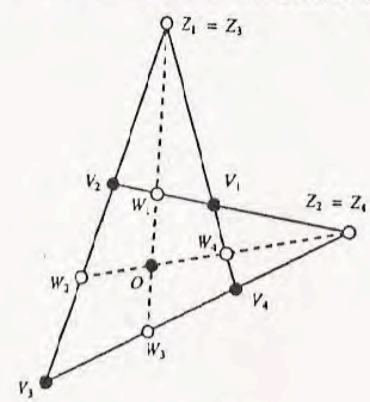


FIGURE 9

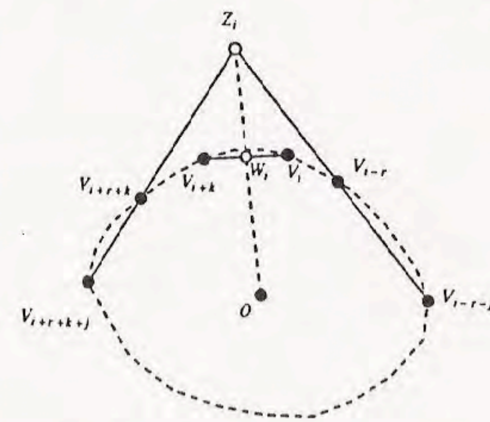
referee pointed out, the assertion (illustrated in Figure 9) yields a simple (Euclidean) proof of the fact that cross-ratios are invariant under projection.

For small values of $n \geqslant 5$, the numbers of non-trivial primitive distinct cases of the main theorem are as follows. (An assertion is primitive if HCF $(n, j, k) = 1$.) Where two different sets of parameters lead to the same assertion, only one is listed. (For example, the case $n = 5$, $j = k = 1$ is the same as $n = 5, j = 1, k = 2$.)

$n = 5$,	two cases:	$j = k = 1; j = k = 2$.
$n = 6$,	two cases:	$j = k = 1; j = 4, k = 1$.
$n = 7$,	six cases:	$j = k = 1; j = k = 2; j = k = 3; j = 1, k = 3;$ $j = 3, k = 2; j = 5, k = 1$.
$n = 8$,	four cases:	$j = k = 1; j = k = 3; j = 2, k = 3; j = 6, k = 1$.
$n = 9$,	six cases:	$j = k = 1; j = k = 2; j = k = 4; j = 1, k = 4;$ $j = 5, k = 2; j = 7, k = 1$.
$n = 10$,	four cases:	$j = k = 1; j = k = 3; j = 4, k = 3; j = 8, k = 1$.

Thus, for $n \geqslant 5$ the number of cases is equal to that of Hoehn's Theorems. For $n = 4$, as remarked above, an extra case arises. So far as we are aware, this is the only anomalous case, a statement that we have checked numerically by computer using *Mathematica*© software for all n up to 13.

FIGURE 10



In fact we can say slightly more. Supposing that we do not insist that the diagonals (which intersect in Z_i) are necessarily contiguous with the diagonal V_iV_{i+k}, and we write

$$Z_i = V_{i-r}V_{i-r-j} \cap V_{i+r+k}V_{i+r+k+j}$$

and

$$W_i = OZ_i \cap V_iV_{i+k}$$

for some $r > 0$ (see Figure 10). Then except in the case where Z_i is a vertex of the n-gon (and the theorem reduces to Ceva's Theorem for n-gons) no relation of type (6) is valid for all n-gons with $r > 0$. Again, this has been verified by computer for all n from 3 to 13.

References

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