Ceva, Menelaus, and the Area Principle

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1. Introduction

The theorems of Ceva and Menelaus [**3**, p. 220], which will be stated shortly, are among the most attractive and useful results in elementary plane geometry. They are easy to state and are quite general in the sense that, for example, Menelaus' Theorem applies to *any* triangle and to *any* transversal that does not pass through a vertex. These theorems, and their proofs, are classical as is reflected in their names. The Greek Menelaus lived in the first century A.D. and the Italian Giovanni Ceva published his theorem (and rediscovered Menelaus' Theorem) in the 17th century. Recently Hoehn [**8**] obtained a new result of a similar kind, showing that the products of five quotients of certain lengths in a pentagram have the value 1.

The purpose of this note is to show that these and other results, and their extensions to general polygons with arbitrarily many sides, are the consequences of a simple idea which we shall call the *area principle*. This principle is illustrated in FIGURE 1, in which P is the point of intersection of the lines BC and A_1A_2. Denoting the lengths of the segments $[A_1, P]$ and $[A_2, P]$ by $|A_1P|$ and $|A_2P|$, and the areas of the triangles $[A_1, B, C]$ and $[A_2, B, C]$ by $|A_1BC|$ and $|A_2BC|$, respectively, the area principle states

$$\frac{|A_1P|}{|A_2P|} = \frac{|A_1BC|}{|A_2BC|}, \tag{1}$$

and this is true whenever the ratios are well-defined, that is, their denominators do not vanish. The validity of (1) does not depend on whether the points A_1 and A_2 are separated by the line BC (FIGURE 1(a)) or not (FIGURE 9(b)). Later we shall refine (1) by assigning signs to the ratios; just for the present we shall consider all areas and lengths as positive.

The proof of (1) is immediate. Clearly each side of this equation is equal to the ratio of the heights of the two triangles on base $[B, C]$. We note that, although some of the well-known proofs of Ceva's Theorem (see for example [**4**, p. 4]) use area arguments, these are distinct from the "area principle."

As a foretaste of the methods to be used, we shall use the area principle to prove three of the theorems mentioned above. Suppose a transversal cuts the lines BC, CA, AB, determined by the named pairs of vertices of the triangle $[A, B, C]$, in points P, Q, R, respectively, and that these three points are distinct from the vertices of the triangle (see FIGURE 2). Then Menelaus' Theorem states that

$$\frac{|BP|}{|PC|} \cdot \frac{|CQ|}{|QA|} \cdot \frac{|AR|}{|RB|} = 1. \tag{2}$$

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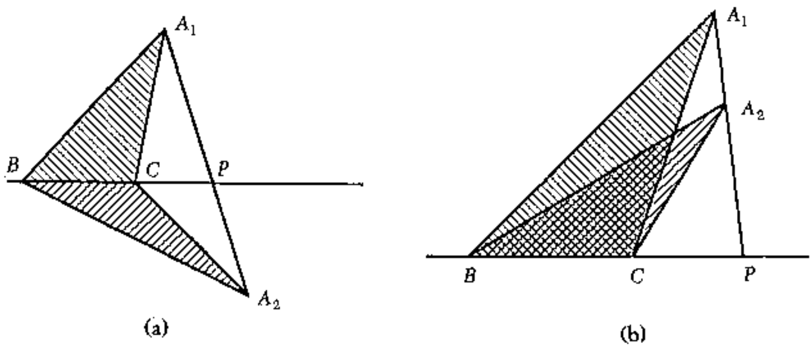


FIGURE 1
The area principle states that $|A_1P|/|A_2P| = |A_1BC|/|A_2BC|$.

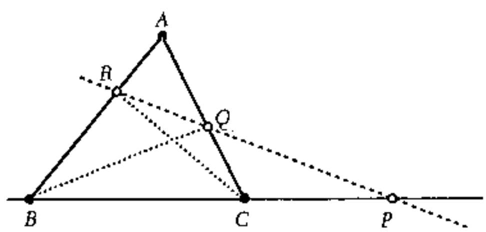


FIGURE 2
Menelaus' Theorem states that $(|BP|/|PC|)\cdot(|CQ|/|QA|)\cdot(|AR|/|RB|) = 1$.

To establish this we apply the area principle to the triangles with base $[R,Q]$, and we see that

$$\frac{|BP|}{|PC|} = \frac{|BRQ|}{|CRQ|}, \qquad \frac{|CQ|}{|QA|} = \frac{|CRQ|}{|ARQ|}, \qquad \frac{|AR|}{|RB|} = \frac{|ARQ|}{|BRQ|}.$$

Substituting these values in (2) the areas of the triangles all cancel, showing that (2) is valid. This proves the theorem.

Let O be a point such that the lines AO, BO, CO meet the opposite sides of the triangle $[A, B, C]$ in P, Q, R, respectively. We shall suppose that these three points are distinct from the vertices A, B, C (see FIGURE 3). Then Ceva's Theorem states that the relationship (2) holds in this case also. To establish this we consider triangles with base $[A, O]$ and then the area principle yields

$$\frac{|BP|}{|PC|} = \frac{|AOB|}{|COA|}.$$

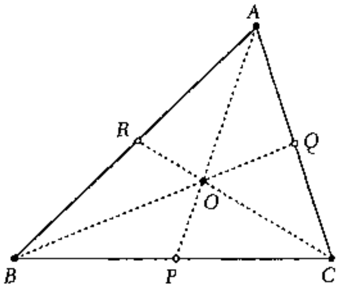


FIGURE 3
Ceva's Theorem states that $(|BP|/|PC|)\cdot(|CQ|/|QA|)\cdot(|AR|/|RB|) = 1$.

Similarly, using triangles with bases $[B, O]$ and $[C, O]$, we have

$$\frac{|CQ|}{|QA|} = \frac{|BOC|}{|AOB|}, \qquad \frac{|AR|}{|RB|} = \frac{|COA|}{|BOC|}.$$

Substituting these values in (2), the areas of the triangles all cancel to yield the value 1, which proves the theorem.

Finally, we consider Hoehn's Theorem, which states that for a pentagon, using the notation indicated in FIGURE 4,

$$\frac{|V_1W_1|}{|W_2V_3|} \cdot \frac{|V_2W_2|}{|W_3V_4|} \cdot \frac{|V_3W_3|}{|W_4V_5|} \cdot \frac{|V_4W_4|}{|W_5V_1|} \cdot \frac{|V_5W_5|}{|W_1V_2|} = 1 \tag{3}$$

and

$$\frac{|V_1W_2|}{|W_1V_3|} \cdot \frac{|V_2W_3|}{|W_2V_4|} \cdot \frac{|V_3W_4|}{|W_3V_5|} \cdot \frac{|V_4W_5|}{|W_4V_1|} \cdot \frac{|V_5W_1|}{|W_5V_2|} = 1. \tag{4}$$

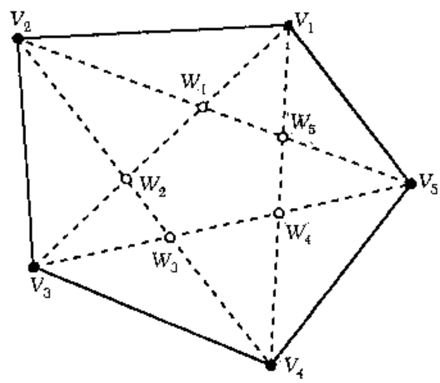


FIGURE 4

Hoehn's Theorem states that

$$\frac{|V_1W_1|}{|W_2V_3|} \cdot \frac{|V_2W_2|}{|W_3V_4|} \cdot \frac{|V_3W_3|}{|W_4V_5|} \cdot \frac{|V_4W_4|}{|W_5V_1|} \cdot \frac{|V_5W_5|}{|W_1V_2|} = 1$$

and

$$\frac{|V_1W_2|}{|W_1V_3|} \cdot \frac{|V_2W_3|}{|W_2V_4|} \cdot \frac{|V_3W_4|}{|W_3V_5|} \cdot \frac{|V_4W_5|}{|W_4V_1|} \cdot \frac{|V_5W_1|}{|W_5V_2|} = 1.$$

In Hoehn's original proof, Menelaus' Theorem was applied to various triangles and transversals in the diagram; here we give a more direct proof using the area principle. Consider the pentagon shown in FIGURE 4. (In this initial treatment we restrict attention to the case in which $[V_1, V_2, V_3, V_4, V_5]$ is a convex pentagon. The more general case, without any assumption of convexity, will be dealt with in Section 4.) Using triangles with base $[V_2, V_4]$, we have

$$\frac{|V_1W_2|}{|W_2V_3|} = \frac{|V_1V_2V_4|}{|V_3V_4V_2|}.$$

Hence $|V_1V_3|/|W_2V_3| = (|V_1W_2| + |W_2V_3|)/|W_2V_3| = (|V_1V_2V_4| + |V_3V_4V_2|)/|V_3V_4V_2| = |V_1V_2V_3V_4|/|V_3V_4V_2|$ where $|V_1V_2V_3V_4|$ is the area of the quadrilateral $[V_1, V_2, V_3, V_4]$. Similarly,

$$\frac{|V_1V_3|}{|V_1W_4|} = \frac{|V_1V_2V_3V_5|}{|V_1V_2V_5|}$$

and so

$$\frac{|V_1W_1|}{|W_2V_3|} = \frac{|V_1V_2V_5|}{|V_1V_2V_3V_5|} \cdot \frac{|V_1V_2V_3V_4|}{|V_3V_4V_2|}.$$

In general, for $i = 1, 2, \ldots, 5$,

$$\frac{|V_i W_i|}{|W_{i+1} V_{i+2}|} = \frac{|V_i V_{i+1} V_{i+4}|}{|V_i V_{i+1} V_{i+2} V_{i+4}|} \cdot \frac{|V_i V_{i+1} V_{i+2} V_{i+3}|}{|V_{i+2} V_{i+3} V_{i+1}|}. \tag{5}$$

Substituting these values in the left side of (3) we see that the areas of the triangles and quadrilaterals all cancel, yielding the value 1 as required. The second assertion (4) of Hoehn's Theorem can be proved in a similar manner.

The pattern of the above proofs will be applied repeatedly: We shall express ratios of lengths as ratios of areas, and then show that, in a product of such ratios, cancellation takes place to yield a constant value.

Before describing the extensions of these results, we put them in a more general context.

2. Affine Geometry and Polygons

Recall that affine geometry [**3**, Chapter 13] is concerned with geometric properties that are affine invariant, which means that they are invariant under *affinities* (that is, non-singular linear transformations combined with translations). Geometrically, such transformations can be thought of as rotations, reflections, translations and shears, or any combination of these. Incidences, ratios of lengths on parallel lines and ratios of areas are preserved under affinities and hence belong to affine geometry; in contrast, lengths, angles and areas do not. All our results belong to affine geometry, but clearly remain valid in the more restrictive Euclidean geometry.

By a *polygon* $P = [V_1, V_2, \ldots, V_n]$ we mean a cyclic sequence of $n \geq 3$ points V_i in the affine plane, together with n closed line segments $S_i = [V_i, V_{i+1}]$. The points V_i are the *vertices* of P and the segments S_i are the *edges* of P. Here and throughout, all subscripts i are reduced modulo n so that they satisfy $1 \leq i \leq n$; moreover, to avoid special cases and degeneracies, we shall always assume that adjacent vertices V_i, V_{i+1} are distinct. A polygon is regarded as oriented and is unchanged by any cyclic permutation of the vertices; however, reversal of the order of the vertices produces a new polygon P', which we shall say is obtained from P by *reversing the orientation*.

A polygon with n vertices, and therefore n edges, is called an *n-gon*. Notice that these polygons are very general; non-adjacent vertices may coincide, and edges may cross or partially or wholly overlap. It is to such polygons that our theorems will apply. Sometimes it is necessary to place further restrictions on the positions of the vertices. These will be mentioned when appropriate.

It is convenient to introduce the notion of a *side* $V_i V_{i+1}$ of a polygon; this is the line containing the edge $[V_i, V_{i+1}]$. A *diagonal* $V_i V_j$ of a polygon is the line defined by two non-adjacent vertices V_i, V_j of P. It is defined if, and only if, V_i and V_j are distinct points[2].

We use uppercase letters $V, X, W, \ldots$, with or without subscripts, for points, and the corresponding lowercase letters $v, x, y, \ldots,$ for their position vectors relative to an arbitrarily chosen origin O. Position vectors can be used, in a perfectly rigorous manner, in affine geometry.

Suppose A, B, C, D are four points such that AB is parallel to CD. (In other words, the directions of the vectors $b - a$ and $d - c$ either coincide or are directly

[2] Notice that the symbols $[A, B]$ and $[A, B, C]$ differ in meaning from the frequently used $[AB] = \operatorname{conv}(A, B)$ and $\operatorname{conv}(A, B, C)$ in that the latter refer to *sets*, whereas the symbols used here also imply an *orientation* of each set.

opposite.) Let λ be the real number defined by $\lambda(d-c)=(b-a)$. Then λ is an affine invariant. It is simply the ratio of the length of $[A, B]$ to that of $[C, D]$, with a plus or a minus sign according to whether these line segments have the same, or opposite directions. We shall denote this ratio by $[A\ B/C\ D]=\lambda$. Clearly, relations such as

$$\left[\frac{A\ B}{C\ D}\right]=-\left[\frac{B\ A}{C\ D}\right]=-\left[\frac{A\ B}{D\ C}\right]=\left[\frac{B\ A}{D\ C}\right]$$

hold, and cancellation is permitted so that, for example,

$$\left[\frac{A\ B}{C\ D}\right]\cdot\left[\frac{C\ D}{E\ F}\right]=\left[\frac{A\ B}{E\ F}\right].$$

These properties follow directly from the definitions.

In particular, if a point W is defined in some prescribed manner on the side V_iV_{i+1} of a polygon P, and is distinct from both V_i and V_{i+1}, then $[V_iW/WV_{i+1}]$ will be referred to as an *edge-ratio* corresponding to that side. In a similar manner, if W lies on a diagonal V_iV_j, then $[V_iW/WV_j]$ is called a *diagonal-ratio*.

Similar considerations apply to areas. In the Euclidean plane we define the signed area $a(A, B, C)$ of a triangle $[A, B, C]$ to be its area (in the usual sense) prefixed by a plus or a minus sign. The $+$ sign is used if the triangle is positively oriented, that is, the vertices are named in a counterclockwise direction; and the $-$ sign is used if the triangle is negatively oriented, that is, the vertices are named in a clockwise direction. For two triangles $[A, B, C]$ and $[D, E, F]$ in the plane, the ratio $a(A, B, C)/a(D, E, F)$ of their signed areas is an affine invariant that will be denoted by $\lambda=[A\ B\ C/D\ E\ F]$. In terms of position vectors, λ is defined by $(a\times b+b\times c+c\times a)=\lambda(d\times e+e\times f+f\times d)$, where $\times$ signifies a vector (cross) product. Since all the points lie in a plane, these product vectors are all parallel. Equivalently, λ is defined by

$$\det\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{bmatrix}=\lambda\det\begin{bmatrix} d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \\ 1 & 1 & 1 \end{bmatrix},$$

where (a_1, a_2) are the coordinates of the point A, that is, the components of the position vector a, etc.

Since cyclic permutation of vertices does not change area, but reversing the order of the vertices changes the sign of the area, relations such as

$$\left[\frac{A\ B\ C}{D\ E\ F}\right]=\left[\frac{B\ C\ A}{D\ E\ F}\right]=\left[\frac{C\ A\ B}{D\ E\ F}\right]=-\left[\frac{B\ A\ C}{D\ E\ F}\right]=\left[\frac{B\ C\ A}{E\ F\ D}\right]=\left[\frac{B\ A\ C}{E\ D\ F}\right]$$

hold. Moreover, as in the case of line segments, cancellation is permitted; for example,

$$\left[\frac{A\ B\ C}{D\ E\ F}\right]\cdot\left[\frac{D\ E\ F}{G\ H\ J}\right]=\left[\frac{A\ B\ C}{G\ H\ J}\right].$$

In this notation the area principle (1) may be written in the slightly more powerful form

$$\left[\frac{A_1P}{A_2P}\right]=\left[\frac{A_1BC}{A_2BC}\right],$$

which takes account of the signed lengths and signed areas of the triangles.

The *signed area* of an n-gon ($n > 3$) can also be defined; we triangulate the n-gon in any way, and orient the triangles *coherently* (see FIGURE 5). This means that whenever two triangles of the triangulation have an edge $[B, C]$ in common, then their orientations are such that they induce opposite directions on this common edge. The area of the polygon is then defined as the sum of the signed areas of the component triangles. For the validity of this definition it is, of course, essential to show that the signed area is independent of the triangulation used. This is a routine calculation, and we do not give details here. In fact, if one interprets the polygon as an oriented curve, its signed area is exactly what one obtains by applying to it the familiar integrals of calculus.

We invite the reader to adapt the proofs given above using signed areas and the new notation, and so provide general proofs for the theorems we discussed. In particular, if

$$\left[\frac{V_iW_i}{W_{i+1}V_{i+2}}\right] = \left[\frac{V_iV_{i+1}V_{i+4}}{V_iV_{i+1}V_{i+2}V_{i+4}}\right] \cdot \left[\frac{V_iV_{i+1}V_{i+2}V_{i+3}}{V_{i+2}V_{i+3}V_{i+1}}\right], \tag{5a}$$

is used instead of (5), a proof of Hoehn's Theorem is obtained that does not depend on the convexity, simplicity or orientation of the pentagon $[V_1, V_2, V_3, V_4, V_5]$.

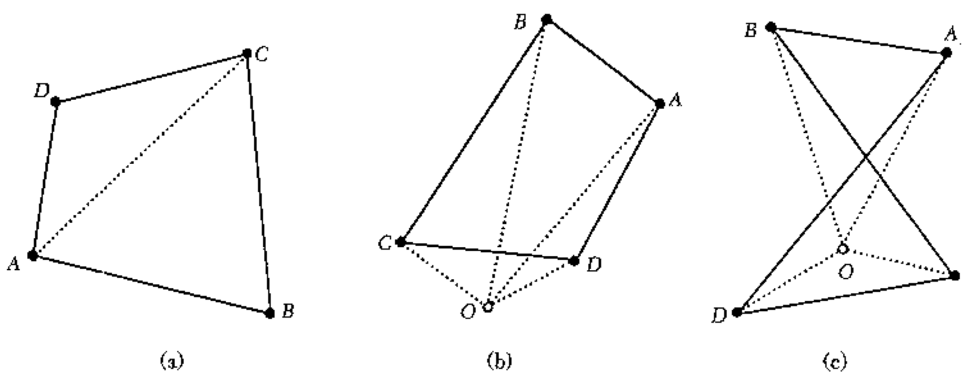


FIGURE 5

Three examples of coherent triangulations of a quadrangle $[A, B, C, D]$. In (a), the quadrangle is triangulated into two triangles $[A, B, C]$ and $[A, C, D]$. In (b) and (c) the triangles are $[O, A, B]$, $[O, B, C]$, $[O, C, D]$, $[O, D, A]$.

3. Generalizations of the Theorems of Ceva and Menelaus

In this and the following section we shall define points W_i on sides or diagonals of an n-gon as points of intersection with other lines. Throughout we shall assume, without further remark, that these points exist (the lines concerned are not parallel) and are distinct from the vertices (so that the relevant edge- and diagonal-ratios are well-defined). This section is concerned with problems of the following type. On each side V_iV_{i+1} of an n-gon, a point W_i is defined in some geometrically meaningful way. Under what circumstances is it possible to make an assertion about the value of the product of the edge-ratios $[V_iW_i/W_iV_{i+1}]$? We shall also consider similar problems for diagonal-ratios.

Our first result of this kind is the generalization of Menelaus' Theorem to n-gons. This is not new—see Section 5. The case $n = 5$ is shown in FIGURE 6.

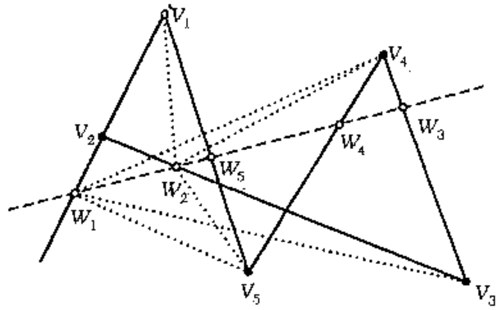


FIGURE 6

Menelaus' Theorem for a pentagon states that

$$\left[\frac{V_1W_1}{W_1V_2}\right]\cdot\left[\frac{V_2W_2}{W_2V_3}\right]\cdot\left[\frac{V_3W_3}{W_3V_4}\right]\cdot\left[\frac{V_4W_4}{W_4V_5}\right]\cdot\left[\frac{V_5W_5}{W_5V_1}\right]=-1.$$

THEOREM 1 (Menelaus' Theorem for n-gons). *Let $P=[V_1,\dots,V_n]$ be an n-gon and suppose that, for $i=1,\dots,n$, a transversal cuts the side V_iV_{i+1} in W_i. Then the product of the edge-ratios is constant, in fact*

$$\prod_{i=1}^{n}\left[\frac{V_iW_i}{W_iV_{i+1}}\right]=(-1)^n \tag{6}$$

for all n-gons P.

In the case where n is odd, the product is negative, which shows that an odd number of intersections of the transversal with the sides V_iV_{i+1} of the n-gon must be *external*, that is, do not lie on the edge $[V_i,V_{i+1}]$. This is, of course, a familiar fact in the case $n=3$; it is known as *Pasch's axiom* [**3**, §12.2]. If n is even, an even number (possibly zero) of intersections of the transversal with the sides of the n-gon must be external.

Proof. Select any two of the points W_i, say W_1, W_2 (see FIGURE 6). Then the area principle for triangles with base $[W_1,W_2]$ yields

$$\left[\frac{V_iW_i}{W_iV_{i+1}}\right]=-\left[\frac{V_iW_1W_2}{V_{i+1}W_1W_2}\right].$$

Substituting for each of the n factors on the left side of (5) we obtain a product of terms each of which is the quotient of the areas of triangles. These cancel to yield the value $(-1)^n$ as required.

A trivial variation of this theorem applies to diagonals instead of edges. If $1<j<n$, and W_i is defined as the intersection of a transversal with the diagonal V_iV_{i+1} for $i=1,\dots,n$, then

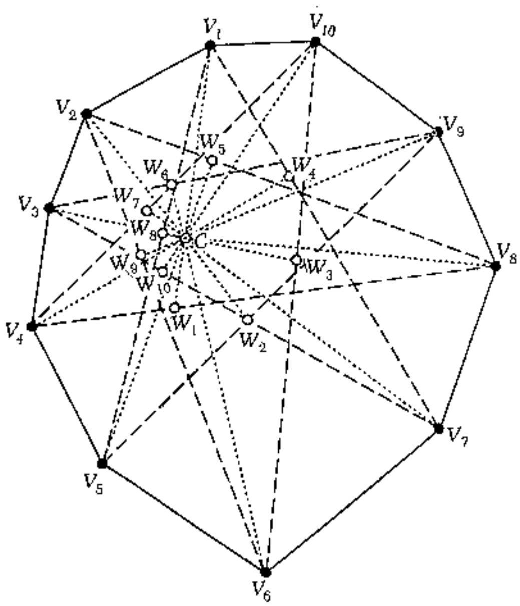
$$\prod_{i=1}^{n}\left[\frac{V_iW_i}{W_iV_{i+j}}\right]=(-1)^n.$$

Despite the apparent generality, this can easily be deduced from Theorem 1 by renaming the vertices $V_1,V_2,V_3,\dots$ as $V_i,V_{i+j},V_{i+2j},\dots$. This renumbering may (if j is not prime to n) split the original polygon into two or more polygons, but this does not affect the result or its proof.

THEOREM 2 (Ceva's Theorem for n-gons). *Let $P=[V_1,\dots,V_n]$ be an arbitrary n-gon, C a given point, and k a positive integer such that $1\le k<n/2$. For $i=1,\dots,n$ let W_i be the intersection of the lines CV_i and $V_{i-k}V_{i+k}$. Then,*

$$\prod_{i=1}^{n}\left[\frac{V_{i-k}W_i}{W_iV_{i+k}}\right]=1. \tag{7}$$

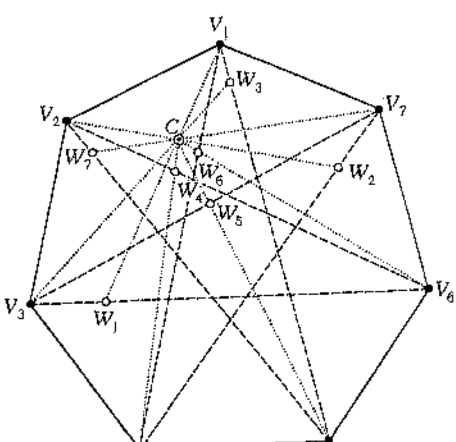
Note that this result concerns diagonal-ratios except in the case n odd and $k=(n-1)/2$. The latter is the only generalization of Ceva's Theorem that we have been able to find in the literature (see, for example, [**6**, p. 86]). However, the result of Theorem 2 is not as general as it appears at first sight. If n and k have highest common factor $d>1$ then the product on the left side of (7) can be split into d products, each with n/d factors, and each of these factors has the value 1. If n and $2k$ are relatively prime then it is possible (by renumbering the vertices as in the case of Menelaus' Theorem) to transform the left side of (7) into a product of n edge-ratios for an appropriate star n-gon. Hence the only genuinely new assertion in Theorem 2 is in the case where n is even and k is prime to n. An example of this case ($n=10, k=3$) is shown in FIGURE 7(a). Other examples appear in FIGURE 7(b) with diagonal-ratios ($n=7, k=2$) and in FIGURE 7(c) with edge-ratios ($n=5, k=2$).



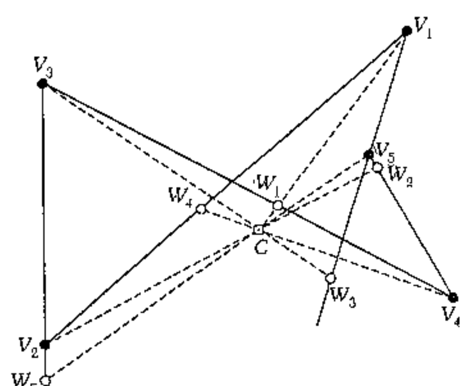
(a)

FIGURE 7

Examples of Ceva's Theorem for n-gons in the cases (a) $n=10, k=3$; (b) $n=7, k=2$; and (c) $n=5, k=2$. In cases (a) and (b) the theorem makes an assertion about diagonal-ratios $[V_iW_{i+3}/W_{i+3}V_{i+6}]$ and $[V_iW_{i+2}/W_{i+2}V_{i+4}]$, respectively, and in (c) an assertion about edge-ratios $[V_iW_{i+3}/W_{i+3}V_{i+1}]$.



(b)



(c)

Proof. The proof follows similar lines to that of Theorem 1. We observe that, applying the area principle to triangles with base $[C, V_i]$ we obtain, for $i = 1, \ldots, n$,

$$\left[\frac{V_{i-k}W_i}{W_iV_{i+k}}\right] = \left[\frac{CV_iV_{i-k}}{CV_{i+k}V_i}\right].$$

Substituting these terms in the left side of (6), we obtain a product of n terms each of which is a quotient of the areas of the triangles. These cancel to yield the value 1 as required.

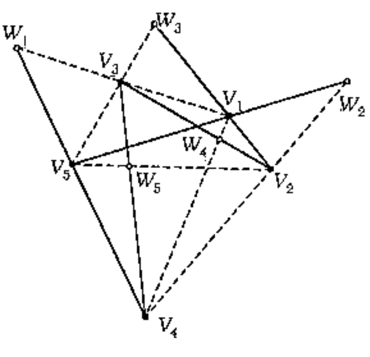
In each of the above theorems we have defined W_i as the intersection of a side or diagonal (a *basis*) with a line CD (the corresponding *transversal*). Menelaus' Theorem corresponds to the case where C and D are fixed points that are not vertices, so the transversal CD is a fixed line for all bases. In Ceva's Theorem C is a fixed point and D is a vertex V_i of the polygon P. Then W_i is the point of intersection of the basis $V_{i-k}V_{i+k}$ with the transversal CV_i. These remarks suggest that there may exist corresponding results for products of edge- or diagonal-ratios when the transversal is determined by two vertices of P. Indeed, we have the following theorem, which appears to be new.

THEOREM 3 (The Selftransversality Theorem). *Let j, r, s be integers distinct* (mod n) *and let W_i be the point of intersection of the basis (side or diagonal) V_iV_{i+j} of the n-gon $P = [V_1, \ldots, V_n]$ with the transversal $V_{i+r}V_{i+s}$. Then a necessary and sufficient condition for*

$$\prod_{i=1}^{n}\left[\frac{V_iW_i}{W_iV_{i+j}}\right] = (-1)^n \tag{8}$$

is that either (i) $n = 2m$ *is even,* $j \equiv m$ *and* $s \equiv r + m$; *or that n is arbitrary and either* (ii) $s \equiv 2r$ *and* $j \equiv 3r$; *or* (iii) $r \equiv 2s$ *and* $j \equiv 3s$. *All congruences are* mod n.

In case (i) the terms in (8) cancel in pairs so the statement of the theorem becomes trivial. Cases (ii) and (iii) are essentially the same since r and s play a symmetrical role in determining the transversal V_{i+r}, V_{i+s}, and so may be interchanged.



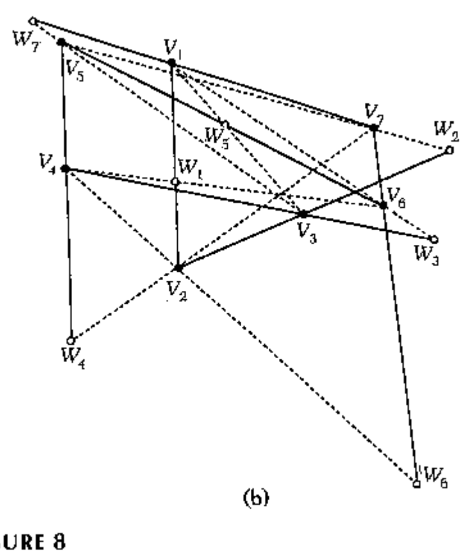


FIGURE 8

Examples of the Selftransversality Theorem for (a) $n = 5, j = 2, r = 3, s = 4$; and (b) $n = 7$, $j = 1, r = 3, s = 5$. In each case the theorem shows that the product of the n ratios $[V_iW_i/W_iV_{i+j}]$ (where $i = 1, \ldots, n$) is -1.

For example, in FIGURE 8(a), $n=5, j=2, r=3, s=4$, and in FIGURE 8(b), $n=7$, $j=1, r=3, s=5$ so each is an instance of case (iii); since n is odd, the sign on the right side of (8) is negative.

Proof. Using the area principle for triangles with base $[V_{i+r}, V_{i+s}]$ and apexes V_i and V_{i+j}, we obtain

$$\left[\frac{V_iW_i}{W_iV_{i+j}}\right] = -\left[\frac{V_iV_{i+r}V_{i+s}}{V_{i+j}V_{i+r}V_{i+s}}\right]. \tag{8a}$$

We substitute these expressions for each of the n factors on the left side of (8) and determine when exactly the same triangles occur in both the numerator and denominator and so their areas (as expressed in terms of determinants) cancel to yield the value ± 1 as required. The term $V_iV_{i+r}V_{i+s}$ in the numerator will cancel with the term $V_{(i+h)+j}V_{(i+h)+r}V_{(i+h)+s}$ in the denominator if, and only if, either

(i) $h \equiv -j,\ r \equiv s-j$ and $s \equiv r-j$; or

(ii) $h \equiv -r,\ r \equiv s-r$ and $s \equiv j-r$; or

(iii) $h \equiv -s,\ s \equiv r-s$ and $r \equiv j-s$.

These alternatives correspond to the three cases given in the statement of the theorem. Notice that each cancellation produces the factor -1 in case (i) and $+1$ in the other two cases, leading to the term $(-1)^n$ on the right side of (8). Thus the theorem is proved.

It is of interest to determine, for a given basis, the corresponding transversals for which the theorem holds. Let us consider case (ii). Writing $j+t$ instead of r and $-u$ instead of s, it is easily verified that the given condition is equivalent to $u \equiv t$ and $2j+3t \equiv 0 \pmod n$. This can be solved explicitly for t (and so r and s determined) in terms of j as follows:

(ii_a) If $n \equiv 3k+1$, then $t \equiv 2kj$ for $j = 1, 2, \ldots, [(n-1)/2]$.

(ii_b) If $n \equiv 3k-1$, then $t \equiv (k-1)j$ for $j = 1, 2, \ldots, [(n-1)/2]$.

(ii_c) If $n \equiv 3k$, then $j \equiv 3i$ and $t \equiv -2i,\ k-2i$ or $2k-2i$, for $i = 1, 2, \ldots, [(k-1)/2]$.

For example, in FIGURE 9 we show the case $n=9$, $k=3$ and $i=1$. The transversals V_2V_3, V_6V_8 and V_5V_9 are indicated by dotted lines and the basis V_1V_4 by a dashed line. All these satisfy the conditions of the theorem. (The other eight bases that occur in (8) are not shown since to indicate these would make the figure so complicated as to be unintelligible.)

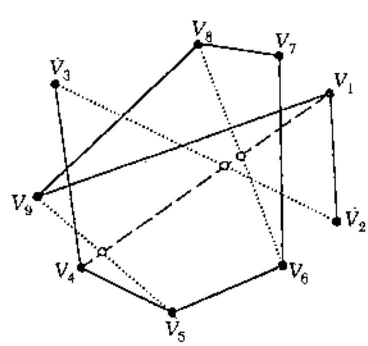


FIGURE 9

The three transversals V_2V_3, V_6V_8, V_5V_9 of the basis V_1V_4 that satisfy the conditions of the Selftransversality Theorem for $n=9$.

4. Generalizations of Hoehn's Theorem

In this section, we consider theorems similar to those in the previous section in that, for every n-gon, the product of n ratios of lengths of line segments has some constant value, namely $+1$ or -1. However, we now consider those cases where, as in Hoehn's Theorem, the line-segments lie on the same line but are not contiguous. Thus we define two points W_i, Z_i on certain edges or diagonals $[V_i, V_j]$ of an n-gon, and consider products of the form

$$\prod_{i=1}^{n}\left[\frac{V_iW_i}{Z_iV_j}\right].$$

THEOREM 4 (Hoehn's First Theorem for n-gons). *Let $P=[V_1,\dots,V_n]$ be a given n-gon and j an integer such that, for each i, the integers $i, i+j, i+2j, i+3j, i+4j$ are distinct* (mod n). *Define W_i (for $i=1,\dots,n$) to be the point of intersection of V_iV_{i+2j} and $V_{i+j}V_{i+3j}$. Then the points W_i and W_{i+j} lie on the line $V_{i+j}V_{i+3j}$ and*

$$\prod_{i=1}^{n}\left\{\frac{V_{i+j}W_i}{W_{i+j}V_{i+3j}}\right\}=1. \tag{9}$$

It will be observed that if $V_i, V_{i+j}, V_{i+2j}, V_{i+3j}, V_{i+4j}$ are not distinct points, then the identity becomes meaningless or trivial. Further, it is only necessary to consider the cases $j=1,2,\dots,[(n-1)/2]$ since other values of j lead to repetitions of the same result.

FIGURE 10 illustrates the theorem for $n=7, j=1,2$ and 3. The original statement (3) of Hoehn's Theorem (see FIGURE 4) corresponds to the case $n=5, j=1$.

Proof. For the first assertion we note that W_i is the intersection of V_iV_{i+2j} and $V_{i+j}V_{i+3j}$, and W_{i+j} is the intersection of $V_{i+j}V_{i+3j}$ and $V_{i+2j}V_{i+4j}$. Hence both of these points lie on $V_{i+j}V_{i+3j}$ as stated.

For the second assertion we note that, as in (5a), using triangles with bases $[V_i, V_{i+2j}]$ and $[V_{i+2j}, V_{i+4j}]$ we obtain

$$\left[\frac{V_{i+j}W_i}{W_{i+j}V_{i+3j}}\right]=\left[\frac{V_iV_{i+j}V_{i+2j}}{V_{i+2j}V_{i+3j}V_{i+4j}}\right]\cdot\left[\frac{V_{i+2j}V_{i+3j}V_{i+4j}V_{i+j}}{V_iV_{i+j}V_{i+2j}V_{i+3j}}\right].$$

Inserting these expressions in the left side of (9) we obtain a product in which the areas of the triangles, as well as the areas of the quadrilaterals, all cancel. Hence the product has the value 1, and the theorem is proved.

THEOREM 5 (Hoehn's Second Theorem for n-gons). *Let $P=[V_1,\dots,V_n]$ be an n-gon, and j,k positive integers such that $j+2k=n$, and for each $i=1,\dots,n$, the integers $i, i+k, i+j, i+j+k$ are distinct* (mod n) *and the integers $i, i+k, i+2k, i+3k$ are also distinct* (mod n). *Define W_i as the intersection of V_iV_{i+k} and $V_{i+j}V_{i+j+k}$. Then the points W_i and W_{i+2k} lie on the line V_iV_{i+k} and*

$$\prod_{i=1}^{n}\left[\frac{V_iW_i}{W_{i+2k}V_{i+k}}\right]=1. \tag{10}$$

Proof. For the first assertion, we note that W_i is the intersection of V_iV_{i+k} and $V_{i+j}V_{i+j+k}$, and W_{i+2k} is the intersection of $V_{i+2k}V_{i+3k}$ and $V_{i+j+2k}V_{i+j+3k}$, which is the same line as V_iV_{i+k} since $j+2k=n$.

For the second assertion, we again use the area principle, as in (5a), for triangles with bases $[V_{i+j}, V_{i+j+k}]$ and $[V_{i+2k}, V_{i+3k}]$ to yield

$$\left[\frac{V_iW_i}{W_{i+2k}V_{i+k}}\right] = \left[\frac{V_iV_{i+j}V_{i+j+k}}{V_{i+k}V_{i+2k}V_{i+3k}}\right] \cdot \left[\frac{V_{i+k}V_{i+2k}V_iV_{i+3k}}{V_iV_{i+j}V_{i+k}V_{i+j+k}}\right].$$

Substituting in the left side of (10) and using the relation $j + 2k = n$ we see that the areas of the triangles, and the areas of the quadrilaterals in the resulting product, all cancel to yield the value 1. This proves the theorem.

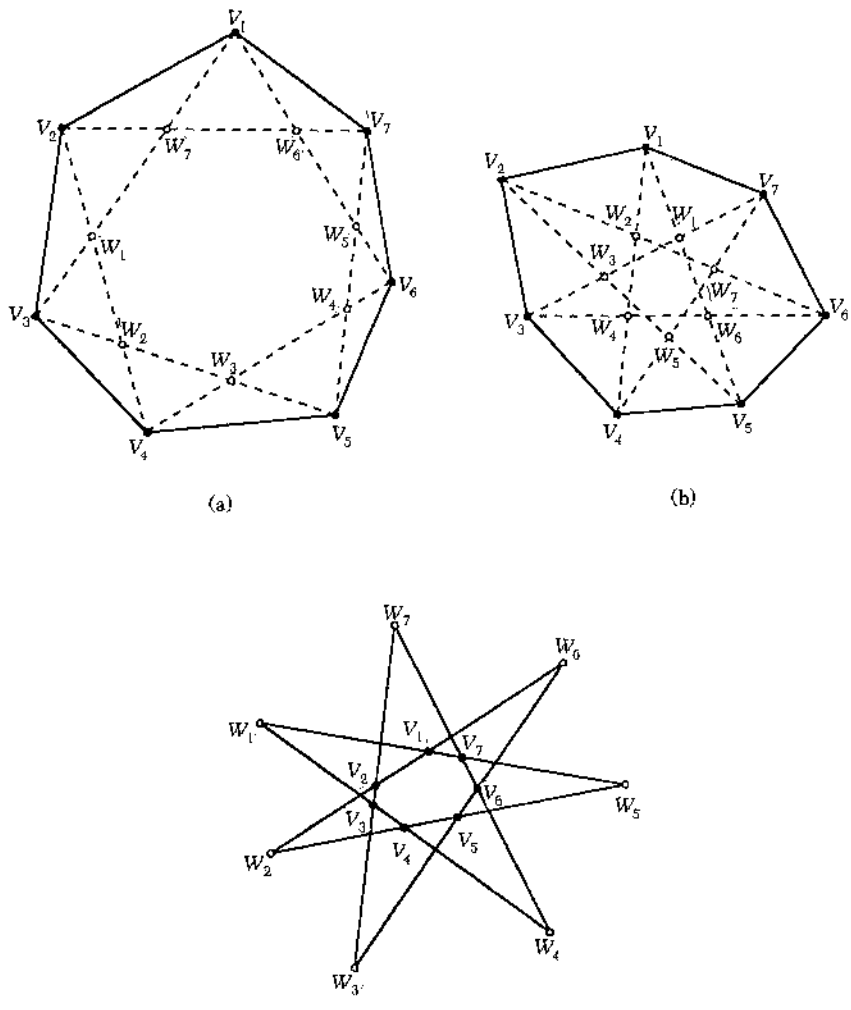


FIGURE 10

Examples of Hoehn's First Theorem for n-gons, for $n = 7$ and (a) $j = 1$; (b) $j = 2$; and (c) $j = 3$. In each case the product of the seven ratios $[V_{i+j}W_i/W_{i+j}V_{i+3j}]$ (for $i = 1, \ldots, 7$) takes the value 1.

The theorem is illustrated in FIGURE 11 for $n = 7$ and $(j, k) = (1, 3)$, $(3, 2)$ and $(5, 1)$. The original statement (4) of Hoehn's Theorem corresponds to the case $n = 5, j = 1, k = 2$.

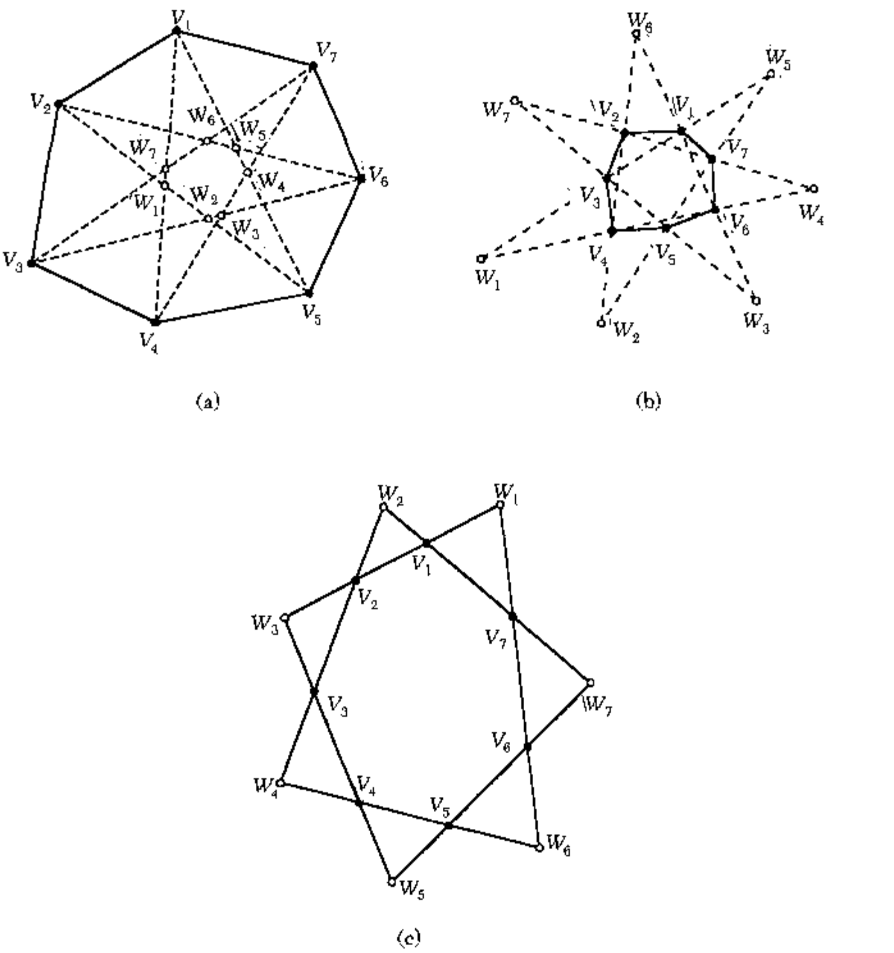


FIGURE 11

Hoehn's Second Theorem for n-gons, for $n = 7$ and (a) $j = 1, k = 3$; (b) $j = 3, k = 2$; and (c) $j = 5, k = 1$. In each case the product of the seven ratios $[V_iW_i/W_{i+2k}V_{i+k}]$ (for $i = 1, \ldots, 7$) takes the value 1.

5. Comments

Menelaus' Theorem for n-gons is not new—it has been known since the early nineteenth century, see [**2**, p. 295], [**6**, p. 75], [**7**, p. 63], [**9**, p. 75]. As mentioned above, special cases of Ceva's Theorem for n-gons have also been established; see, for example, [**6**, p. 86], [**7**, p. 64].

The most interesting and unexpected feature of the results of this paper is *not* that the various products of edge- and diagonal-ratios are equal to $+1$ or -1, but that the values of these products are independent of the polygon P from which we start the construction (subject to the restrictions stated at the beginning of Section 3). In view of this, one might suppose that all products of n ratios in an n-gon have the same property; however, this is not the case. For example, in FIGURE 4, the value of

$$\prod_{i=1}^{5}\left[\frac{W_iW_{i+1}}{V_iV_{i+2}}\right] \tag{11}$$

depends on the pentagon P which is chosen—in spite of the fact that such a product appears, at first sight, to be very similar to those that occur in the original statement of Hoehn's Theorem. That (11) is not constant can be shown by a little experimentation with numerical examples. In fact, we suspect that the product (11) attains its maximum value when $P=[V_1,\ldots,V_5]$ is an affine image of a regular pentagon. We have no proof of this statement, and a discussion of problems of this nature does not seem appropriate here.

It is not difficult to see that Theorems 4 and 5 cover all the cases mentioned in the introduction to Section 4, namely those in which a product of n factors takes a constant value that does not depend on the choice of the original polygon P. If we extend our investigation to consider products of $2n, 3n, 4n, \ldots$ factors (each of which is the quotient of lengths of line-segments in an n-gon) then many more possibilities arise. The reader may wish to investigate these. However, the results we have stated here will illustrate the versatility and power of the area principle in proving results of this nature.

The case $n=4, j=2, k=1$ of Theorem 5 deserves special mention. It is illustrated in FIGURE 12, and it is clear that it can be interpreted as stating that the product of certain ratios of lengths in a complete quadrilateral equals 1. Although complete quadrilaterals have been thoroughly investigated for two centuries, we were unable to find any mention of this particular result.

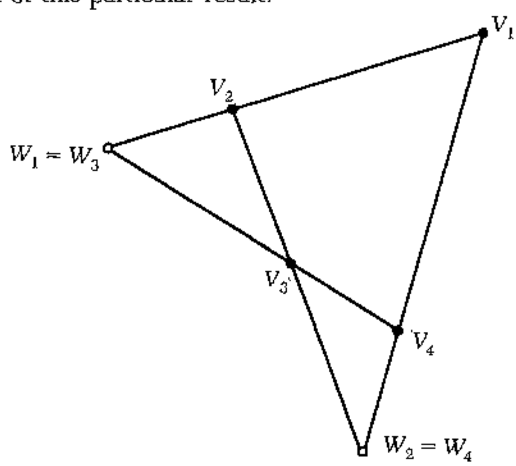


FIGURE 12

The case $n=4, j=2, k=1$ of Hoehn's Second Theorem. The product of the four ratios $[V_iW_i/W_{i+2}V_{i+1}]$ (for $i=1,2,3,4$) equals 1.

Since all the theorems in this paper are so straightforward and their proofs so elementary, it is surprising that they were not discovered two or three centuries ago. This may be partly explained by the fact that, unlike us, earlier authors did not have the advantages of modern technology. When we started this investigation we were, of course, familiar with Ceva's and Menelaus' Theorems for triangles; it was Hoehn's Theorem that suggested to us that it might be worthwhile to investigate products of other ratios in n-gons with $n > 3$. Using a simple program in Mathematica® we were able to accurately calculate the values of circular products of various ratios for large numbers of n-gons, and the results suggested our Theorems 1 to 5. (Later on we discovered that the n-gonal forms of Ceva's and Menelaus' Theorems are known and can be found in the literature. However, so far we were unable to find any result that can be interpreted as even a particular case of our Selftransversality Theorem.) We observed that the "area principle" was an easy method of proving (in the traditional sense) all the results that had been discovered empirically. Further, it suggested other possibilities as well as enabling us to state the theorems for n-gons with arbitrary n. It also led to higher-dimensional generalizations, which will be presented elsewhere.

Note. Since the completion of the manuscript we have learned from Baptist [**1**, p. 61] that the "area principle" was used—without any special name—in Crelle's proof of Ceva's Theorem [**5**]. But its general utility was not noticed and so it was, for all practical purposes, completely forgotten.

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Carl B. Allendoerfer Awards 1995

The Carl B. Allendoerfer Awards, established in 1976, are made to authors of expository articles published in *Mathematics Magazine*. The awards are named for Carl B. Allendoerfer, a distinguished mathematician at the University of Washington and President of the Mathematical Association of America, 1959–1960.

Lee Badger
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***Mathematics Magazine* 67 (1994), 83–91.**

This interesting article combines a famous problem in probability with statistical methods of data analysis that can demonstrate fraud—or what the author more charitably calls "hoaxes." The example is concrete, the mathematics is rich and detailed, and the material is accessible to students with no more than a calculus background. There are broader lessons for the reader too: rigged data are a fact of life, but so are the statistical tools that can detect them.

Tristam Needham
"The Geometry of Harmonic Functions"
***Mathematics Magazine* 67 (1994), 92–108.**

The geometric approach of this carefully crafted and well-written article is enlightening; there is much here for the reader to learn about the interplay between geometry and analysis. In particular, we see how material almost invariably treated by analytic methods can be rethought—perhaps in ways closer to those used when the ideas were being developed. The author also helps us to understand what something means and not just why it might be possible to prove it true. In this article, geometry becomes a powerful tool for conveying meaning.