HOW TO CONVEXIFY A POLYGON

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Dedicated to Paul Erdös, on the sixtieth anniversary of his problem that started the whole topic

Assume that we are given a simple polygon P (that is, a polygon without selfintersections) which is nonconvex. Then there is a line L determined by vertices A and B of P such that P is contained in one of the closed halfspaces determined by L, but the segment [A, B] meets no points of P other than A and B. Figure 1(a) illustrates this situation. We shall call (A, B) an *exposed pair* of vertices of P. Then [A, B] together with one of the polygonal arcs of P determined by A and B forms a closed polygon, which encloses the other arc of P determined by A and B. Call these two arcs P_1 and P_2. The polygon Q = f(P; A, B) is obtained by "flipping" (reflecting) the arc P_2 about L, and joining it with the unmodified arc P_1. This is illustrated in Figure 1(b).

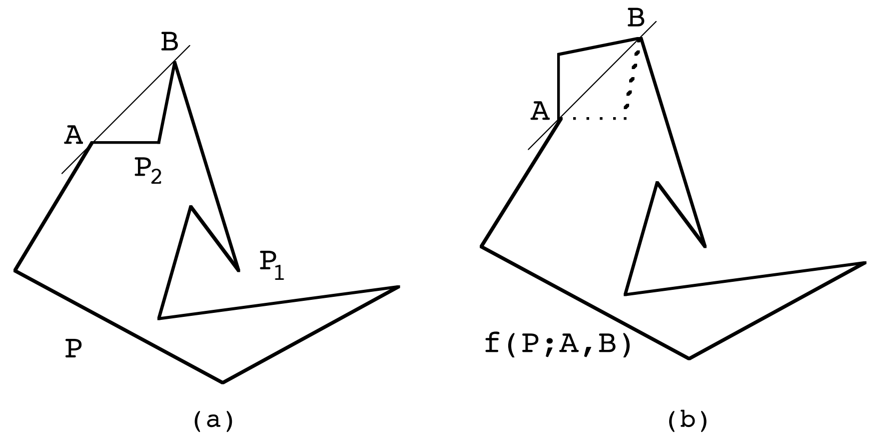

Figure 1. The transformation of a non-convex polygon by "flipping" of an arc determined by an exposed pair of vertices.

Theorem 1. Every simple polygon can be transformed into a convex polygon by a finite sequence of flips about lines determined at each stage by exposed pairs.

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Theorem 1 is illustrated by the example in Figure 2.

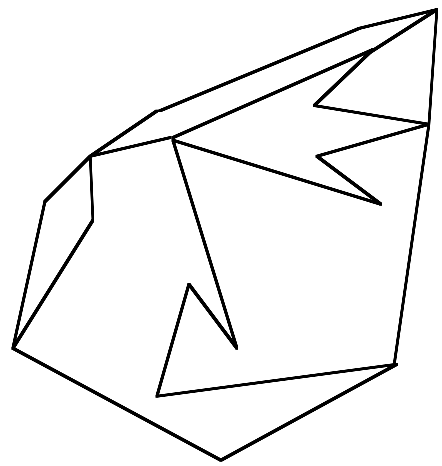


Figure 2. A convex polygon obtained by a finite number of flips from the polygon in Figure 1(b).

Proof. Let us denote the given polygon by $P = P_0$; if P_i has already been found and is nonconvex, let $P_{i+1} = f(P_i; A_i,B_i)$, where (A_i, B_i) is an exposed pair of P_i. For definiteness, let us at each step choose such an exposed pair (A_i, B_i) that the increase in the area enclosed by the polygon is as large as possible. Since the perimeter of all polygons P_i is the same, all P_i's are contained in a fixed circle. It follows that if the sequence of distinct P_i's is infinite (thus contradicting the assertion of the theorem) one can choose a subsequence of the polygons such that corresponding vertices of the polygons form sequences that converge to points of the limit polygon P^*. Due to the choices of the exposed pairs as maximizing the area, the polygon P^* is convex; each vertex of P^* is a limit point of vertices of the P_i's, but some of the limits may be interior points of sides of P^*. Moreover, since each flip either increases or leaves unchanged the distance from a vertex to any point inside the polygon, it follows that P^* is, in fact, the limit of the complete sequence of polygons P_i, without the need to select a convergent subsequence. We shall now show that P^* is obtained, in fact, already after a finite number of flips.

Each vertex V_k of P^* (that is, a point of P^* at which the two sides incident with it form an angle strictly less that $180°$) is a limit of vertices $V_{k,i}$ of the polygons P_i. Therefore there exist a positive ε small enough so that a circle C, centered at V and of radius ε, can be separated by a straight line $L(V_k)$ from the family of circles of radius ε

centered at the limit points of all the other vertices of the polygons P_i. If $V_{k,i(k)}$ is the first among the vertices $V_{k,i}$ inside C, then $V_{k,i(k)}$ will coincide will all the later $V_{k,i}$'s since the existence of the separating line $L(V_k)$ guarantees that it will not belong to an arc that will be flipped. Hence $V_k = V_{k,i(k)}$ is immobilized after i(k) steps, thus all of the vertices of P* are reached already after i* = max{ i(k) } steps. Since the limit points of the polygons P_i that converge to non-vertex points of P* are also immobilized after i* steps, it follows that the convexification is complete after the finite number i* of flips. ◊

One of the reasons for bringing up the topic of convexification is its strange history. As far as I know, it began (more or less) as a problem posed by Paul Erdös in 1935 [2]. Since then it has been solved four times, with none of the solvers aware of the others, and none except the first aware of Erdös's problem. The first solution is that of B. de Sz.–Nagy [6] in 1939. He begins by showing that the problem as posed by Erdös needs to be amended: Erdös asked whether a finite number of steps will convexify every simple polygon, where a "step" consists of *simultaneously* carrying out the flips about *all* exposed pairs of vertices. Sz.–Nagy observed that such a step may lead to non-simple polygons (see Figure 3), and changed the problem to require that flips be carried out one at a time. He than proceeded to solve it; the proof given above is essentially the one in [6]. Our Theorem 1 next surfaces in apparently simultaneous papers by Reshetnyak [5] and by Yusupov [7] in 1957; neither mentions any sources or references for the question, and the proofs — like those to be mentioned — differ in details from the above but follows similar ideas.

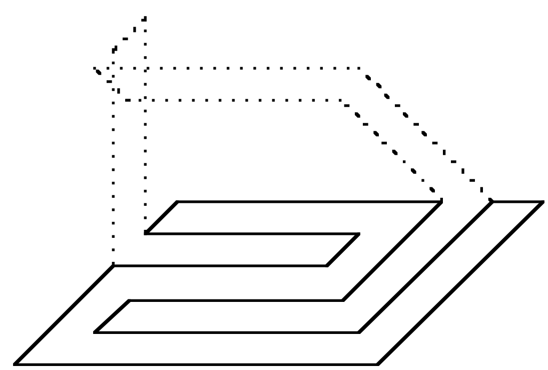


Figure 3. An example in which simultaneous flips result in a non-simple polygon.

Finally, a proof by R. H. Bing and N. D. Kazarinoff was published in two venues: on pages 30 – 34 of Kazarinoff's book [3] and in the joint paper [1], both published in 1961; an abstract [4] announcing their work appeared already in 1959. No sources or references are given in any of these publications, except that in [1] (but not in [3]) it is mentioned in a concluding remark (that possibly was added in proof) that the proof of Theorem 1 given in [6] is invalid; there is no basis for this claim. In all three publications by Bing and Kazarinoff it is conjectured that the convexification of every polygon with n sides is achieved after at most $2n$ flips. In [3], the discussion of the topic end with the following line:

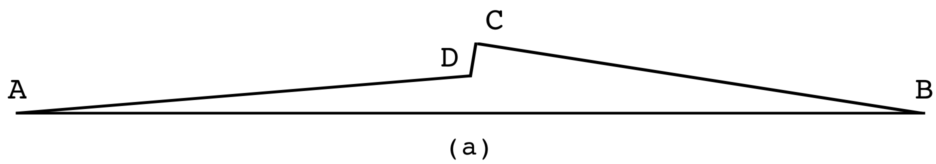
"Can you prove or disprove this conjecture? Paul Erdös did."

I am not aware of the reason for this statement, and I do not know what Erdös did in this context; there appears to be no further mention of the convexification question in Erdös' writings after [2].

There was one further development that should be reported, with rather sad outcome. In 1973 or 1974, R. R. Joss and R. W. Shannon (at that time graduate students at the University of Washington) found a simple counterexample to the Bing–Kazarinoff conjecture; it will be given below, as Theorem 2. They also proved what we shall formulate as Theorem 3, which is a different way to convexify a simple polygon. They sent a preprint to Kazarinoff, but he answered that their discussion of the conjecture, and of the underlying flips, is "totally confusing", and that they do not deal at all with the process he and Bing had in mind. This apparently discouraged Joss and Shannon sufficiently so that they never published their results. (Although both were mathematically talented, and obtained their Ph.D. from the University of Washington, neither seems to be mathematically active — at least as far as publications go — at the present time.) Since it seems a shame that their results be lost, as the second aim of this paper I shall present the two achievements of Joss and Shannon.

Theorem 2. For any positive integer m there exist simple quadrangles that cannot be convexified by fewer that m flips.

Outline of the proof. Consider the quadrangle P with vertices $A = (0, 0)$, $B = (2, 0)$, $C = (2 - \cos 2\varphi, \sin 2\varphi)$, $D = (\cos \varphi, \sin \varphi)$, where φ is a small angle. See Figure 4(a). Then P has only one exposed pair, A and C, and all derived polygons will also have only a single possible exposed pair. Since the images of C and D remain on unit circles centered at B and A (see Figure 4(b)), the number of flips needed to reach a convex polygon will clearly increase without bound with decreasing angle φ. ◊



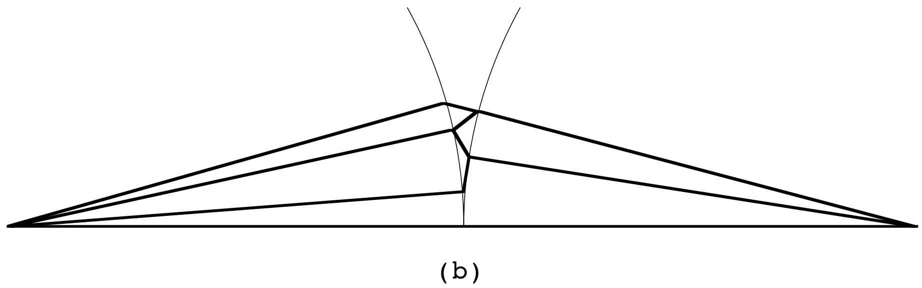


Figure 4. Illustration of the construction of Joss and Shannon.

In order to formulate the second result of Joss and Shannon, define a "flipturn" in the same way as a flip (that is, starting from an exposed pair of vertices A and B of a polygon P), but instead of reflecting the smaller arc of P in the line determined by A and B, we transform that arc by a halfturn about the midpoint of [A, B]; see Figure 5.

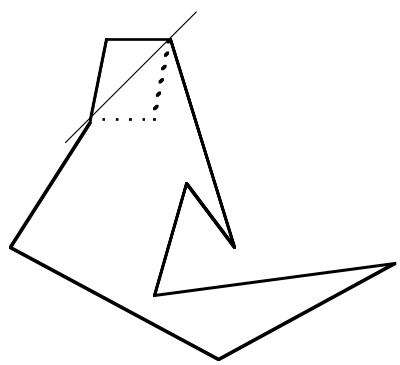


Figure 5. A flipturn determined by the exposed pair in Figure 1.

In contrast to the situation described by Theorems 1 and 2, we now have:

Theorem 3. Any polygon with n sides can be convexified by a sequence of at most (n-1)! flipturns.

Proof. If Q results from P by a flipturn, the sides of P and Q when considered as vectors consist of precisely the same vectors, and differ only in their cyclic order. Since there are at most (n-1)! cyclic orders, and since every flipturn strictly increases the area so that no permutation of the vectors can appear twice, it follows that there are at most (n-1)! steps prior to reaching a convex polygon. ◊

It seems rather obvious that the bound (n-1)! in Theorem 3 is much too high. Joss and Shannon concluded their manuscript by conjecturing that $\frac{1}{4}n^2$ flipturns are always sufficient. This conjecture is still open.

Another open question is whether one can start with a polygon which is not necessarily simple, and convexify it in a finite number of steps (either in the flips version, or in the flipturn version). Concerning the former, there is a remark in [BK] saying that "simplicity of the polygon is not necessary". However, no justification of this claim was given; it seems likely that at least some restrictions have to be placed on the polygons under consideration.

References.

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[2] P. Erdös, Problem 3763. Amer. Math. Monthly 42(1935), p. 627.

[3] N. D. Kazarinoff, *Analytic Inequalities*. Holt, Rinehart & Winston, New York 1961.

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[5] Yu. G. Reshetnyak, On a method of transforming a non-convex polygonal line into a convex one. [In Russian]. Uspehi Mat. Nauk (N.S.) 12(1957), No 3(75), pp. 189 – 191.

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