THE ANGLE-SIDE RECIPROCITY OF QUADRANGLES

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Everybody knows that triangles are either equilateral, or isosceles, or scalene — depending on whether all sides have the same length, or two have the same length different from that of the third side, or the lengths of the three sides are all different. Equally well known is the result, which goes back to Euclid, that precisely the same classification of triangles results if one considers how many of the angles are equal. In connection with a geometry-for-teachers class there arose the question whether something analogous is true for quadrangles. The negative answer was obvious at once: squares have all sides equal as well as all angles, but rectangles have only angles equal while rhombi have only sides equal.

This situation brings one to a slightly different formulation of the question. Concerning the equality or inequality of the (lengths of the) sides of a quadrangle, there are seven distinct possibilities:

(1) all sides are equal;
(2) three sides are equal, different from the fourth;
(3) two pairs of adjacent sides are equal;
(4) two pairs of opposite sides are equal;
(5) one pair of adjacent sides are equal, the other two are different from these and from each other;
(6) one pair of opposite sides are equal, the other two are different from these and from each other;
(7) all four sides are different.

Exactly the same seven possibilities arise with respect to the angles, and now one can ask:

Which of the 49 paired conditions can be realized by a convex quadrangle?

The answer is that *precisely twenty* of the 49 possibilities can be realized. Specifically, Table 1 shows which are the pairs that correspond to quadrangles, and Figure 1 shows representatives of these types (cued to the letters in Table 1.)

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Side	Angle type						
type	(1)	(2)	(3)	(4)	(5)	(6)	(7)
(1)	A			B			
(2)			C		D		E
(3)		F				G	
(4)	H			J			
(5)		K			L	M	N
(6)			P		Q		R
(7)		S			T	U	V

Table 1.

In Figure 1 equal angles are indicated by dots of the same kind; unmarked angles are different from marked ones and among themselves. Equal sides are marked by the same number of crossing dashes; unmarked sides are different from marked ones and among themselves.

The proof that this is in fact the solution of the question posed above it quite elementary (though somewhat time consuming !!!), but it provides a good opportunity for students to try finding answers to questions they have not previously encountered. Most of my class did reasonably well, although no one did a perfect job; more about that later.

As soon as I wrote out the above answer, I was struck by one aspect of Table 1: *There is complete reciprocity between the sides and the angles*, as is evident in the symmetry with respect to the main diagonal of the entries in Table 1. I have no explanation for this fact, and I do not know whether it is something that "accidentally" happens for quadrangles (just as it happens for triangles ?!), or whether there is a general result here trying to get our attention. I conjecture that this reciprocity is a general fact, valid for polygons of any number of sides. Clearly, before spending much effort trying to prove this conjecture, it would be reasonable to experiment with pentagons; however, due to the fact that there are 144 paired possibilities to be considered, I have not done so.

The situation becomes even more intriguing on account of two considerations that support the conjecture. First, the reciprocity just mentioned may be thought of as arising by some duality or polarity that is often said to exist between sides and angles of polygons (especially, convex polygons). However, if there is such a duality that can be used to prove that sides and angles of polygons have to play completely equivalent rôles — I am not familiar with it. But on the other hand, such a polarity does exist on the sphere, and leads to a proof of the spherical analogue of our conjecture.

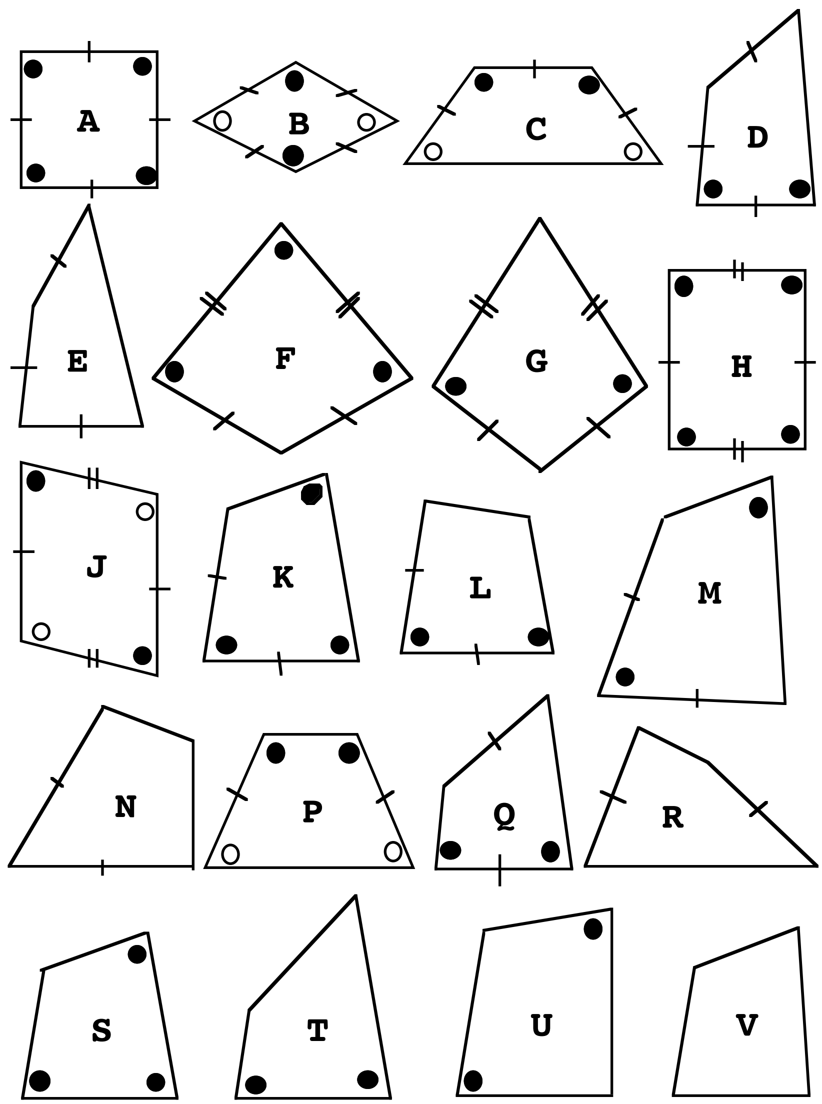

Figure 1.

Second, there is a refinement that could (and should) be applied to the original question. Consider, for example, the situation that corresponds to the case (2,5) of Table 1 (that is, alternative (2) for the sides, and alternative (5) for the angles). It is easily seen that there are, in fact, three distinct possibilities that arise in that pairing, depending on the mutual position of the three equal sides and the two adjacent equal angles. As it turns out, two of these possibilities cannot be realized, and only one leads to a convex quadrangle, specifically, a quadrangle of type D. In many of the 49 cases there are such subcases, each of which needs to be examined separately before one can conclude that no realization is possible. (A large part of the errors committed by students was due to the neglect of such variants; I hope that it provided them with a lesson concerning the necessity to examine all logical possibilities.) The only case where there are two possibilities that are both realizable is (5,5). But the most interesting aspect is that *the symmetry between the behavior of angles and of sides with respect to realizability by quadrangles persists even with this more refined point of view.* Naturally, this extends greatly the number of possibilities that need to be examined if one wishes to tackle pentagons. It may be noted in passing that even for triangles the case of two equal sides and two equal angles leads to two combinatorially distinct possibilities, only one of which can be realized (by isosceles triangles); the combination $a = b \neq c$ and $\alpha = \gamma \neq \beta$ in unrealizable.

Except for a few isolated cases, I have not examined pentagons. Let's hope that some readers will either have the fortitude to investigate the 144 cases (of the original version) directly, or the insight to find a less labor-intensive way to decide what happens to pentagons — or to all polygons. In fact, it may well be that the refined version is more easily decidable.

To conclude, here is an even more quantitative version of the conjecture in general terms. Let $0 < a \leq b \leq c \leq d \leq \ldots$ and $0 < A \leq B \leq C \leq D \leq \ldots$ denote real numbers, unrestricted except for their relative sizes. For each $n \geq 3$, let S denote a circular sequence of $2n$ numbers, where the odd-numbered positions are occupied by distinct lower-case letters, and the even-numbered positions are occupied by the upper-case letters. We shall say that a convex n-gon P *oddly* [*evenly*] *realizes* S if the relative sizes of the sides of P are the same as the relative sizes of the numbers in the odd [even] positions in S, and the relative sizes of the angles of P are the same as the relative sizes of the numbers in the even [odd] positions in S.

Conjecture. A sequence S can be oddly realized if and only if it can be evenly realized.