GEOMBINATORICS
4(1995), 77 - 79.

THE PARABOLIC 3-SPACE

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Several years ago I was teaching our geometry course for future teachers. The examples used in our textbook to illustrate the meaning and properties of axioms and axiom systems consisted mostly of various finite models. The students did not really warm up to such "geometries" and, in any case, finite models ceased being relevant once we started discussing order, separation and distance properties. This led me to look for, and invent, several kinds of plane geometries that are in some sense a more realistic medium for studying the axiomatic basis of Euclidean geometry. These systems, and a number of their properties, are described in Grünbaum & Mycielski [1990]. One of these models is of special intrinsic interest; since it is also relevant to the topic of this note, I shall briefly describe it.

For lack of a better name let's call this geometry the "parabolic plane" Π^2. The points of Π^2 coincide with the points of the usual Euclidean plane E^2, coordinatized by pairs (x,y) of real numbers. The lines of Π^2 are of two kinds: they are either (i) vertical Euclidean lines, that is, sets of the form $\{ (x,y) : x = c \}$ for some constant c; or else (ii) translates of the parabola $y = x^2$, that is, sets of the form $\{ (x,y) : y = (x-a)^2 + b \}$ for some reals a and b. (See Figure 1 for prototypes of the two kinds of "lines".) Incidence, order, betweenness, separation, parallelism and related concepts are defined in the natural way. Then it is easy to prove that the usual axioms are satisfied, and to see that the parabolic plane is a nice object for study. (As an illustration we show in Figure 2 how to find the midpoint M of a segment AB, using the fact that diagonals of a parallelogram — such as ACBD — bisect each other.) If one wishes to advance farther, a distance d(A,B) between points $A = (x_1,y_1)$ and $B = (x_2,y_2)$ may be introduced by the (somewhat unexpected) expression

$$d(A,B) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2 - x_1^2 + x_2^2)^2}.$$

With this definition it becomes possible (in appropriate axiomatic systems for E^2) to show that Π^2 satisfies **all** the axioms of the Euclidean plane. In other words, the parabolic plane Π^2 is just an unconventional model of the usual Euclidean plane !

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1 Research supported in part by NSF grant DMS-9300657.

The mystery of why this happens is easily cleared up by observing that the mapping f of Π^2 onto E^2 given by $f(x,y) = (x, y - x^2)$ is an isometric homeomorphism between Π^2 and E^2, which maps the lines of Π^2 onto the lines of E^2.

After a hiatus of several years, I am again teaching the geometry-for-teachers course. In contrast to many other books at this level, our text this time (Kay [1993]) presents an axiomatic approach to the 3-dimensional geometry. Rather naturally, this leads to the question whether one could devise a 3-dimensional analog of the parabolic plane. It turns out that this is very easy to do, and it leads with little effort to an unconventional model — which we call the "parabolic space" Π^3 — of the Euclidean 3-space E^3.

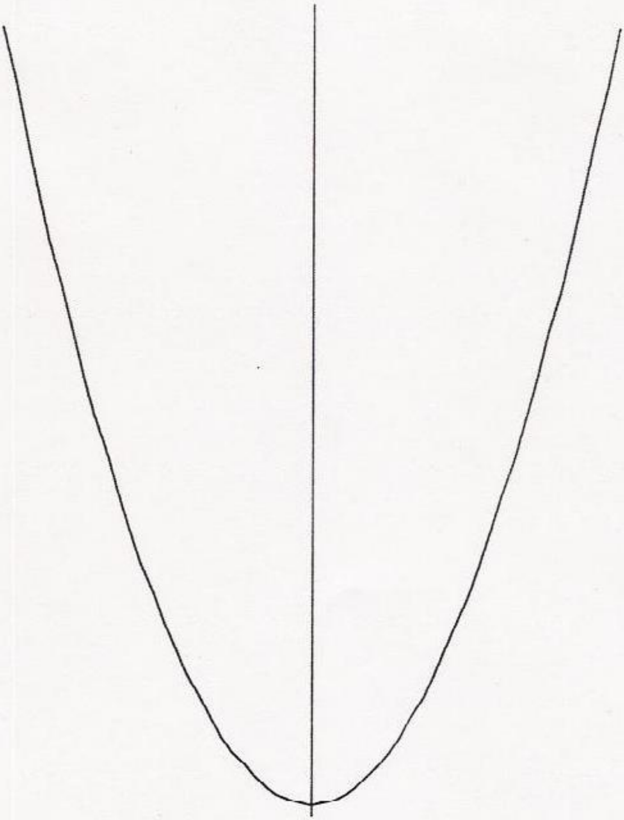

Figure 1.

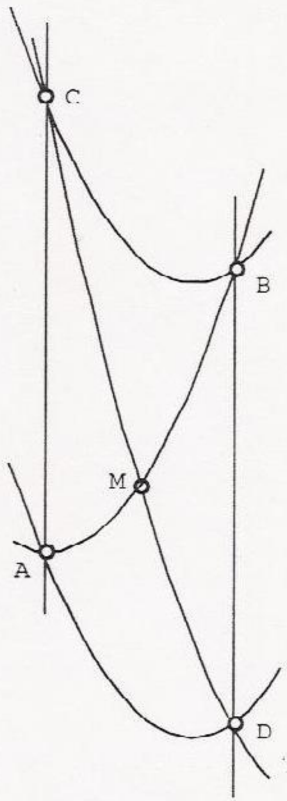

Figure 2.

The method we follow is patterned after the one used for the parabolic plane. The points of Π^3 are all the points of the Euclidean space E^3, coordinatized by triplets (x,y,z) of real numbers. The planes of Π^3 are of two kinds: either (i) the vertical planes of E^3, that is, sets of the type $\{ (x,y,z) : ax + by + c = 0 \}$ for suitable constants a, b, c; or else (ii) translates of the paraboloid of revolution $z = x^2 + y^2$, that is, sets of the type $\{ (x,y,z) : z = (x - a)^2 + (y - b)^2 + c \}$ for suitable reals a, b, c. Lines of Π^3 are defined as intersections of distinct but non-parallel planes of Π^3.

The verification of the various incidence, betweenness, separation and parallelism properties is quite simple. However, I have to admit that some of the results represent properties of which I have not been aware previously. The one that surprized me most is the following easily provable fact: Any intersection of two (non-coaxial) translates of a given paraboloid of revolution is a parabola congruent to the one used to generate the paraboloid.

In analogy to the planar case, the parabolic space Π^3 can be made into a model of the Euclidean 3-space if it is given a suitable metric. This can be accomplished by defining

$$d((x_1, y_1, z_1), (x_2, y_2, z_2)) =$$

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2 - x_1{}^2 + x_2{}^2 - y_1{}^2 + y_2{}^2)^2}.$$

The proof of the fact that Π^3 is a model of E^3 follows at once by observing that the mapping f from Π^3 to E^3 given by $f(x, y, z) = (x, y, z - x^2 - y^2)$ is an isometric homeomorphism between Π^3 and E^3, which sends the planes of Π^3 onto the planes of E^3.

The "parabolic space" Π^3 can provide interesting exercises for the students, as well as unexpected insights for their teachers.

References.

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