From: "The Lighter Side of Mathematics",
Proc. Eugène Strens Memorial Conference,
R. K. Guy and R. E, Woodrow, eds.
Math. Assoc. of America, Washington, D.C. 1994 Pp. 35 - 48.

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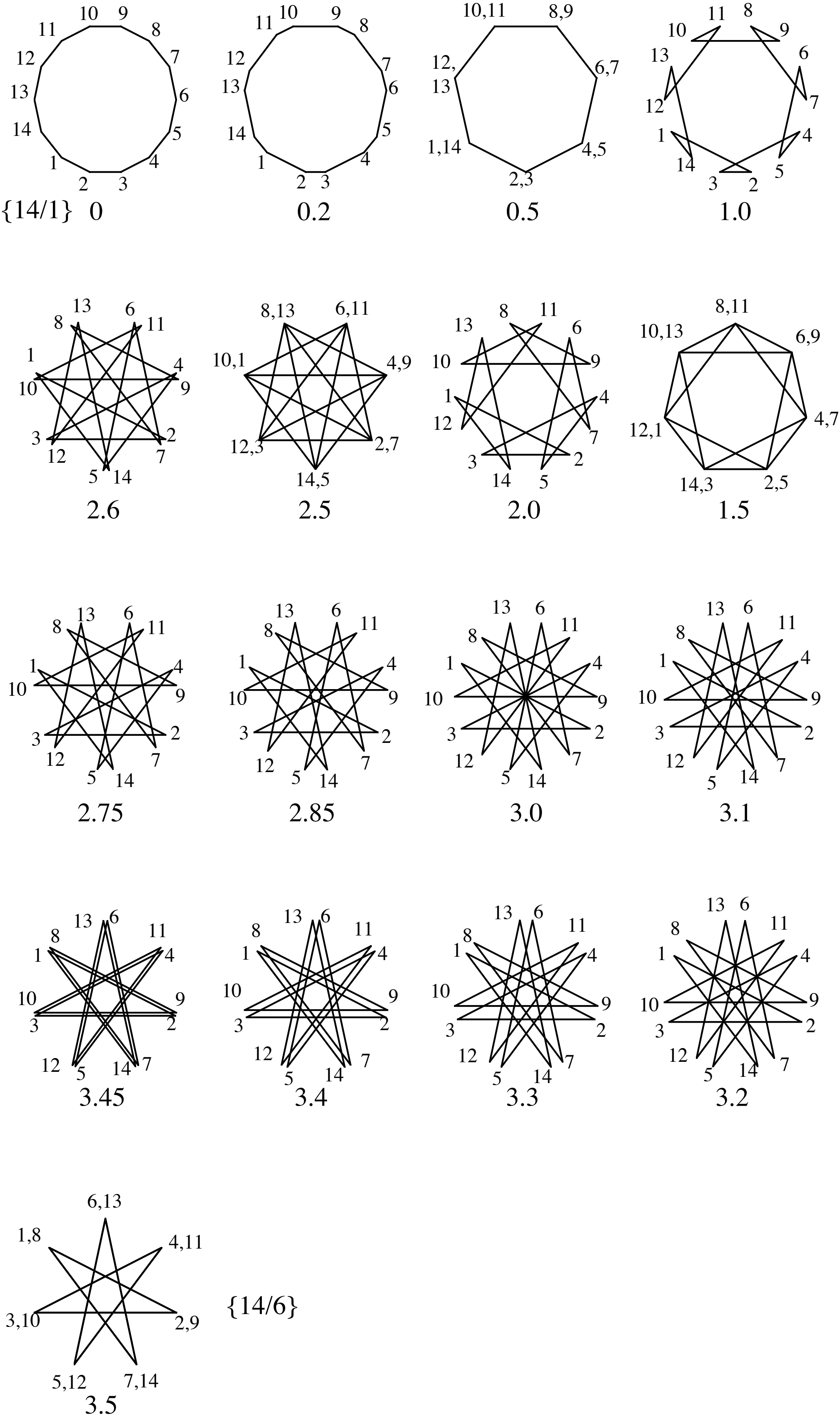
Metamorphoses of polygons

The first three illustrations of this note show "metamorphoses" of polygons -- sequences of polygons gradually changing from one regular 14-gon to another regular 14-gon. While the "star"-polygons that arise at the intermediate steps can be enjoyed for their unfamiliar but attractive shapes, there is quite a lot of mathematics that can be appreciated at the same time. I should hasten to add that there is nothing difficult or deep in the mathematical aspects of these metamorphoses; in fact, most of the assertions I shall make are so obvious that any formal proofs would only obscure the situation.

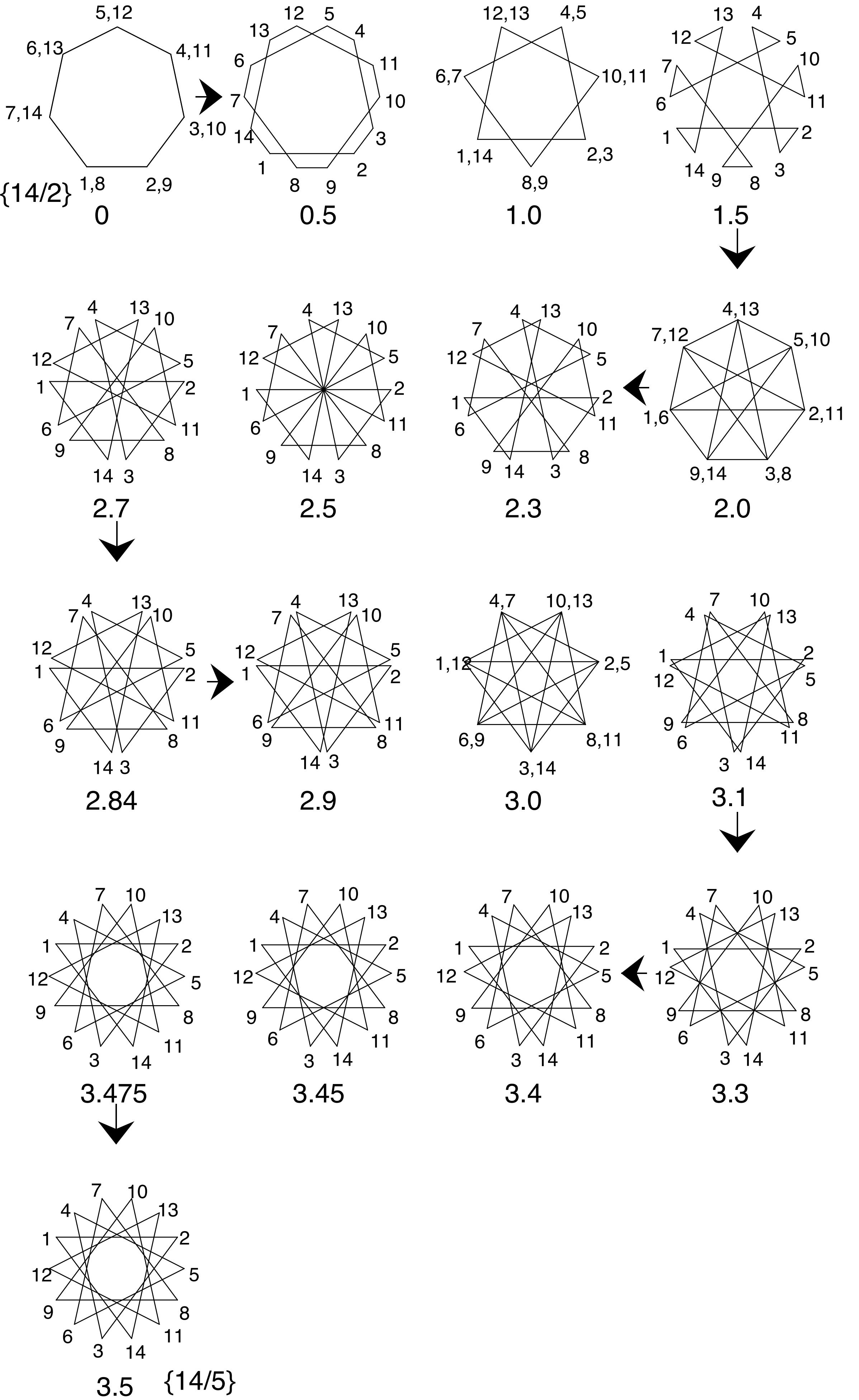
First, a brief explanation of the diagrams in Figures 1, 2 and 3. Each begins and ends with a regular polygon, that is, with a polygon in which all vertices are alike, and all edges (sides) are alike. The intermediate polygons are "regular" to a lesser degree -- only the vertices of each are alike, while the edges are of two kinds. In each of the three sequences, a finite number of polygons is shown; however, they are only instances that happen to have been selected from among families of polygons that change in shape continuously, reaching through gradual change from one of the two extreme specimens to the other.

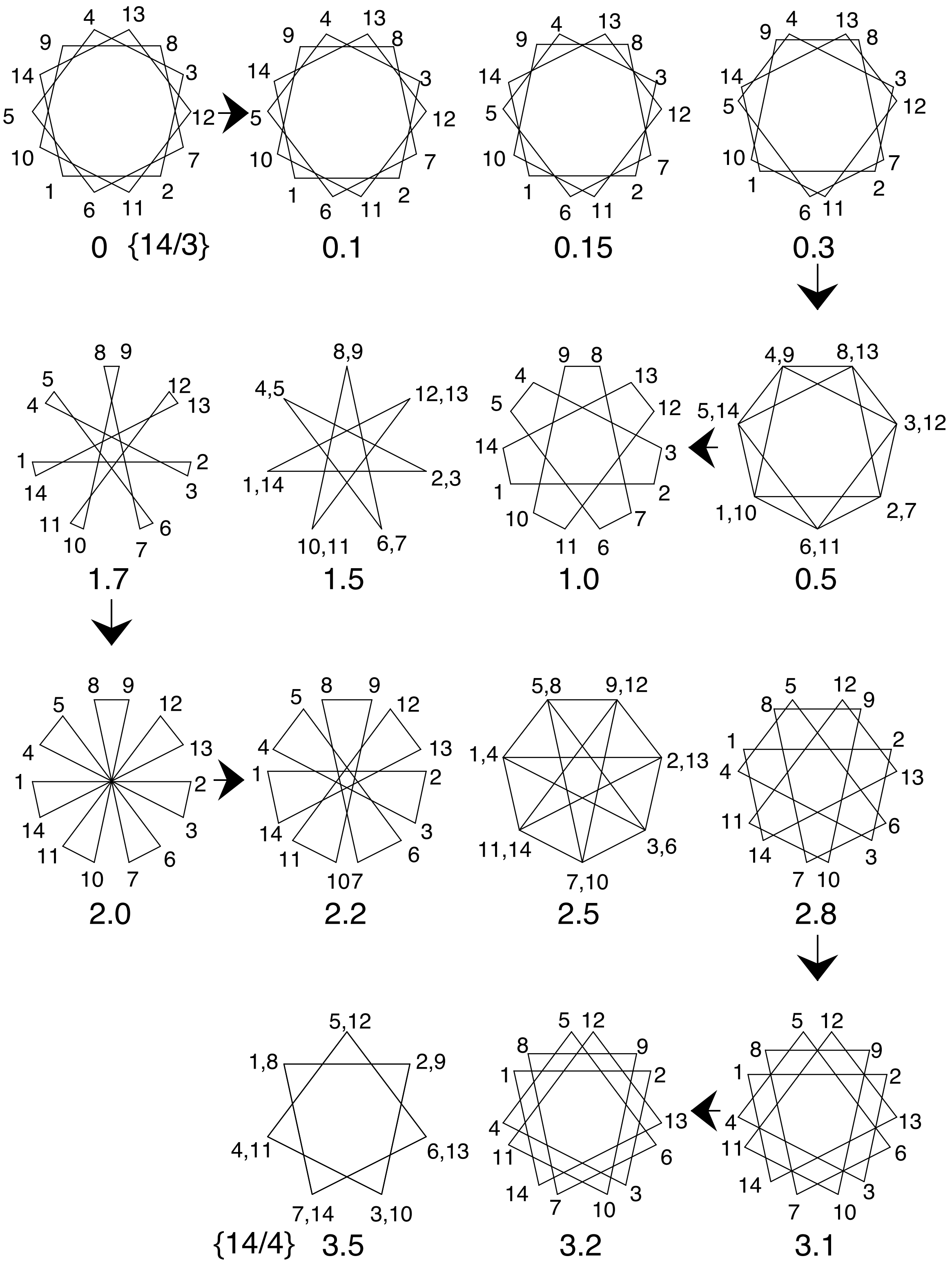
Now, the first mathematical point to be made is that all the polygons shown in the diagrams are 14-gons, despite the seemingly obvious presence of heptagons. Clearly, this calls for some explanation, and it brings up errors made almost two centuries ago and perpetuated ever since. We better start with some definitions.

Given an integer $n > 1$, an n-gon P is any collection of n points $A_1, A_2, ..., A_{n-1}, A_n$, called **vertices** of P, together with the n segments $A_1A_2, A_2A_3, ..., A_{n-1}A_n, A_nA_1$, called the **edges** of P. In general, the vertices can be points in any setting in which it makes sense to talk about segments; in the present note we shall restrict ourselves, without exception, to the traditional Euclidean plane. The edges are straight line segments; however, since the coincidence of two consecutive vertices has not been excluded, some of the segments that form edges can be reduced to single points. The definition of "polygon" does not preclude this from happening; neither does it preclude many other kinds of coincidence or overlap. In fact, one could even admit the case in which all



{14/1}
0
0.2
0.5
1.0
2.6
2.5
2.0
1.5
2.75
2.85
3.0
3.1
3.45
3.4
3.3
3.2
3.5
{14/6}


{14/2}
0
0.5
1.0
1.5
2.0
2.3
2.5
2.7
2.84
2.9
3.0
3.1
3.3
3.4
3.45
3.475
3.5
{14/5}


0 {14/3}
0.1
0.15
0.3
0.5
1.0
1.5
1.7
2.0
2.2
2.5
2.8
3.1
3.2
{14/4} 3.5

vertices coincide, though it should be granted that such a polygon is not of great interest; in order to avoid repetitious exceptions, we shall exclude such polygons from all further considerations.

Although some earlier writers made hints in the direction of such a general definition of polygons, the first explicit statement appears in Meister [7] -- an intriguing work, with which posterity has dealt very poorly (as will be explained below). The same definition reappeared in Poinsot's often-quoted paper [9], and has since become standard -- except that some authors balk at admitting consecutive vertices that coincide, while other workers forbid the coincidence of any vertices, or even the placement of a vertex at any point of an edge (other than the two of which it is an endpoint). One hint why such a negative attitude is unjustified appears in our diagrams: if polygons with coinciding vertices were banned from the discourse, our continuous families of polygons would shatter into many fragments (which would then be unrelated), and some of the extreme polygons would also be ruled out of existence. Clearly, the universe becomes more orderly if vertices may coincide.

Next, we should consider what is meant by "regular" polygon. The idea, from time immemorial, was that a polygon is regular if all its angles are congruent, and if all its edges are congruent. As can be seen by the example of the cross-shaped 12-gon that is the boundary of the union of five congruent squares, this definition has to be taken with a grain of salt: nobody would like to consider that polygon as regular. Following Möbius [8], it became customary to understand the requirements just stated in a stricter sense than conveyed by the words used, namely as additionally requiring that the two edges which determine an angle be correspondingly congruent to the two edges determining any other of the angles, and analogously, that the two angles at the endpoints of each edge be correspondingly congruent and equally placed on every other edge. Clearly, this eliminates the unwanted examples. The same goal can be achieved by considering the polygon and the plane as **oriented**, so that angles can be positive as well as negative, and inserting the appropriate requirement in the definition.

However, a much more elegant way relies on symmetries, that is, isometric mappings (congruences) that may bring a polygon to coincide with itself. The above definitions (in the restricted versions) can be rephrased, equivalently, in each of the following two forms:

(1) A polygon P is regular if and only if

(a) for each pair of vertices of P there is a symmetry of P that maps the first onto the second; and

(b) for each pair of edges of P there is a symmetry of P that maps the first onto the second.

(2) A polygon P is regular if and only if for each pair of flags of P there is a symmetry of P that maps the first onto the second.

Although formally (1) seems to be more restrictive than the previous definition, and (2) more restrictive than (1), in fact all these definitions are equivalent.

By well-known general arguments, all symmetries of a regular polygon (or any other polygon) form a **group of symmetries**, in which the group operation is composition of the isometries. Condition (a) expresses the transitivity of the group of symmetries on the set of vertices, condition (b) transitivity on the set of edges. Any polygon satisfying (a) is called **isogonal**, and any polygon satisfying (b) is said to be **isotoxal**. The mathematical fact visually expressed by Figures 1, 2 and 3 is the possibility of connecting certain pairs of regular n-gons by a continuous family of isogonal n-gons.

But here again we run into the need for some explanations, since practically every text that mentions regular star-polyhedra states that each can be denoted by a symbol $\{n/d\}$, where n and d are **relatively prime**, a condition that clearly does not apply to such expressions as $\{14/2\}$. The rather dismaying story is as follows.

When starting his investigation of regular polygons, Poinsot [9] used definitions of polygons and regular polygons equivalent to the ones given above. He went on to say (with different words, but quite correctly) that if $d \leq n/2$ is a positive integer relatively prime to n, and if n points equidistributed on a circle are connected to each other by segments each of which spans d of the arcs determined by the points, a regular polygon is obtained; this is the polygon usually denoted by $\{n/d\}$. After illustrating this construction with a few examples, Poinsot goes on to say (again, quite correctly) that if n points are equidistributed on a circle, and are connected by segments each of which spans d of the arcs, but with n and d having a common factor $k > 1$, then no regular polygon is obtained. Instead of a single polygon, one obtains a family consisting of k regular polygons of type $\{\frac{n}{k}/\frac{d}{k}\}$.

The logical error committed by Poinsot, and repeated ever since in all the texts, is the assumption that the statements of the preceding paragraph prove the non-existence of regular polygons (satisfying the requirements of the definitions) but corresponding to values of n and d that are not relatively prime. In fact, to see the fallacy of that assumption and to actually construct the polygons in all cases, all one has to do is to start with a point on the circle, connect it to a second point by a segment spanning an arc which is d/n times the length of the perimeter of the circle, connect that point to a third in the same way, and so on, until the n^{th} step, which closes the circuit of edges of the polygon. It is obvious that the difference between n and d relatively prime or not is expressed by the fact that in the former case the n^{th} step will be the first time the starting point is met again, while in the latter case it is the k^{th} such encounter. Although in that case k-tuples of vertices of the polygon come to lie at the same point, they all are still distinct as vertices. Following the edges of {n/d} one goes around the center of the polygon d times before closing the circuit -- regardless of whether n and d are relatively prime or not. (If d = n/2, the polygon appears as a segment covered n times; since these polygons have certain special properties, in some contexts they need to be considered separately.)

We note in passing that such "unconventional" regular polygons as {6/2}, {8/2}, etc. can be used to generate "unconventional" regular polyhedra -- but exploring this direction in the present paper would lead us too far afield.

It is important to understand that, for example, the regular polygon {20/4} (in which the 20 vertices are located by fours on each of five points, which are the vertices of a regular pentagon) consists of one circuit of 20 edges winding four times around the pentagon -- and not of four superimposed pentagons. The error just mentioned is one of many made by Edmund Hess (see [5], page 632) in his study of isogonal polygons (which will be discussed below).

The mistakes of Poinsot and Hess are even more striking in view of the fact that Meister [7], writing at a much earlier date, saw the situation correctly, and explained and illustrated it in great detail -- including the diagrams of all ten regular 20-gons {20/1}, {20/2}, ..., {20/9}, {20/10}. But instead of gratitude, Meister reaped slander: it appears that the only person who read Meister's masterpiece during the first 200 or so years after its publication was the historian of mathematics Sigmund Günther; unfortunately, Günther ([4], pp. 45-46) misquotes Meister's explicit statements and ascribes to Meister the same (erroneous) conclusions that were reached by Poinsot. All later writers (such as Brückner [1]), if they mention Meister at all, quote from Günther, thus missing the deeper understanding achieved by Meister, and helping perpetuate Poinsot's fallacy.

With these explanations, the mysteries and misgivings concerning the endpoints of each of the metamorphoses are removed, and it is time to clarify the construction of the intermediate polygons. The idea is as follows. We start, as appropriate according to the above explanations, by taking, for $j = 1, 2, ..., n$, on a circle C points A_j that determine with a fixed radius R of C an angle of $\frac{2\pi d}{n} \cdot j$, and connecting each A_j to A_{j+1} by a segment. This yields a regular polygon $\{n/d\}$. From now on, we shall assume that n is even since, if n is odd, it is easy to see that every isogonal n-gon must be regular, hence of no interest in the present context. The continuous families visualized in the diagrams arise by taking a parameter $t \geq 0$, and locating A_j so that the angle to the radius R is $\frac{2\pi}{n} \cdot (j \cdot d + (-1)^j t)$; in other words, the vertices of the starting regular polygon are alternately moved ahead or backwards from their original position, all through the same distance. Clearly, an isogonal polygon is obtained regardless of the value of t; however, a short calculation shows that when $t = n/4$ the isogonal polygon is, in fact the regular polygon $\{n/e\}$, where $e = \frac{n}{2} - d$. We take this value of t as the end of our metamorphosis; if larger values of t are used, one obtains the same kinds of isogonal polygons as before, but in the opposite order, till one reaches the starting regular polygon $\{n/d\}$ at $t = n/2$.

In the diagrams of Figures 1, 2 and 3 the values of d are 1, 2 and 3, respectively; the value of t is indicated near each of the polygons.

The metamorphoses just described are most interesting for such values of n which are twice an odd prime -- this is the reason the illustrations deal with $n = 14$. The reader may find it amusing to investigate the families that result for some other values of n that are equal to twice an odd number. It is not hard to show that the procedure just explained yields all possible isogonal polygons in these cases. However, if n is divisible by 4 another metamorphosis is possible. It is illustrated in Figure 4 for $n = 4$. We leave it to the reader to investigate what are the analogues of the sequence in Figure 4 for other values of n that are divisible by 4, and to formulate a complete description of the isogonal polygons possible in these cases.

The polygons in Figure 1, 2 and 3 are meant to show all possible **types** of isogonal 14-gons. Naturally, such a statement makes sense only if we agree on a definition of type -- in other words, if we provide criteria allowing us to distinguish between polygons of types that are considered distinct. This task is less simple that it may appear at first blush, and the details are, in fact, largely a matter of convenience. The (somewhat redundant) criteria we adopted that two isogonal polygons have to satisfy in order to be considered of the same types are the following, all formulated under the assumption that a fixed correspondence has been chosen between their vertices:

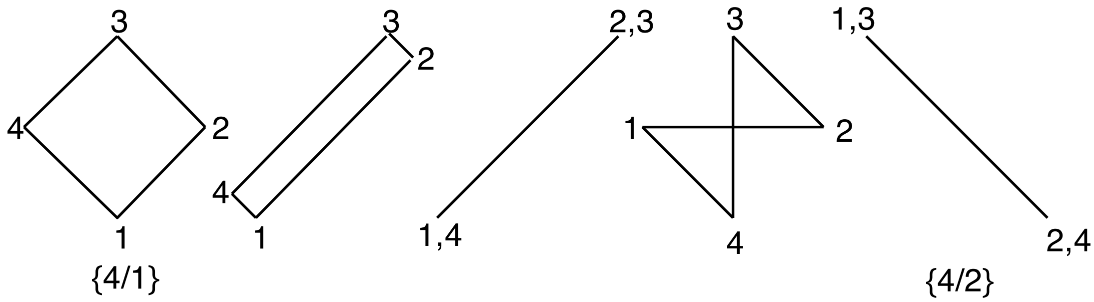


Figure 4. Metamorphosis of {4/1} to {4/2}, through isogonal polygons.

(i) on the circumcircle of each, corresponding vertices have to coincide in the the same sets, and follow in the same order;

(ii) corresponding edges meet or fail to meet in the same way in both, and, on each edge, the order of intersections by the other edges is the same as on the corresponding edge in the other polygon;

(iii) if three or more edges of one polygon meet at one point, the same is true for the corresponding edges of the other;

(iv) the symmetry groups of the two have to be isomorphic, and act on the polygons in the same way.

With this definition, it is easy to verify that Figures 1, 2 and 3 show all types of isogonal 14-gons, except the regular polygon {14/7}. Similarly, in Figure 4 are shown all types of isogonal 4-gons.

The metamorphoses in our diagrams can be used to illustrate the concepts of winding number and rotation number, and to clarify the distinction between them. For both concepts it is necessary to orient the polygon, and we shall assume that an orientation has been chosen in each case. The **rotation number** r of a polygon P can be defined as the sum of the **deflections** at the vertices of P, measured in units of the full angle; the deflection at a vertex A_j is the angle through which the extension of the incoming edge has to be turned in order to coincide with the outgoing edge, the angle being chosen to lie between -1/2 and 1/2 of the full angle. (We recall that counterclockwise angles are counted as positive, clockwise ones as negative.) It follows that the rotation number is **undefined** if an edge overlaps with the following one (since there is no way to decide whether the deflection is 1/2 or -1/2), or if two consecutive vertices coincide (since the single-point edge does not determine any direction). The first eventuality happens, for example, for regular polygons {n/d} where d = n/2; the second occurs, in a very important role, among isogonal polygons, as we shall see shortly. Regardless of the relative primeness of n and d, the rotation number of a regular polygon {n/d}, with $1 \le d < n/2$, is either d or -d, depending on the orientation.

While the rotation number of a polygon depends only on the polygon itself, the winding number depends on a reference point as well. If O is a point, the winding number w(P,O) with respect to O of a polygon P can be computed by following a suitable ray X (or any curve, for that matter) from O to points sufficiently far, so as to reach outside a circle enclosing the whole polygon; each time the ray crosses the polygon we obtain a contribution to the winding number, the contribution being 1 if the direction of the polygon at the crossing is from right to left when looking in the direction of the ray, and -1 otherwise. For this to work, O should not lie on any line determined by the edges of P, and X should not pass through any vertex of P. It is well known that the winding number does not depend on the particular ray X chosen, that the value of w(P,O) is the same as w(P,O*) if the segment OO* meets no edge of P, and that the definition of w(P,O) can be extended by continuity to all points of the plane except those on the edges of P. Additional information about the rotation and winding numbers, as well as some other functions associated with polygons, can be found in Grünbaum & Shephard [3].

Both the rotation number and the winding number change sign if the direction of the polygon is reversed. Hence it is in many cases convenient to assume that the orientation is such that one of these numbers is nonnegative. However, as the examples in Figures 1, 2, 3 show, it is not in all cases possible to choose the orientation so that both numbers are positive.

In connection with regular polygons, it is customary to call the absolute value of the winding number of a polygon P with respect to its center O the **density** of P. It is well known and easily shown that the density of {n/d} is d; this holds regardless of the relative primeness of n and d, except that the density of {n/d} is not defined if d = n/2. The winding numbers of isogonal polygons are also usually considered with respect to the center of the polygon; hence the winding number (with respect to the center) of an isogonal polygon is not defined if some of the edges of the polygon pass through the center.

It is of interest to follow the values of the rotation and winding numbers as we advance in each of the metamorphoses in Figures 1, 2 and 3. When t satisfies $0 \leq t < 1/2$, the rotation number has constant value $\pm d$ (the sign depending on the orientation); at $t = 1/2$ pairs of consecutive vertices coincide, and the rotation number is undefined. For t with $1/2 < t \leq n/4$ (in the present case $n = 14$) the rotation number is $\pm(d - \frac{n}{2})$. In contrast, for t with $0 \leq t < n/4 - d/2$ the winding number with respect to the center is $\pm d$, and for $n/4 - d/2 < t \leq n/4$ the winding number is $\pm(d - \frac{n}{2})$; for $t = n/4 - d/2$ the winding number with respect to the center is not defined. We see that in the interval $1/2 < t < n/4 - d/2$ the rotation number and the winding number have opposite signs, regardless of the orientation. It should be mentioned that although an analogous situation occurs with isogonal n-gons when n is divisible by 4, the second part of Figure 4 shows a new phenomenon: the winding number is undefined, and the rotation number is 0.

Isotoxal n-gons, that is polygons in which the symmetries act transitively on the edges, behave analogously to the isogonal n-gons; they also present a variety of interesting shapes, transitional between regular polygons $\{n/d\}$ and $\{n/e\}$, where $e = \frac{n}{2} - d$. In Figures 5, 6 and 7 we show representative isotoxal 14-gons, with $d = 1, 2$ or 3, respectively. We shall not dwell in detail on their construction or properties, but would like to point out two aspects. First, the behaviour regarding rotation numbers and winding numbers is simpler (and less interesting) then among isogonal polygons: the values of the two numbers coincide whenever defined, and in each of the metamorphoses, there is a single polygon for which they are not defined; at that stage, the common value changes from $\pm d$ to $\pm(d - \frac{n}{2})$.

The second aspect worth mentioning is the duality between isogonal and isotoxal polygons. In the older literature (and some of the newer as well) much is made of duality among polygons, without bothering to point out the limitations of the assertions. It is well known that in the projective plane there is a complete duality between points and lines, that is, a correspondence between points and lines that preserves incidences. Due probably in part to the widespread semantic confusion caused by the venerable tradition of using the word "line" to denote both the infinite lines (of the Euclidean or projective planes) and the finite segments of straight lines, many writers have blithely discoursed on the duality of polygons in general, and regular and other special polygons in particular. This is especially visible in the writings of Hess [5] and Brückner [1], who have at length discussed the duality between isogonal and isotoxal polygons. We shall not repeat here the details of the objections that can be raised against duality among polygons (or polyhedra) in general, since their polyhedral formulation can be found in Grünbaum & Shephard [2]. It is clear that there is nothing wrong concerning the duality of convex polygons and polyhedra; also, for regular polygons and polyhedra, and for many polygons that have a "center", one can give

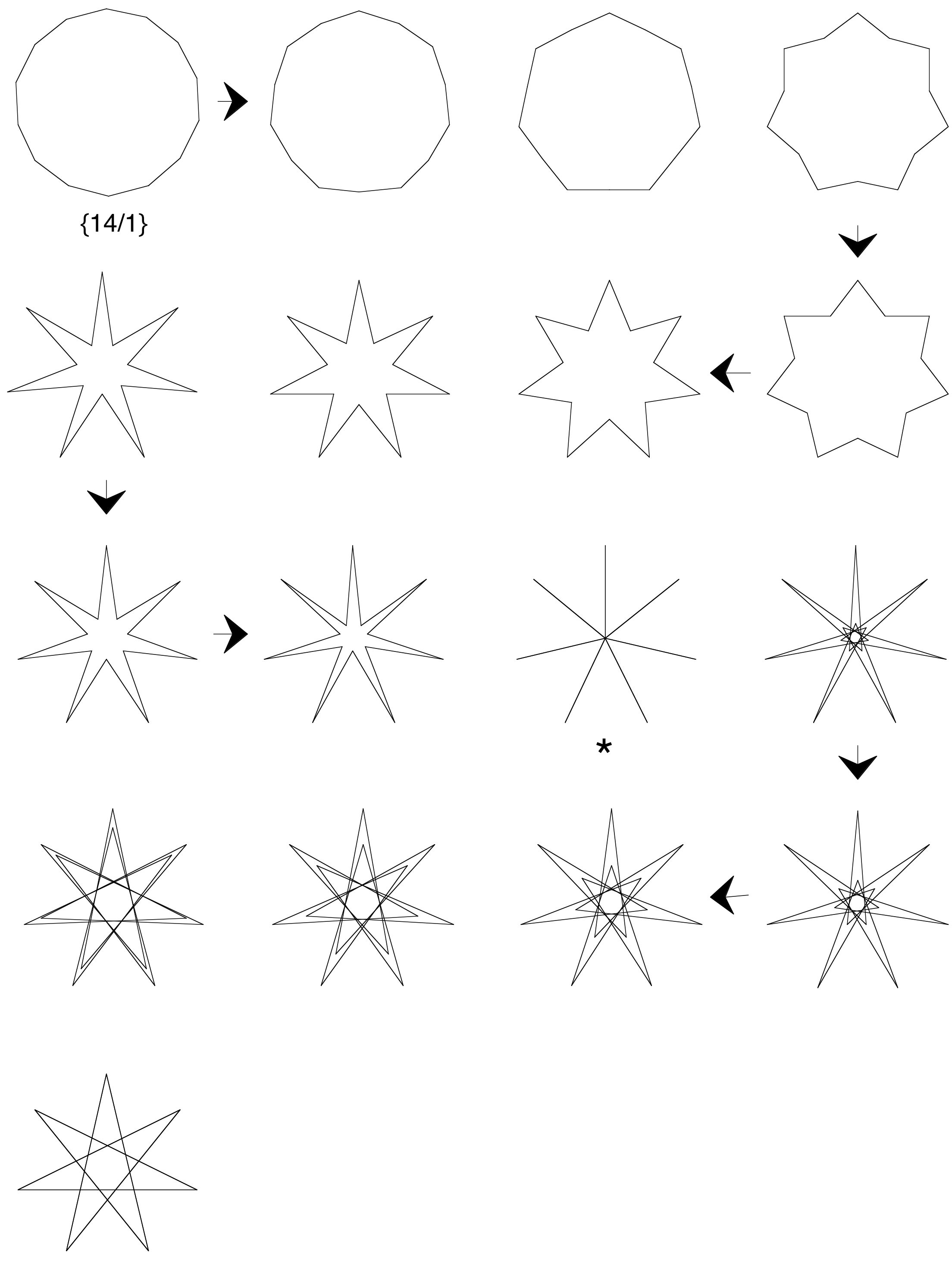


Figure 5. Metamorphosis of {14/1} to {14/6}, through isotoxal polygons.

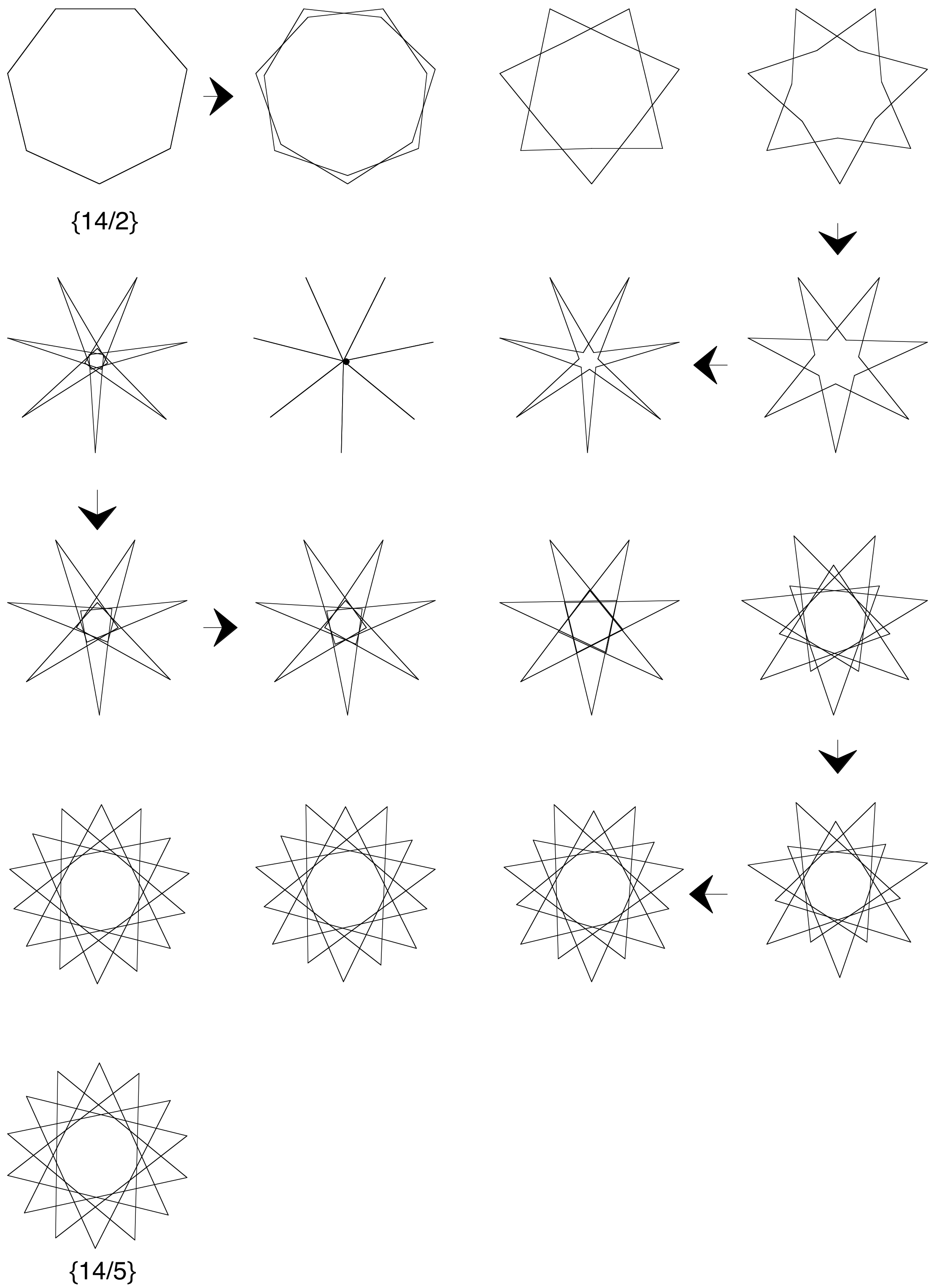


Figure 6. Metamorphosis of {14/2} to {14/5}, through isotoxal polygons.

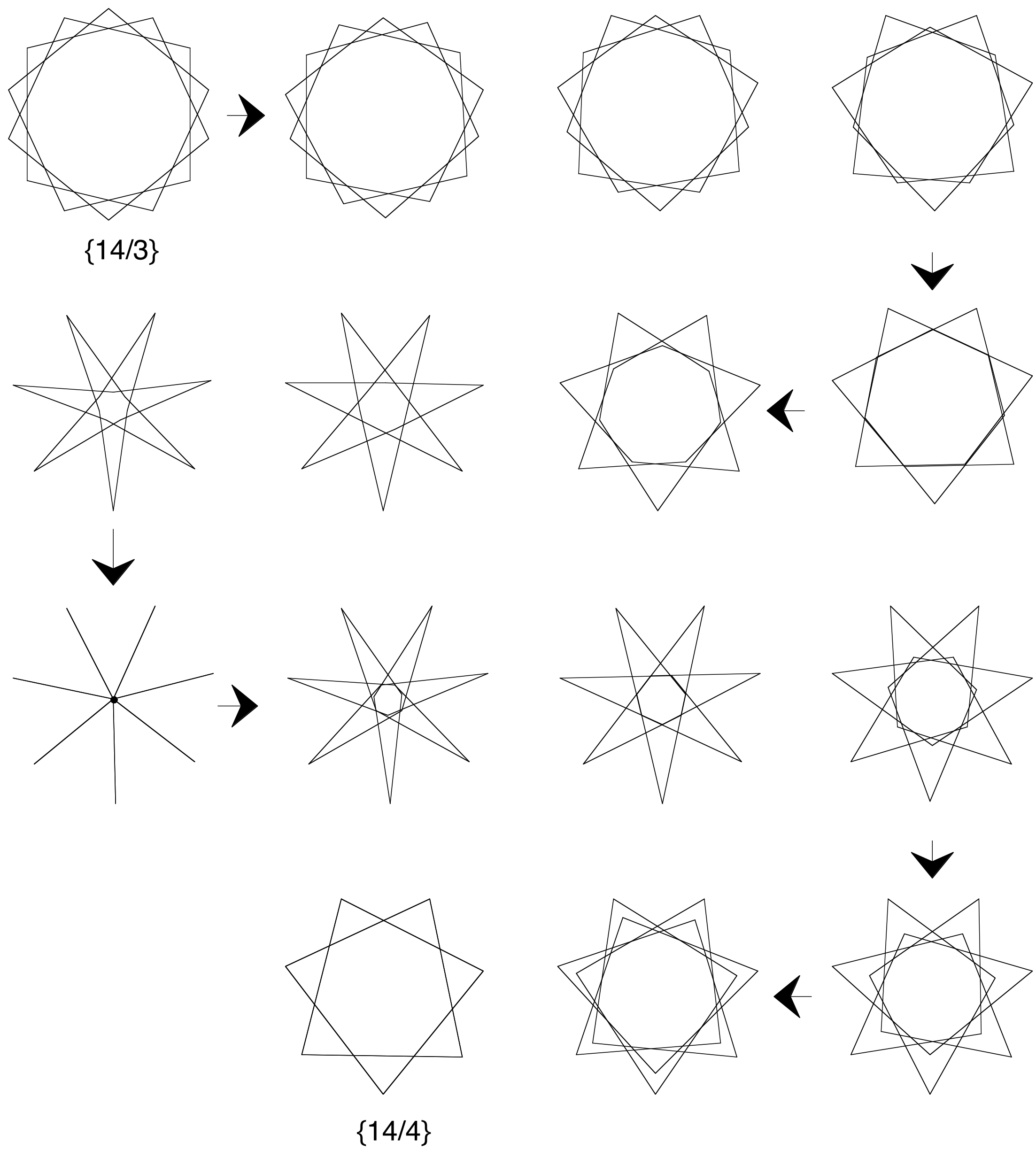


Figure 7. Metamorphosis of {14/3} to {14/4}, through isotoxal polygons.

reasonably satisfactory definitions of duality. But the diagrams of the metamorphoses shown in the present paper quite clearly show what can go wrong in these dualities. The polygons in Figures 1 and 5 correspond to each other by such a duality (in fact, by reciprocation in a suitable circle) in all cases except one: the isogonal 14-gon for $t = 3$ does not admit a dual isotoxal polygon under that duality, and the isotoxal polygon marked by an asterisk does not correspond to any isogonal polygon under the same duality. Similar is the situation in the "dual" pairs of metamorphoses in Figures 2 and 6, and in Figures 3 and 7. Even more blatant is the lack of duality between the isogonal 4-gons shown in Figure 4, and the isotoxal 4-gons shown in Figure 8.

In conclusion, here are some historical remarks. I am not sure about the origin of the winding numbers (which play a rather prominent role in the calculation of areas), but they were certainly defined and used by Meister [7]; Steinitz [10], p. 4, assigns to Meister the priority of their definition. The rotation numbers were also introduced rather early, by Wiener in his frequently mentioned but apparently rarely read work [12]. Wiener used the word "Art" (German for "kind") instead of rotation. Unfortunately for Wiener, and for mathematics, Hess [5] (mis)appropriated the term "Art" for a different concept which has not turned out to be useful; however, since Hess (and later Brückner [1]) used the term in the modified sense, Wiener's original concept was effectively forgotten. It was independently rediscovered only much later, by Whitney [11] in 1936; rotation numbers play an important role in topological considerations, as well as in the classification of polygons (see Mehlhorn & Yap [6]).

The work of Hess [5], which we have already mentioned several times, sets out to investigate isogonal and isotoxal polygons at length. Unfortunately, it is essentially devoid of any worthwhile results or insights, mainly due to his insistence of classifying the polygons by their "Art" according to his definition of this term. There seems to have been no later investigation of these interesting types of polygons, besides Brückner's [1] account of some of Hess's statements. It is my hope that the present note will make the isogonal and isotoxal polygons more accessible, and will lead to their use as examples of various concepts and misconceptions. But above all I hope that there may be more appreciation given to their visually appealing qualities.

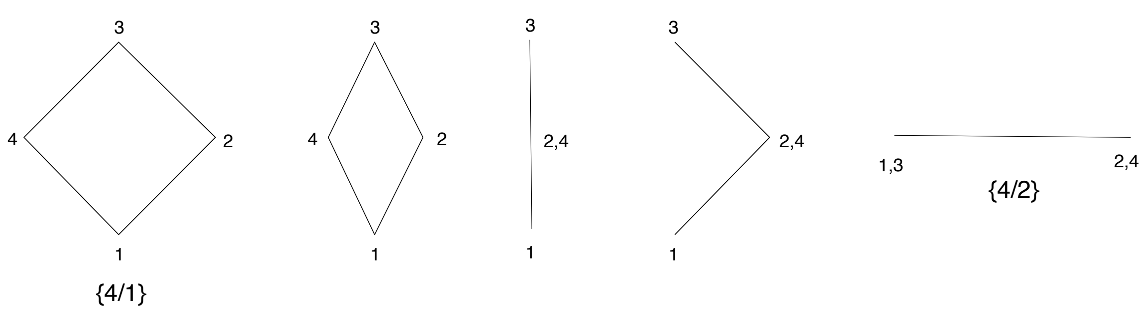


Figure 8. Metamorphosis of {4/1} to {4/2}, through isotoxal polygons.

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