Polyhedra with hollow faces.

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1. Introduction. The Original Sin in the theory of polyhedra goes back to Euclid, and through Kepler, Poinsot, Cauchy and many others continues to afflict all the work on this topic (including that of the present author). It arises from the fact that the traditional usage of the term "regular polyhedra" was, and is, contrary to syntax and to logic: the words seem to imply that we are dealing, among the objects we call "polyhedra", with those special ones that deserve to be called "regular". But at each stage — Euclid, Kepler, Poinsot, Hess, Brückner, ... — the writers failed to define what are the "polyhedra" among which they are finding the "regular" ones. True, we now know what are the convex polyhedra, which we think are the polyhedra Euclid had in mind; hence there is no stigma attached to the use of a term like "regular convex polyhedron". But where in the literature do we find acceptable definitions of polyhedra that could be specialized to give the "regular Kepler-Poinsot polyhedra"? For these, a better expression would be to say that they are "regular-polyhedra" — a distinct kind of objects, constructed according to more or less explicit procedures, and without any connection to what the separate parts of that ungainly word may mean.

One aim of the present paper is to provide a usable and intrinsically consistent foundation for a geometric theory of polyhedra which need not be convex; in fact, as defined here, they can be very general. An effort was made to formulate the relevant concepts is such a way as to "legitimize" not only the regular and uniform nonconvex polyhedra, but also the various interesting kinds of polyhedra introduced by Hess and Brückner, as well as other polyhedra described below and believed to be new.

Another aim of this paper is to atone for a cardinal sin committed by Poinsot and all later writers on regular polyhedra (again including the present author): Poinsot's stated definitions of "polygon" and "regular polygon" do not tally with his actual usage of the terms. In the theory of polyhedra this inconsistency causes needless restrictions and loss of contact with other mathematical disciplines, thus leading to the widespread impression that the theory of (nonconvex) polyhedra is an obscure and arcane deadwater. As will become clear in the sequel, this impression is unjustified; in fact, one may imagine that an understanding of these general polyhedra may well inspire and contribute to new results in graph theory, the study of maps on manifolds, and topology in general.

The present paper is meant to be an introduction to a topic for which no satisfactory exposition is available, but on which much that is inaccurate and misleading has been written. Hence I hope that the reader will excuse the details, and the slow pace. A great effort

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was devoted to making the exposition precise and clear, without being pedantic. Before starting on the chosen topic, a few general considerations seem appropriate.

In the study of convex polyhedra it makes no essential difference whether a polyhedron is understood as a "solid" or as the collection of its faces. The combinatorial aspects are most often related to the facial structure, while other properties are conveniently studied in the general setting of convex bodies; since the transition between the two points of view is easy, in each specific problem the most appropriate one may be used without causing confusion or complications. For nonconvex polyhedra, however, even the most cursory examination shows that the choice between these points of view is of critical importance. For better or worse, throughout this paper we are guided by the idea that a polyhedron is an object formed and determined by the polygons that are its faces, and we relegate to another occasion the study of nonconvex polyhedral solids; for some steps in the latter direction see [21].

With this understanding, it is convenient to distinguish three main levels of generality of nonconvex polyhedra. The smallest departure from convexity occurs among **acoptic** polyhedra (from the Greek κοπτω, to cut); these are polyhedra in which the faces are planar, simple polygonal regions, and different faces intersect only along common edges or vertices. Next in generality are polyhedra in which faces may be selfintersecting planar polygons (such as the pentagrams in some of Kepler's polyhedra), and different faces may intersect and cross in various ways. Finally, there are polyhedra which may have nonplanar faces. There exists no systematic presentation of a theory of polyhedra at any of these levels of generality, and it was originally our intention to discuss here all three levels. However, constraints of time, space and energy led us to restrict the attention to the middle level, and to polyhedra in Euclidean 3-space. We hope that the reader will consult some of [2], [17], [18], [20], [30], [31] in order to see at least a few of the results concerning acoptic polyhedra, and the references mentioned in Section 7 for polyhedra whose faces need not be planar. Each of these directions presents ample opportunities for further research.

We start by clarifying (in Section 2) the definition of polygons which we shall use in later sections as building blocks of polyhedra. In Section 3 we define a class of nonconvex polyhedra, which may be called "hollow-faced polyhedra", and illustrate the definition by several examples. This concept of polyhedron is then used to discuss isogonal polyhedra (Section 4), regular polyhedra (Section 5), and polyhedra we call "noble" (Section 6). (A polyhedron is called noble if it is both isogonal and isohedra; these polyhedra were last studied in a very long paper by Max Brückner in 1906.) Some general remarks and problems are collected in Section 7.

Part of this material was presented at courses given by the author at the University of Washington in 1990 and 1992. Acoptic polyhedra have been considered in joint work with G. C. Shephard (in the papers mentioned above, and in ongoing research), as were aspects of hollow-faced polyhedra which are not discussed here and which will be presented in separate papers.

2. Polygons. Polyhedra, as the term is understood here, are collections of polygons. Hence we lay a foundation for the study of polyhedra by giving a precise definition of "polygon". The definition is more general than the traditional interpretation, but it appears to be the one most appropriate for the present purposes. Moreover, this method of defining polygons will serve as a pattern in the more complicated definition of polyhedra. The central point is that just as a function cannot be considered as given if one knows only its range, so for a polygon it is not sufficient to know only what points lie on its vertices and edges. We require information about the "internal structure" of that set of points. Although other approaches are possible, here we attain this goal by first defining "abstract polygons", and then considering "geometric polygons" as "realizations" of the abstract ones in a Euclidean (or other) setting. In practice, it is convenient to use oriented as well as unoriented polygons. We shall limit ourselves to unoriented polygons, since oriented ones may be easily derived from them, in each case in just two (oppositely directed) ways.

An **abstract polygon** $\mathcal{A}$ is a fixed Eulerian **circuit** $\mathcal{C}$ in a simple Eulerian graph $\mathcal{G}$ (that is, a graph with no loops or multiple edges, and with vertices of even valence).

In greater detail: The abstract polygon $\mathcal{A}$ is a system consisting of a set $\mathcal{V}$ of distinct elements, and a cyclically ordered set $\mathcal{C}$ of distinct unordered pairs $\mathcal{E}$ of distinct elements of $\mathcal{V}$, such that

(i) for each pair $\mathcal{E}$ in $\mathcal{C}$ one element of the pair $\mathcal{E}$ is shared with the preceding pair $\mathcal{E}^*$, and the other with the following pair $\mathcal{E}^{**}$; these pairs $\mathcal{E}^*$ and $\mathcal{E}^{**}$ are said to be **adjacent** to the pair $\mathcal{E}$.

(ii) no subsystem of $\mathcal{A}$ which has the same adjacent pairs has property (i).

Note that this implies that card $\mathcal{V} \geq 2$. The definition may be easily modified to apply to infinite polygons $\mathcal{A}$; we do not dwell on this since, except for a brief mention in Section 5, we restrict attention to finite polygons.

The elements of $\mathcal{V}$ are the "vertices" of the polygon, and the pairs represent the "edges" of the polygon. The departure from tradition is evident in the fact that a "vertex" can be visited more than once — but the adjacency relation among the edges is determined once and for all, and each edge is adjacent to precisely two other edges. This is the "structure" to which reference was made earlier.

Abstract polygons can be graphically represented by (undirected) graphs (without loops or multiple edges) provided a labelling is indicated to show which is the cyclic order in which the edges are traversed. If the number of elements of $\mathcal{V}$ is the same as the number of pairs in $\mathcal{C}$, so that each vertex is visited just once, the polygon is said to be **unicursal**.

A **geometric polygon** (or **polygon** for short) $\mathcal{P}$ is the image of an abstract polygon $\mathcal{A}$ under a map ϕ which associates with each element of $\mathcal{V}$ a point (**vertex** of $\mathcal{P}$) in the Euclidean d-space E^d, and with each pair from $\mathcal{C}$ the segment (**edge** of $\mathcal{P}$) having as endpoints the images of the elements of $\mathcal{A}$ that constitute the pair. If $\mathcal{P}$ has n edges we shall call it an **n-gon**.

Clearly, this definition of polygon generalizes naively understood polygons in several respects. First, the polygon may be in a space of arbitrarily large dimension; however, except for brief mentions of skew polygons in Section 5 and 7, each polygon we consider will be a subset of a plane — although when forming polyhedra, the planes of the various polygons will in general be distinct. Second, various kinds of coincidences may occur among the images of the elements of $\mathcal{V}$ and of $\mathcal{C}$, even if $\mathcal{A}$ is unicursal. This includes the possibility that adjacent or other vertices coincide, or that edges overlap or pass through vertices; if any kinds of coincidences are to be excluded in the formulation of a definition or result, this has to be stated explicitly. Third, $\mathcal{P}$ may be contained in a line. This possibility can be eliminated by imposing the requirement that the polygon be **full-dimensional**, that is, not contained in a straight line. Without further mention, all polygons considered here as faces of polyhedra will be assumed full-dimensional.

In Figures 1 and 2 are shown several polygons, meant to illustrate the concepts introduced above. In most situations, there is no need to distinguish polygons that can be mapped onto each other by a similarity; hence we may speak of "the square", etc.

Since an abstract polygon $\mathcal{A}$ is combinatorially just a circuit, the group of incidence-preserving automorphisms of a unicursal n-gon is the **dihedral group** dn, which consists of n stepwise advances along the circuit, together with n maps which reverse the direction. In all other cases, the automorphism group of an n-gon is a (proper or improper) subgroup of dn.

A **symmetry** of the geometric polygon $\mathcal{P}$ is a pair (σ,π) where σ is an isometry of the plane and π an edge-preserving permutation of the vertices of $\mathcal{P}$, such that π and σ commute on the set of vertices of $\mathcal{P}$. The symmetry group of any (geometric) polygon with n edges is clearly (isomorphic to) a subgroup of dn. A polygon is called **isogonal** [**isotoxal**, or **regular**] if its symmetry group acts transitively on its vertices [edges, flags consisting of a vertex and an edge incident with it]. It is easily seen that a regular polygon is unicursal, and that:

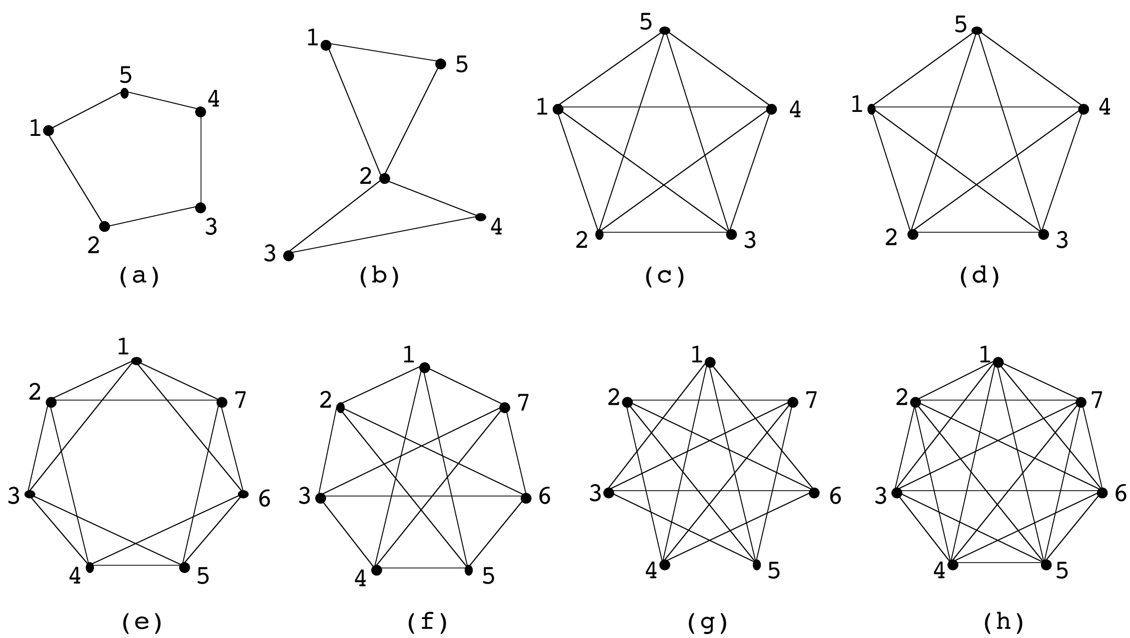


Figure 1. Examples of polygons. The 5-gon in (a) can be described by the cyclic sequence of vertices 1,2,3,4,5(1); the last label is in parentheses to indicate that it coincides with the first and closes the cycle. Other symbols for the same polygon can be obtained by reversing the listing, or by starting with any other vertex, in either of the two directions. The 6-gon in (b) can have either of the structures 1,2,3,4,2,5(1) or 1,2,4,3,2,5(1). If the 10-gon in (c) is interpreted as 1,2,4,5,2,3,5,1,3,4(1) and that in (d) as 1,2,5,1,4,5,3,4,2,3(1), they are clearly distinct (non-similar), but the underlying abstract 10-gons are isomorphic. The diagram in (e) leads to two 14-gons constructed analogously: 1,2,4,5,7,1,3,4,6,7,2,3,5,6(1) and 1,2,7,1,6,7,5,6,4,5,3,4,2,3(1) which have non-isomorphic underlying abstract polygons. However, they are isomorphic to the two polygons that can be derived in the same way from (f) and to the two derivable from (g). The 21-gon in (h) can be interpreted in many ways; a very symmetric one is 1,2,4,7,1,3,6,7,2,5,6,1,4,5,7,3,4,6,2,3,5(1). All the diagrams in (c) to (g) can also be interpreted as polygons in several less symmetric ways.

(i) A polygon is regular if and only if it is isogonal and isotoxal.

(ii) A polygon is regular if and only if its symmetry group is isomorphic to d_n.

For any regular n-gon, all the vertices are obtained from any one vertex by rotations through an angle of $2\pi d/n$ and its multiples, where without loss of generality we can assume that d is an integer such that $1 \le d \le n/2$. Different values of d yield different (that is, non-similar) polygons. A polygon corresponding to n and d shall be denoted by the "Schläfli symbol" $\{n/d\}$; however, $\{n/1\}$ is usually simplified to $\{n\}$. In a suitable system of coordinates the vertices of the polygon $\{n/d\}$ can be chosen as points with coordinates

$$\{(\cos \frac{2d\pi j}{n}, \sin \frac{2d\pi j}{n}) \mid j = 0, 1, ..., n-1\} .$$

Hence, for each $n \ge 2$, there exist precisely $[n/2]$ distinct regular n-gons (usually represented by the symbols $\{n/d\}$, $d = 1, 2, ..., [n/2]$) such that no two are similar, and every regular n-gon is similar to one of them.

Figure 2 shows all regular n-gons with $n \le 10$. In this figure and some of the following ones, the numerals j near the vertices are the labels of the vertices. Although

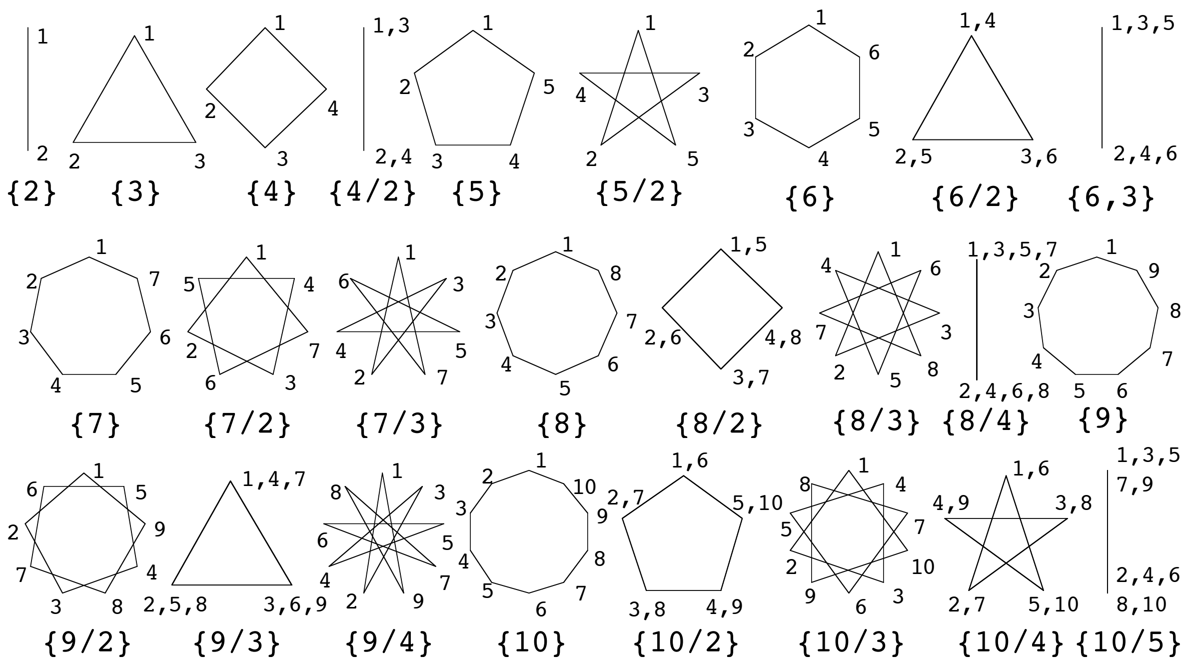


Figure 2. Representatives of all the different types of regular polygons with at most 10 vertices. The type {n/d} is indicated under each polygon, and the vertices are to be visited in the numerical order of their labels.

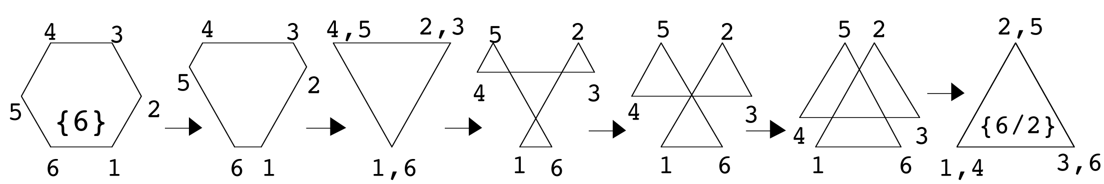


Figure 3. An isotopy between regular polygons {6} and {6/2}, via isogonal hexagons.

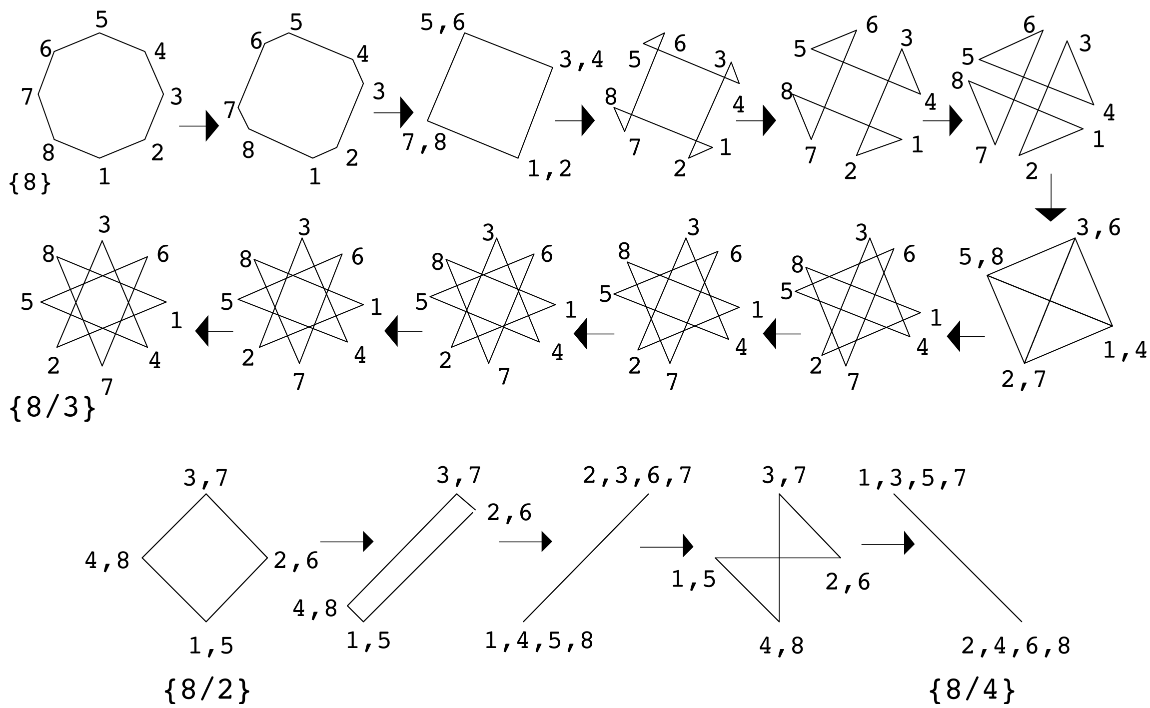


Figure 4. An isotopy between {8} and {8/3}, and between {8/2} and {8/4}, via isogonal octagons.

each regular polygon is unicursal, if n and d are not relatively prime then several distinct vertices of the underlying abstract polygons are represented by the same point in the geometric polygon. It should be emphasized that each part of Figure 2 represents a single polygon. For example, $\{6/2\}$ is a hexagon whose vertices happen to coincide in pairs; it is not a pair of triangles. This is rather vividly demonstrated in Figure 3, which shows the "metamorphosis" (an isotopy via isogonal polygons) of $\{6\}$ to $\{6/2\}$. Analogous are the metamorphoses for all $n \equiv 2 \pmod 4$; there are $[n/4]$ different isotopies, from $\{n/d\}$ with $1 \le d \le [n/4]$, to $\{n/e\}$, where $e = n/2 - d$. In case $n \equiv 0 \pmod 4$ the metamorphoses are slightly different; this is illustrated for $n = 8$ in Figure 4. For more details, including an illustration in case $n = 14$ and analogous illustrations of metamorphoses between regular polygons via isotoxal polygons, see [16]; some remarks in this context are in Section 7.

3. Polyhedra. Now we proceed to use polygons in order to construct polyhedra. We start with abstract polyhedra; note that our use of this term differs from the meaning it has in other papers in the present volume.

An **edge-sharing family** $\mathcal{P}$ is a triplet $(\mathcal{V},\mathcal{E},\mathcal{F})$ consisting of a nonempty set $\mathcal{V}$ of **vertices**, a set $\mathcal{E}$ of **edges**, and a set $\mathcal{F}$ of **faces**, with a relation of incidence satisfying the conditions below:

(i) each E in $\mathcal{E}$ is incident with a pair of distinct vertices in $\mathcal{V}$;

(ii) each F in $\mathcal{F}$ is an abstract unoriented polygon, with vertices in $\mathcal{V}$ and edges in $\mathcal{E}$;

(iii) each edge E in $\mathcal{E}$ is an edge of a positive even number of distinct faces in $\mathcal{F}$; these faces are arranged in pairs, and each member of a pair is said to be **adjacent** to the other member.

If $\mathcal{P} = (\mathcal{V},\mathcal{E},\mathcal{F})$ is an edge-sharing family and V is a vertex of $\mathcal{P}$, the **vertex figure** $C(V)$ of $\mathcal{P}$ at V is a collection of vertices and edges defined as follows: If $F \in \mathcal{F}$ and if $\{V,V_1\}$ and $\{V,V_2\}$ are two adjacent edges of F, then $C(V)$ contains V_1 and V_2 as vertices, and $\{V_1,V_2\}$ as an edge incident with V_1 and V_2. If V_i, V_j and V_k are vertices of $C(V)$, while $\{V_i,V_j\}$ and $\{V_j,V_k\}$ are edges of $C(V)$ arising from faces F* and F**, then $\{V_i,V_j\}$ and $\{V_j,V_k\}$ are said to be adjacent edges of $C(V)$ if and only if the faces F* and F** are adjacent faces of $\mathcal{P}$.

Now we are ready for the definition. An **abstract polyhedron** is an edge-sharing family $\mathcal{P} = (\mathcal{V},\mathcal{E},\mathcal{F})$ such that:

(i) For each $V \in \mathcal{V}$ the vertex figure $C(V)$ is an abstract polygon;

(ii) No subfamily of $\mathcal{P}$ with the same adjacencies has property (i).

A **geometric polyhedron**, or **polyhedron** for short, is the image of an abstract polyhedron in the d-dimensional Euclidean space E^d, such that vertices are mapped into points, and faces into (geometric) polygons (hence edges into segments, possibly of length 0). Each such geometric polyhedron is said to be a **realization**, or a **representation**, of the abstract polyhedron; in turn, the abstract polyhedron is said to be **underlying** the geometric polyhedron. Note that, in general, there is no requirement that the faces be planar polygons. A geometric polyhedron is called **epipedal** if each of its faces is a planar polygon, and it is called **full-dimensional** if it is not contained in any plane. Traditionally, epipedal polyhedra were the only polyhedra studied; however, polyhedra that are not epipedal can be interesting as well as useful in the study of epipedal polyhedra (see Section 5). Since most of our discussion will concern epipedal polyhedra, unless the opposite is explicitly stated by "polyhedron" we shall mean "full-dimensional epipedal polyhedron".

The polyhedra described by the above restrictions will be called **hollow-faced**. Two hollow-faced polyhedra are said to be **combinatorially equivalent** if they are realizations of the same abstract polyhedron, and **isomorphic** if they are combinatorially equivalent in such a way that the same coincidences of corresponding vertices, as well as the same crossings or coincidences of corresponding edges, take place in both. A polyhedron (abstract or geometric) is called **oriented** if it faces are oriented polygons, such that each edge is oriented in opposite directions in the pair of adjacent polygons that contain it. A polyhedron is **orientable** if its faces can be assigned orientations in such a way that the polyhedron is oriented. Unless the contrary is explicitly mentioned, whenever we present the faces of an orientable polyhedron by lists of vertices, these will be selected in such a way as to make the polyhedron oriented.

Symmetries of polyhedra are defined in complete analogy to those of polygons, as are concepts of **isogonal** or **isotoxal** polyhedra. A polyhedron is **isohedral** [**regular**] if its faces [flags consisting of a vertex, an edge, and a face, all mutually incident] form one orbit under its symmetries.

An abstract polyhedron is **unicursal** if it has unicursal faces and vertex figures. Each unicursal polyhedron $\mathcal{P}$ can be interpreted as a **map**, that is, a cell-complex decomposition of a compact 2-manifold, by considering each of the faces of $\mathcal{P}$ as the boundary of a 2-cell. If $\mathcal{P}$ is orientable so is the map, and the genus of the map is taken as the genus of

$\mathcal{P}$. Abstract polyhedra that are not unicursal correspond to cell complexes more general than 2-manifolds.

It is clear that polyhedra as defined here are generalizations of the usual concept of polyhedra. If the faces of a polyhedron are simple polygons, they determine polygonal regions; if these regions have no unsuitable intersections we have an acoptic polyhedron. This is the basis of constructing cardboard models of such polyhedra. In most illustrations of polyhedra (even non-acoptic ones) the same approach is used; however, for polyhedra with selfintersections its usefulness is limited at best, and at times can be badly misleading. Our illustrations will rely on "skeletal" representations, which show only the segments that form the faces of the polyhedra in question.

A few examples will illustrate the concepts introduced above. The polyhedron $\mathcal{P}$ shown in Figure 5a and described in the caption, is isogonal and oriented; three of its faces are selfintersecting quadrangles. The map $M(\mathcal{P})$ associated with $\mathcal{P}$ is toroidal, the genus of $\mathcal{P}$ is 1.

Next, consider the polyhedron $\mathcal{P}$ shown in Figure 6, and described in its caption. It is oriented and isogonal, and is combinatorially equivalent (but not isomorphic) to the (uniform, Archimedean) truncated cube (which is usually denoted by (3.8.8)); thus the map of the polyhedron is spherical, and so the genus is 0. We note that while it may require some effort to understand the structure of this polyhedron, this task is much easier if we consider the isotopy of the isogonal family of polyhedra to which it belongs (shown in Figure 10 and discussed in Section 4); the labels in one of the diagrams in Figure 10 correspond to the labels in Figure 6(a).

The isogonal and isohedral polyhedron $\mathcal{P}$ shown in Figure 7 is not orientable; its Euler characteristic is –4. Note that some pairs of faces (such as the first two) have two disjoint edges in common, as well as all four vertices. Neither of these coincidences is forbidden by the definition of polyhedra.

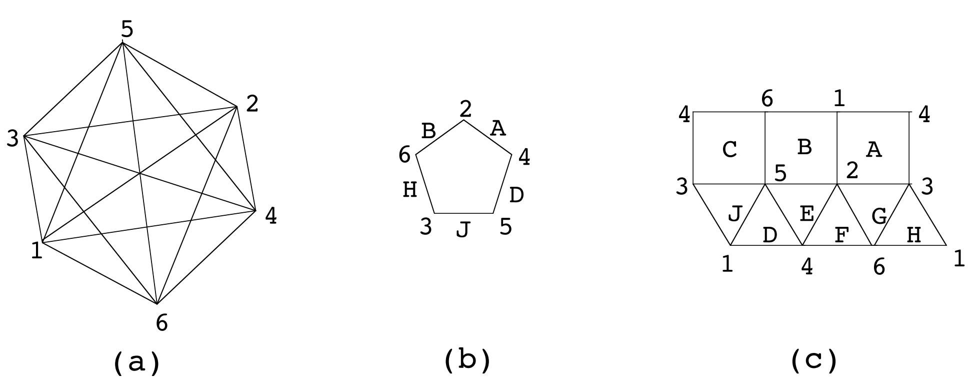
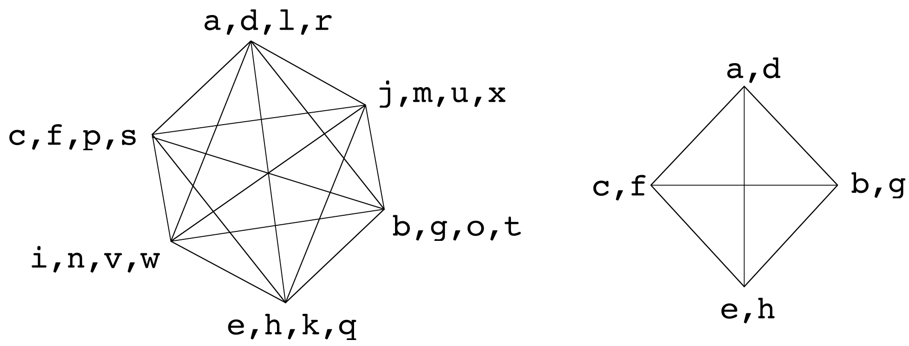


Figure 5. The vertices of the polyhedron $\mathcal{P}$ shown in (a) are labelled 1,2,3,4,5,6; they coincide with the vertices of a regular octahedron. The nine faces of $\mathcal{P}$ are A = 1,2,3,4(1) B = 2,1,6,5(2); C = 4,3,5,6(4); D = 1,4,5(1); E = 2,5,4(2); F = 2,4,6(2); G = 2,6,3(2); H = 1,3,6(1); J = 1,5,3(1); it follows that $\mathcal{P}$ is isogonal. The symmetry group of $\mathcal{P}$ is isomorphic to $[2^+,6]$ in the notation of Coxeter-Moser [8]. The map associated with $\mathcal{P}$, shown in (c), is a torus, hence $\mathcal{P}$ has genus 1. The vertex figure of $\mathcal{P}$ at the vertex 1 is the abstract polygon 2,4,5,3,6(2) schematically shown in (b).



(a) (b)

Figure 6. Quadruplets of vertices of the polyhedron $\mathcal{P}$ coincide with each vertex of a regular octahedron; their labels are a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w,x, as indicated in (a). The faces of $\mathcal{P}$ are a,b,c,d,e,f,g,h(a); a,h,i,j,k,l,m,n(a); k,o,p,q,r,s,t,l(k); q,x,w,e,d,v,u,r(q); b,n,m,t,s,u,v,c(b); f,w,x,p,o,j,i,g(f); a,n,b(a); c,v,d(c); e,w,f(e); g,i,h(g); j,o,k(j); l,t,m(l); p,x,q(p); r,u,s(r). The shape of one of the octagons is indicated in (b), and the other five octagons are congruent to it; all eight triangles are equilateral. All vertex figures are triangles. The symmetry group is isomorphic to [3,4].

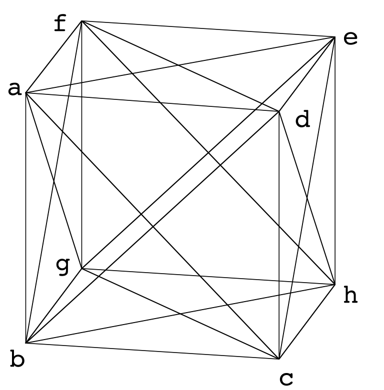


Figure 7.

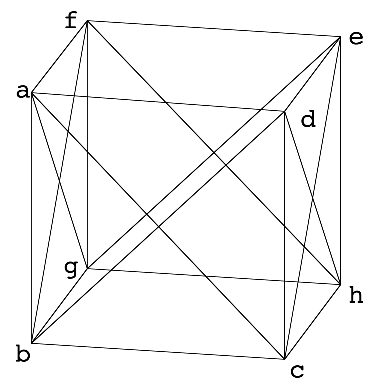


Figure 8.

Figure 7. The eight vertices of $\mathcal{P}$ are at the vertices of a cube. The twelve faces are a,b,d,c(a); a,c,b,d(a); a,b,f,g(a); a,f,b,g(a); a,d,f,e(a); a,e,d,f(a); b,c,g,h(b); b,g,c,h(b); c,d,h,e(c); c,e,d,h(c); e,f,h,g(e); e,g,f,h(e). All faces are congruent selfintersecting quadrangles. This polyhedron $\mathcal{P}$ is isogonal and isohedral, but it is not orientable; its Euler characteristic is –4, and its symmetry group is isomorphic to [3,4].

Figure 8. The vertices of this polyhedron $\mathcal{P}$ are the same as in Figure 7; its faces are a,c,b,d(a); a,d,h,g(a); a,f,h,c(a); a,g,b,f(a); b,c,e,f(b); b,g,e,d(b); ,h,d,e(c); e,g,h,f(e). This $\mathcal{P}$ is oriented and isogonal, but not isohedral; its symmetry group is isomorphic to [2,4]. The associated map is one of the regular maps on the torus.

The map corresponding to the polyhedron $\mathcal{P}$ in Figure 8 is a regular map on the torus. In other words, the underlying polyhedron of $\mathcal{P}$ is regular; however, this particular geometric realization is isogonal but not regular. It is not known whether any realization of this map by a full-dimensional, epipedal geometric polyhedron is a regular polyhedron.

A family of non-unicursal polyhedra, which we call **corrugated pyramids**, is illustrated in Figure 9. A pyramid over any non-unicursal polygon provides another example of a non-unicursal polyhedron. More interesting examples will be encountered in Section 6.

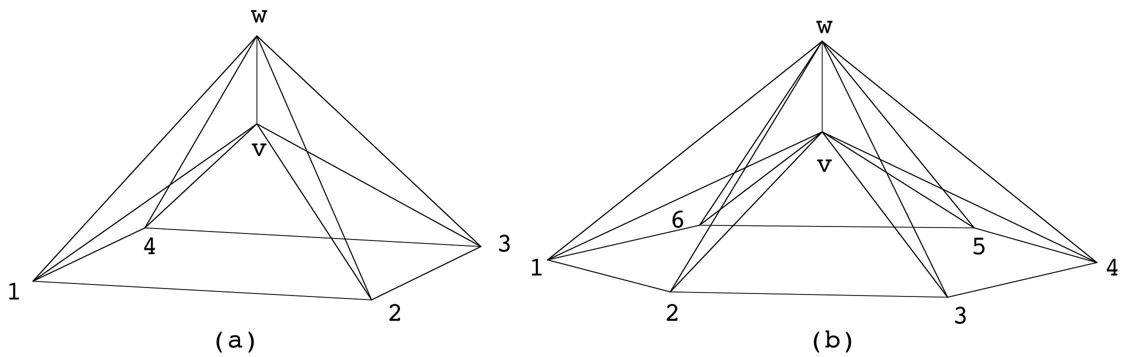


Figure 9. Examples of corrugated pyramids: 4-sided in (a), 6-sided in (b). In general, a 2m-sided corrugated pyramid has as basis a (2m)-gon with vertices 1,2,...,2m(1), and two additional vertices v and w. Besides the basis, its faces are j,v,w(j) for j = 1,2,...,2m, as well as 2j,2j+1,v(2j) and 2j-1,2j,w(2j-1) for j = 1,2,...,m. For each j = 1,2,...,m the faces j,v,w(j) and j+m,v,w(j+m) are adjacent along the common edge v,w. The corrugated pyramids are not unicursal since the vertex figures at v and w are not unicursal polygons. The 2m-sided corrugated pyramid is orientable for odd m, non-orientable for even m. The lists of faces just given are not arranged to show orientability.

4. Isogonal polyhedra. We now turn to the consideration of some special hollow-faced polyhedra. As mentioned in Section 2, for each n, all isogonal n-gons fit into a finite number of families within which the members are related by isotopy (via isogonal n-gons). Isogonal polyhedra can be arranged in families of a similar nature. (For convex isogonal polyhedra such isotopies have been described by Robertson [29] and others.) An example of such an isotopy is shown in Figure 10; this extends the isotopy on page 92 of [29]. It should be noted that the last polyhedron in the isotopy is inverse to (that is, negative of) the starting polyhedron; hence to get to the original polyhedron (and not a mirror image of it) an extension of the isotopy is necessary. In either variant it can be shown that every realization (as isogonal polyhedron) of the abstract polyhedron which underlies those in Figure 10 has the same symmetry group [3,4] as those in Figure 10, and belongs to the same isotopy arc as shown there (or its extension, as just mentioned).

Similar isotopies exist for other types of isogonal polyhedra, as well as for isohedral polyhedra. Most of them have not been investigated in detail, and the topological character of their realization space is not known. Also, in only a few cases other than the one shown in Figure 10 is it known how many disjoint isotopic families there are of isogonal (or of isohedral) polyhedra which are realizations of the same abstract polyhedron. Since in the case of n-gons (for even n) the number of isotopic families grows linearly with n, prisms over such polygons show that there is no finite upper bound on the number of isotopic families of isogonal polyhedra combinatorially equivalent to each other.

While it is obvious that the vertices of a uniform polyhedron must all be located at sets of points which form one orbit under symmetries, this does not limit the types of isogonal polyhedra to the traditional ones since in the present context several vertices may be represented by the same point. As an example of such a "new" type consider the following polyhedron $\mathcal{P}$ which is not merely isogonal but uniform (since all its faces are regular polygons). The n = 12m vertices of $\mathcal{P}$ are located at the vertices of a uniform antiprism based on a regular (3m)-gon {3m}. Each vertex of the antiprism is the image of two vertices of p. A sketch of p (for m = 2) is shown in Figure 11. The peculiar labelling is meant to make obvious the formation of the faces, the claim of isogonality, and the

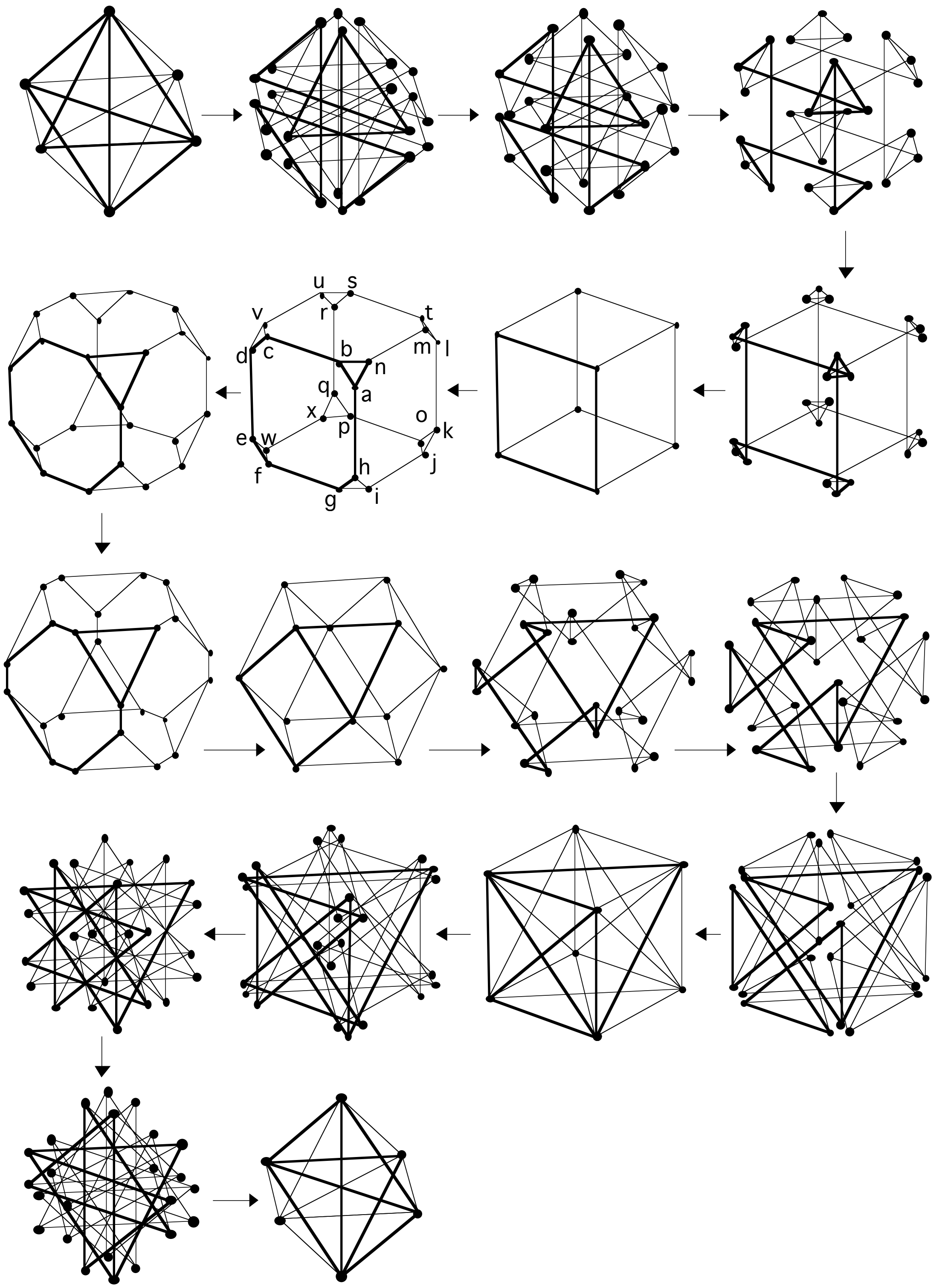


Figure 10. An illustration of the isotopy connecting all isogonal polyhedra isomorphic with the truncated cube; note that at certain stages pairs, triplets or quadruplets of vertices are represented by the same point. In particular, the polyhedra appearing to be a cube, an octahedron and a cuboctahedron are not isomorphic to these classical polyhedra. Only the polyhedra strictly between the "cube" and the "cuboctahedron" are convex. One representative of each orbit of faces is emphasized.

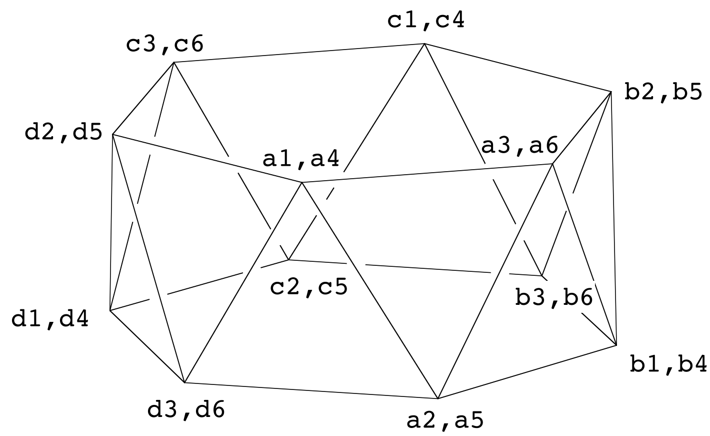


Figure 11. A uniform polyhedron p of genus 5, all faces of which are regular hexagons: four faces of type {6}, and twelve faces of type {6/2}. Details of the construction are indicated in the text. The symmetry group is the direct product of $[2^+,12]$ and the cyclic group c2.

generalization to all values of m. The "bases" are regular hexagons {6} (regular polygons {3m} in general); the two upper bases are a1,a6,b5,c4,c3,d2(a1) and a4,a3,b2,c1,c6,d5(a4), while the two lower bases are a2,d3,d4,c5,b6,b1(a2) and a5,d6,d1,c2,b3,b4(a5). All "mantle faces" are regular hexagons {6/2}, formed by vertices with labels (disregarding the letters) 1,2,3,4,5,6(1) if they have edges on upper bases, and 6,5,4,3,2,1(6) if they have edges on lower bases. The polyhedron $\mathcal{P}$ is oriented and of genus 5 (in the general case the genus is 3m - 1).

Many other types of isogonal (or even uniform) polyhedra exist. There seems to be no detailed information available on the possibilities. It is even not known whether every isogonal type is combinatorially equivalent to a uniform polyhedron — a fact that is well known for convex polyhedra. Prime candidates for isogonal polyhedra not isomorphic to uniform ones are the various isogonal polyhedra of positive genus described in [17].

5. Regular polyhedra. Turning now to the question what are the regular polyhedra that can be constructed with hollow faces, we note first that they all must "look like" the "old" regular polyhedra; however, this does not preclude the existence of infinitely many "new" ones. (See Section 7, as well as [15], for remarks about the "new" regular polyhedra.) We start by recalling that the four well-known Kepler-Poinsot polyhedra are, in fact, polyhedra with hollow faces, and are best understood as such. Next, there is a set of eight new regular polyhedra which are obtainable by a certain construction from the Platonic and Kepler-Poinsot polyhedra (other than the cube). To obtain one of the new polyhedra, we replace each vertex v of the original polyhedron by two vertices, v^+ and v^-. The faces of the new polyhedra are obtained by going around the original faces, but alternating between + and - labelled vertices. Since the number of edges of each face of P is odd, to complete a face of the new polyhedron we need to go twice around the old one. Hence if P was the regular polyhedron {n,k}, with n odd, the new polyhedron is {(2n)/2,k}. (We use the natural extension of the familiar Schläfli notation.) The polyhedra {6/2,3} and {6/2,4} are shown in Figure 12 together with the corresponding regular maps. Here and in later captions maps are identified by a symbol of the type W#e.f, which means that it is entry f among maps with e edges listed in the catalogue of S. E. Wilson [32]. Although this construction does not work for the cube, there exists a regular polyhedron $\{8/2,3\}$. It is shown in Figure 13, and its regularity is easy to verify.

One way to get an understanding of the method used in constructing the polyhedra in Figure 12 and the six analogous ones is the following. Start with any one of the classical regular polyhedra (except the cube). Replace every n-gonal face $\{n/d\}$ by a regular skew prismatic (2n)-gon (that is, $\{2.n^{\alpha}/d\}$ in the notation of [14]), identifying the edges of adjacent polygons in the obvious way. It is clear that the polygons do not really fit -- but the fit improves as α decreases; in the limit, "squashing" the prismatic polygons to $\alpha = 0$, we get the polyhedra as described above.

The reason for bringing up this explanation (and the skew polygons that appear in it) is that a similar construction applied to the regular octahedron leads to infinitely many new regular polyhedra. We start with the regular octahedron $\{3,4\}$, and replace (abstractly) each triangle by a regular helical polygon $\{\infty^{\alpha,\beta}\}$ (see [14]) with $\beta = 2\pi/3$ (that is, by an infinite triangular helix); adjacent helices should wind in opposite directions. direction. This clearly results in a regular polyhedron — admittedly, an abstract one, and one which uses infinite skew helices as faces. Selecting any integer $d \geq 2$ that is not a multiple of 3, identifying in each helical polygon vertices that are at distance 3d along the helix from each other, and finally squashing the helices to $\alpha = 0$ we get an epipedal, regular polyhedron $\{(3d)/d,4\}$. For $d = 2$ we obtain again the polyhedron $\{6/2,4\}$ shown in Figure 12(b). The smallest new polyhedron $\{12/4,4\}$ is obtained for $d = 4$; it is shown, together with the corresponding map, in Figure 14.

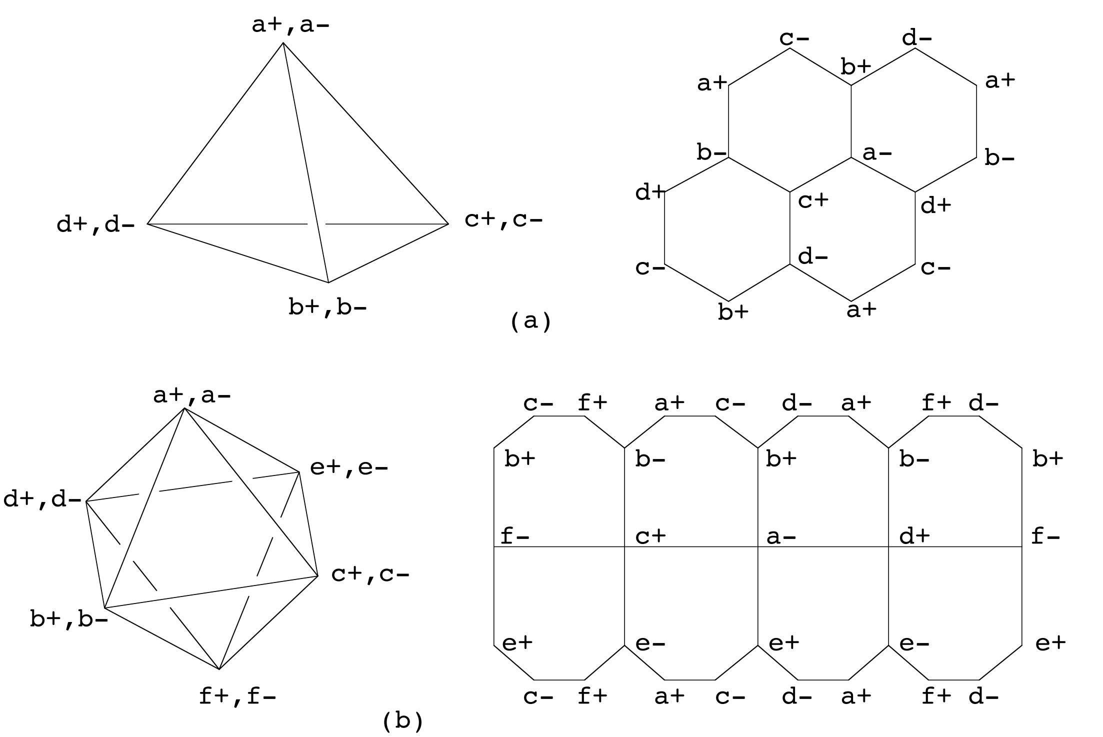


Figure 12. Two regular polyhedra with hollow faces, and the corresponding regular maps. The construction is explained in the text. The polyhedron in (a) is $\{6/2,3\}$ with symmetry group isomorphic to $c2 \otimes [3,3]$ and map W#12.3 of genus 1, that in (b) is $\{6/2,4\}$ with symmetry group isomorphic to $c2 \otimes [3,4]$ and map W#24.27 of genus 3.

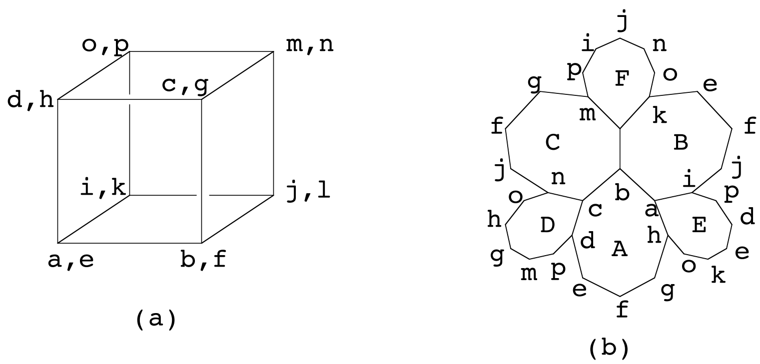


Figure 13. (a) An orientable regular polyhedron {8/2, 3} with 16 vertices and six octagonal faces: A = a,b,c,d,e,f,g,h(a); B = a,i,j,f,e,k,l,b(a); C = b,l,m,g,f,j,n,c(b); D = c,n,o,h,g,m,p,d(c); E = a,h,o,k,e,d,p,i(a); F = i,p,m,l,k,o,n,j(i). (b) A topologically regular map (W#24.21) of genus 2, which is isomorphic to the regular polyhedron in (a).

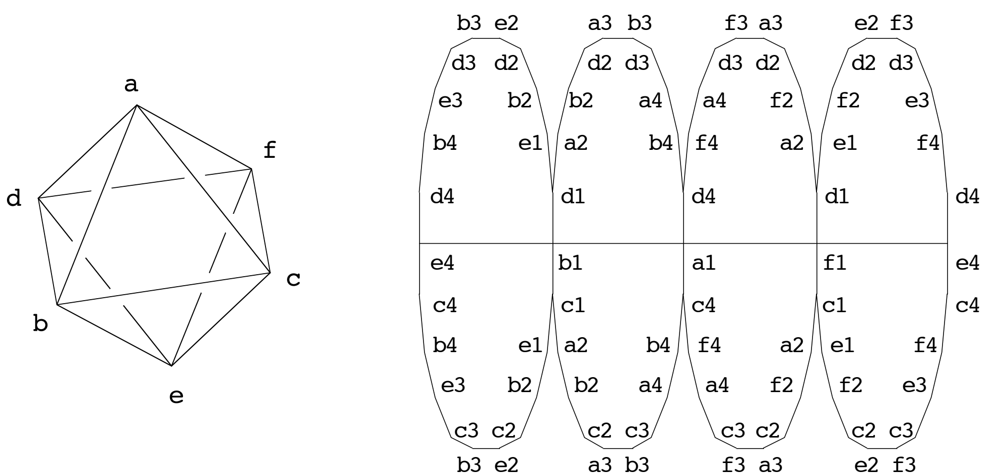


Figure 14. The regular polyhedron {12/4,4} obtained by the construction explained in the text. Each letter near a vertex in the diagram at left stands for labels of four vertices, distinguished by subscripts 1,2,3,4. The corresponding map is W#48.45, of genus 9.

Although this method applies only to the octahedron, there exist other regular polyhedra in which faces are of the type {(k·d)/d}. As an example, a polyhedron {12/3,3} is shown in Figure 15. On the other hand, it seems that not every symbol of the type {(a·b)/b,c} corresponds to a regular polyhedron. For example, there appears to exist no hollow-faced polyhedron {9/3,3}.

We conclude this brief introduction to regular polyhedra by noting that the Schläfli symbols we use are inadequate to characterize these polyhedra. For example, the polyhedron shown in Figure 16 has symbol {6/2,3}, the same as the polyhedron in Figure 12(a) — although the former has 12 faces and the latter only 4. Thus, there is a need for a better notation.

Additional comments concerning regular polyhedra will be made in Section 7.

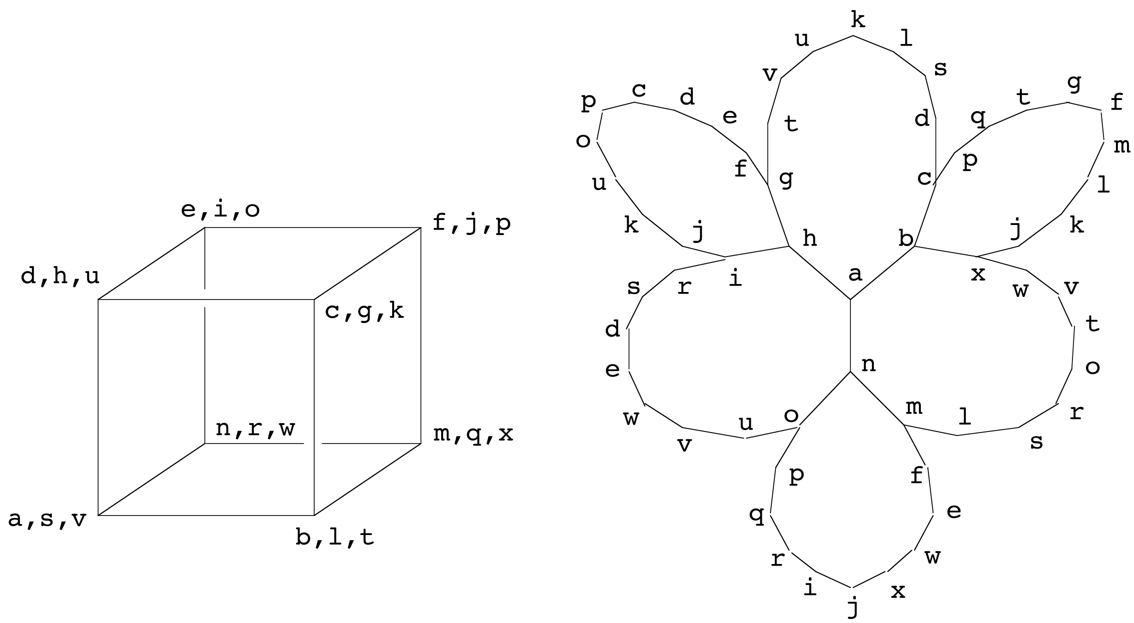


Figure 15. A regular polyhedron {12/3,3} and the corresponding map W#36.13 of genus 4.

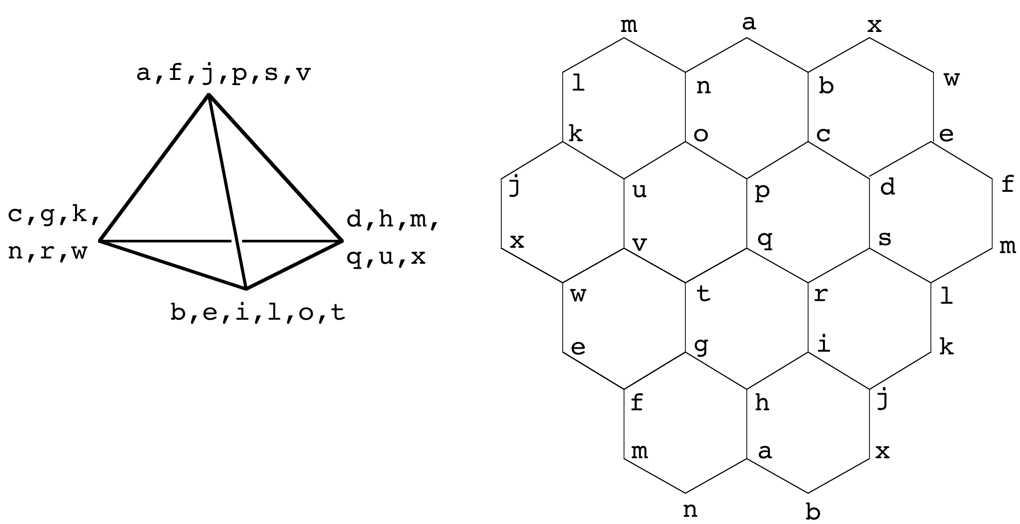


Figure 16. Another orientable regular polyhedron {6/2,3}. It has 12 faces and 24 vertices, and the corresponding map is W#36.12 of genus 1.

6. Noble polyhedra. The last topic we shall discuss concerns an interesting family of polyhedra that briefly emerged in the works of Hess [24], [25], [26], and Brückner [4], [5], [6], [7], but seems to have been completely forgotten since the early years of this century: the hollow-faced polyhedra which are both isogonal and isohedral. In order to simplify the syntax, we shall call such polyhedra "noble". Naturally, all regular polyhedra are noble, as are all (necessarily convex) "sphenoids" (that is, tetrahedra with all faces mutually congruent). In fact, among convex polyhedra these are the only possibilities. However, among hollow-faced polyhedra there are many additional noble polyhedra, and their complete enumeration with the definition of polyhedron adopted here is still outstanding. The papers by Hess and Brückner may have choked off the investigation of noble polyhedra by their vagueness, inconsistency and errors. Some examples of these shortcomings will be given below, while the full discussion must await another occasion.

We begin with the polyhedra that appear to have been discovered by Hess, called "stephanoids" by Hess and Brückner. The name comes from the Greek word for "crown", and was chosen by Hess on account of the appearance of these polyhedra. They form two infinite families, related to the prisms and antiprisms, which we shall call **prismatic** (resp. **antiprismatic**) **crown** polyhedra. Combinatorially, each such polyhedron depends on three positive integer parameters. To obtain the prismatic crown polyhedron PC(n,p,q), where $2p - n < 2q < p < n$, we start from an n-sided prism with vertices $a_1,..., a_n, b_1,..., b_n$ (see Figure 17(a)), and for $j = 1,..., n$ we form the selfintersecting quadrangles $a_j,b_{j+q},a_{j+p},b_{j+p-q}(a_j)$ and $b_j,a_{j+q},b_{j+p},a_{j+p-q}(b_j)$. It is easily checked that these 2n faces indeed define a polyhedron, that the polyhedron is orientable of genus 1, and that it is both isogonal and isohedral. The antiprismatic crown polyhedron AC(n,p,q), where $2p-n < q < p < n$ and q is odd, results from the n-sided antiprism with vertices of the mantle labelled consecutively $a_1,..., a_{2n}$ (see Figure 17(b)) by constructing all the quadrangles $a_j,a_{j+q},a_{j+2p},a_{j+2p-q}(a_j)$, for $j = 1,...,2n$. The resulting polyhedron is again orientable of

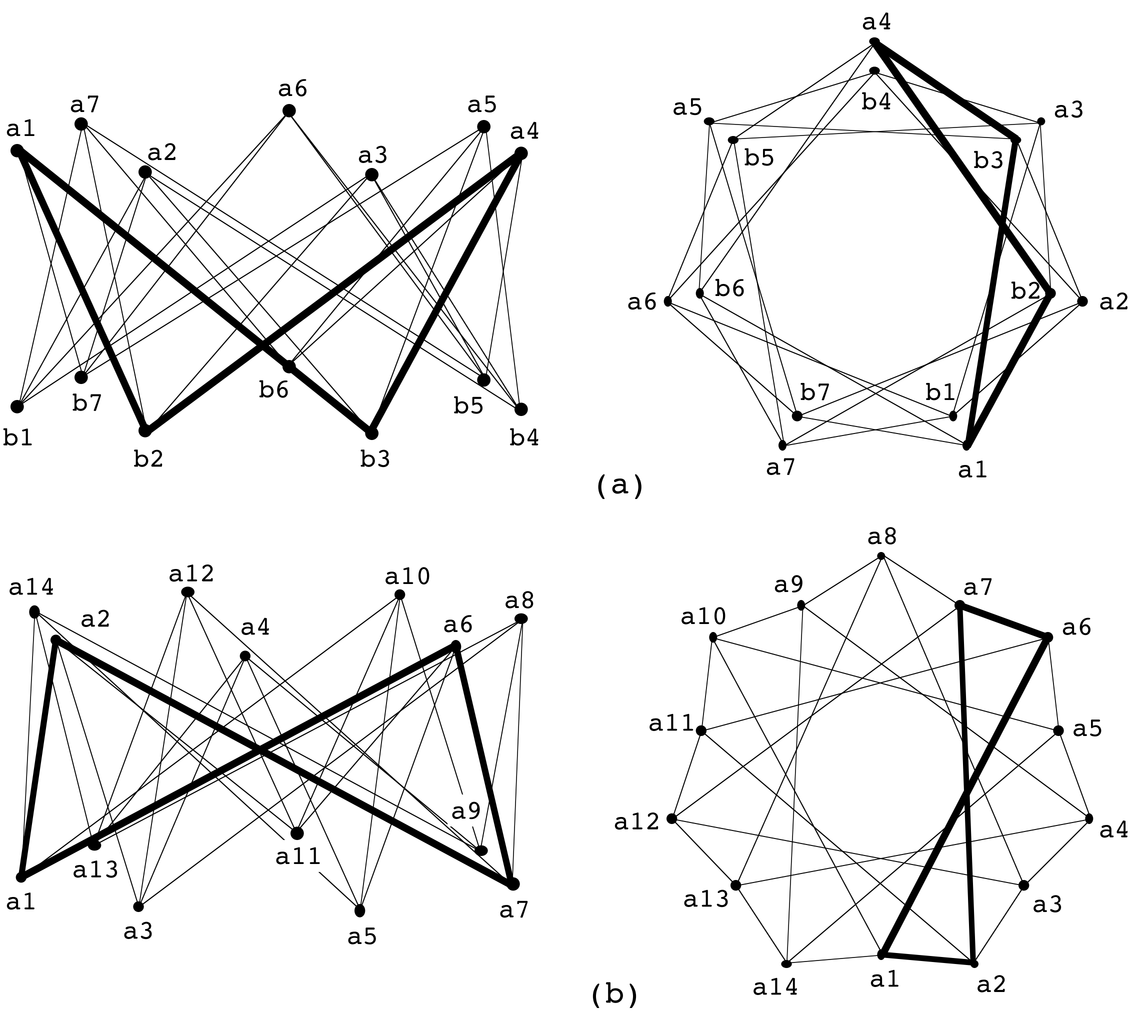


Figure 17. Crown polyhedra, PC(7,3,1) in (a) and AP(7,3,2) in (b), each in two perspective views; the views at right are from a point on the axis of the polyhedron, not very high above the polyhedron itself.

genus 1, and it is isogonal and isohedral. The crown polyhedra depend also on a continuous parameter, determined by the metric proportions of the starting prism or antiprism. While all crown polyhedra are isomorphic to their polars (polarity with respect to a unit sphere with the same center), for suitable choices of the continuous parameter the polar polyhedron is congruent to the original. More interestingly, however, for some sets of the discrete parameters (for even p in the prismatic case, or even $p - q$ in the antiprismatic case), the continuous parameter can be chosen so that the polar polyhedron is identical with the starting one. A polyhedron which is selfpolar in this most restricted sense will be called **autopolar**. Clearly, no convex polyhedron is autopolar.

Other noble polyhedra arise if each of the quadrangles that form a crown polyhedron is decomposed into two triangles by one of its diagonals (either the short one, or the long one, in all faces). We obtain a triangle-faced noble polyhedron which we call a prismatic or antiprismatic **wreath** polyhedron; its vertices are 6-valent, and it is orientable of genus 1. Using a notation analogous to the one for crown polyhedra, the examples shown in Figure 18 are PW(7,3,1) and AW(7,3,2). Since the vertices of wreath polyhedra are 6-valent, infinitely many other noble polyhedra may be obtained by applying to wreath polyhedra with even n an analogue of the construction using helical polygons described in Section 5.

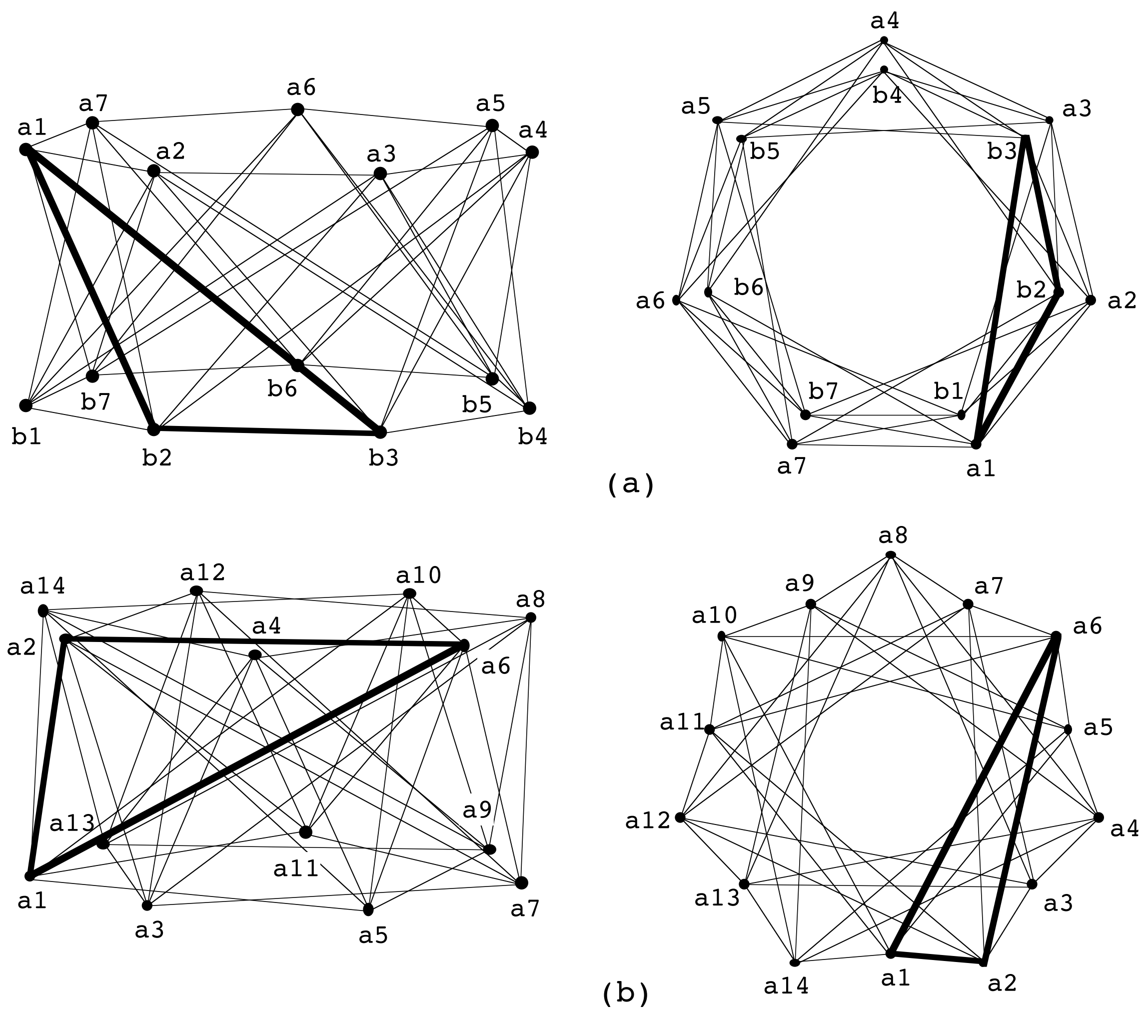


Figure 18. Two wreath polyhedra, each shown in two perspective views. (a) A prismatic polyhedron PW(7,3,1); (b) an antiprismatic polyhedron AW(7,3,2).

An additional infinite class of noble polyhedra, the family of V-faced polyhedra, is illustrated in Figure 19. The vertices of V-faced polyhedra are situated at the vertices of (skew) antiprismatic and prismatic polygons (see [14]); their faces are quadrangles in which one pair of nonadjacent vertices is represented by the same point. In the notation used in Figure 19, the faces are $a_j, a_{j+1}, b_j, a_{j-1}(a_j)$ and $b_j, b_{j+1}, a_j, b_{j-1}(b_j)$, for $j = 1,2,\ldots,n$.

We next show a few examples of noble polyhedra related to regular or Archimedean polyhedra, in hope that they will whet the reader's appetite for a more thorough investigation of such polyhedra. Figure 20(a) shows an autopolar noble polyhedron that has 20 hexagonal faces (one of which is indicated by heavy lines) and 20 vertices. Figure 20(b) can be interpreted as illustrating several remarkable noble polyhedra; we denote the first two by $\mathcal{P}_1$ and $\mathcal{P}_2$. We start with 12 points at the vertices of a regular icosahedron as vertices, and as edges we take all the segments determined by the vertices, except the diameters of the icosahedron; see Figure 20(b). The faces of $\mathcal{P}_1$ and $\mathcal{P}_2$ are the 10-gons shown in Figure 1(c) and 1(d), respectively; the edges of one face are heavily drawn in Figure 20(b). It is easily checked that both $\mathcal{P}_1$ and $\mathcal{P}_2$ are autopolar. To obtain two other noble polyhedra, we interpret each point v which is a vertex of the icosahedron in Figure 20(b) as representing two vertices v^+ and v^- of the polyhedron we are constructing. Each of $\mathcal{P}_3$ and $\mathcal{P}_4$ has 24 isogonal decagons as faces. One face of $\mathcal{P}_3$ is $b^+,c^-,e^+,f^-,c^+,d^-,f^+,b^-,d^+,e^-(b^+)$, another is obtained from this by interchanging + and -; note that the vertices of either of these faces are at points that in the original icosahedron were adjacent to one vertex. The other 22 faces of $\mathcal{P}_3$ are obtained analogously, from points adjacent to other vertices of the original icosahedron. The polyhedron $\mathcal{P}_4$ is obtained similarly, except that the faces are now all congruent to $b^+,c^-,f^+,b^-,e^+,f^-,d^+,e^-,c^+,d^-(b^+)$. The isogonality and isohedrality of the polyhedra is easily established in all cases.

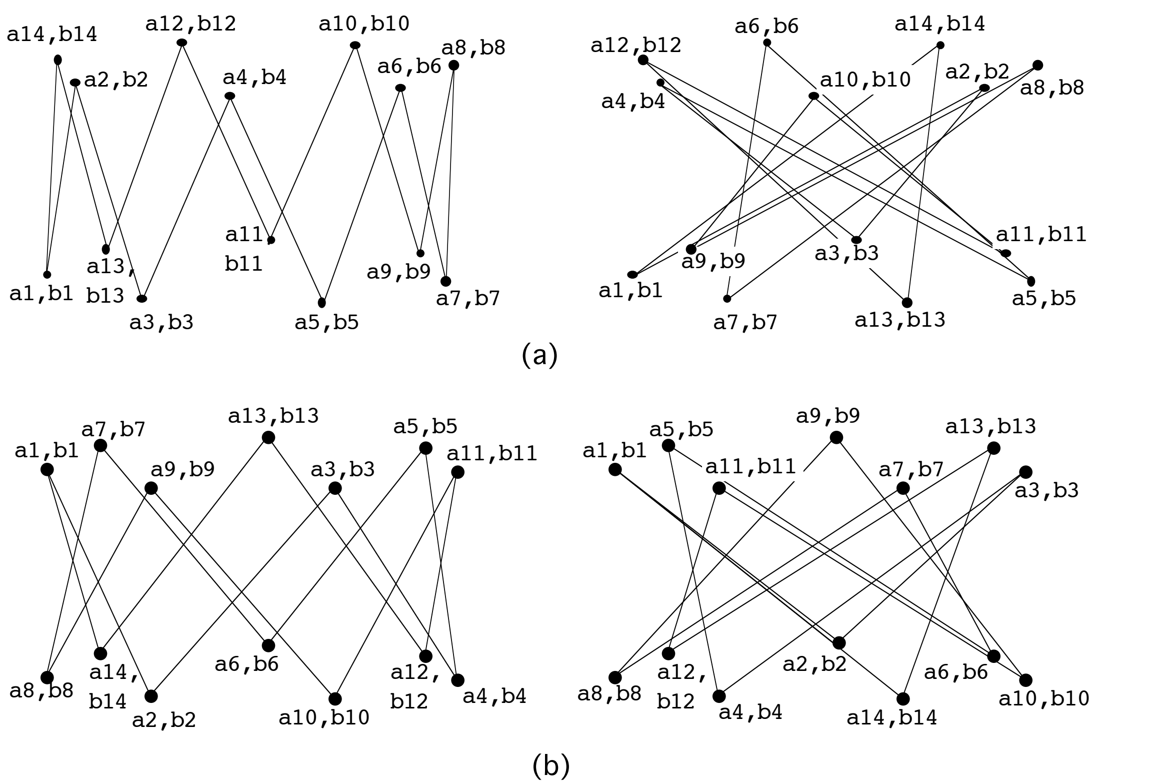


Figure 19. Antiprismatic and prismatic V-faced polyhedra in which each face is a quadrangle with two nonadjacent vertices at the same point.

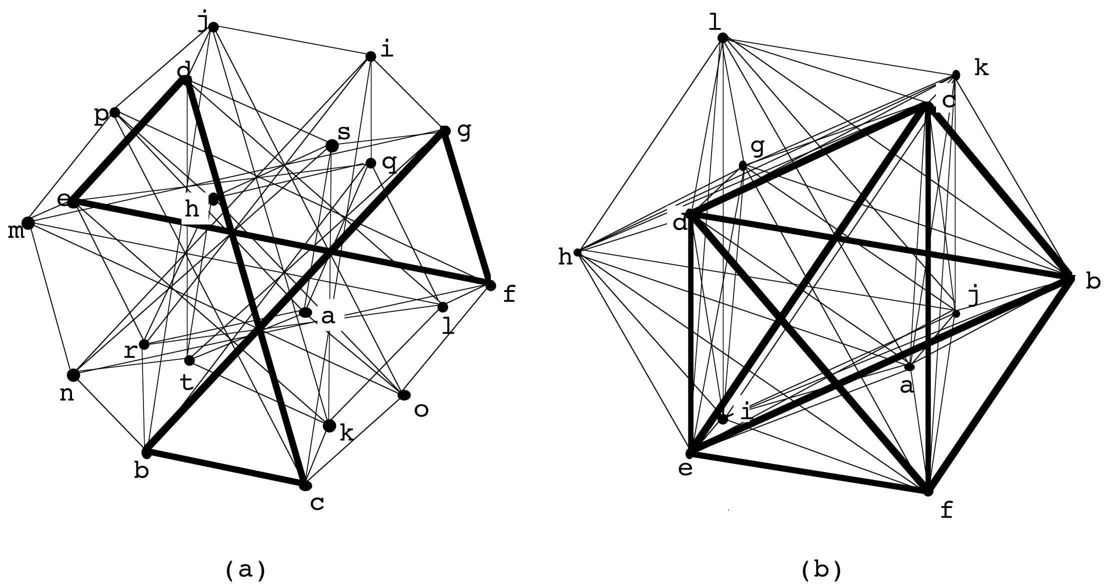


Figure 20. (a) An autopolar noble polyhedron described by Brückner [6], with 20 vertices and 20 hexagonal faces. (b) A sketch of two noble polyhedra with 24 vertices and 24 decagonal faces. Details are given in the text.

Our final examples of noble polyhedra are shown in Figure 21. They can most easily be understood as being constructed on a rhombicuboctahedral scaffold (indicated by the dotted lines). The edges of all four are the same, but their organization into faces is different; the faces of the first and third polyhedron are polygons of rotation number 0, those of the second and fourth have rotation number ±2. Details of the construction are indicated in the caption.

7. Remarks and comments. The desirability and necessity of formulating a consistent theory of polygons and polyhedra in which selfintersections are permitted becomes obvious when one tries to find existing writings on this topic. Meister [27] started developing such a theory in 1769, but the influence of his writings — often quoted but apparently hardly ever looked at — was negligible. At least in part this is due to a misrepresentation of his definitions by later authors, such as Günther [22] and Brückner [4]; see [15] and [16] for details. In fact, our definition is nothing but Meister's put in more modern language. Brückner's book, which is supposed to be a survey of previous knowledge concerning nonconvex polyhedra, is so deeply flawed precisely in presenting this topic, that it is useless as a basis for any research on hollow-faced polyhedra. The worst offense is the book's internal inconsistency, even concerning concepts as basic as the definition a polygon or polyhedron. While one may try to excuse the vagueness and imprecision as consequences of the period in which [4] was written, on closer inspection it turns out that the book is inconsistent, as are most writings on the topic of polyhedra more general than convex polyhedra. This lack of precise concepts becomes particularly aggravating, and impedes any resolution of basic questions, when one is considering regular polyhedra or noble ones. A detailed description of the faults has no place here, but we have to mention the fact that on page 2 of [4] a polygon is defined by requiring (among other properties) that each endpoint of each edge is the endpoint of one and only one other edge; however, later on (for example, on page 215) he considers a 10-gon in which pairs of vertices coincide; this is the polygon in

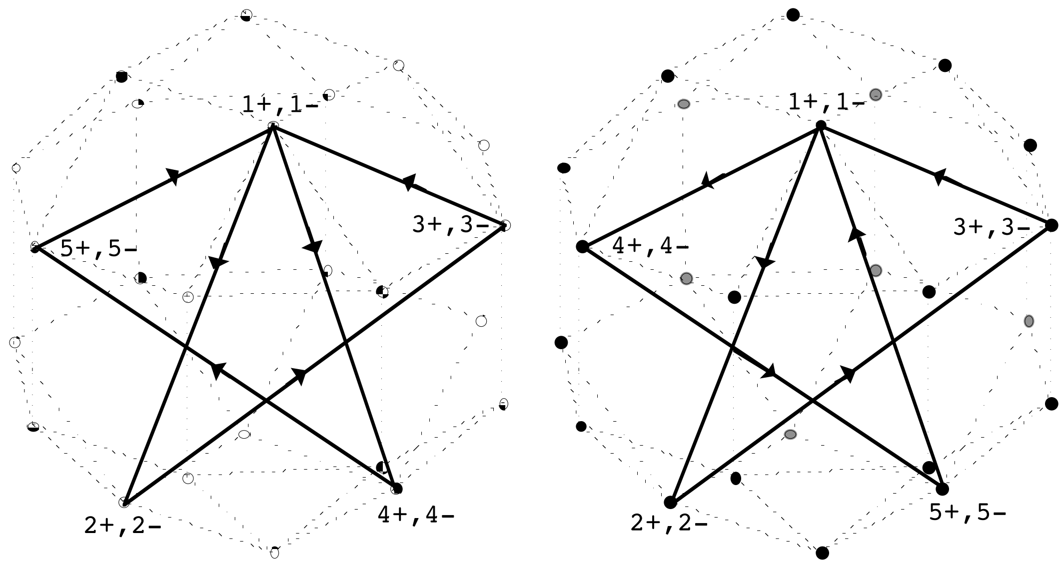


Figure 21. Schemes for the construction of four noble polyhedra. For the first two, the vertices are those of the rhombicuboctahedron (which is shown by dotted lines) One face of each polyhedron is 1,2,3,1,4,5(1); note the difference in labelling of the two cases. As an aid to visualization, the selected face was shown as an oriented polygon. The other 23 faces are obtained by symmetries of the rhombicuboctahedron. Two additional noble polyhedra result if every vertex of the rhombicuboctahedron stands for two vertices of the noble polyhedron; the two vertices are distinguished by + or - in their labels. One face of each polyhedron is 1+,2-,3+,1-,4+,5-(1+); the other 47 faces are obtained by symmetries (including those interchanging + and -).

our Figure 1(d). It is interesting to note that Brückner never mentions the 10-gon in our Figure 1(c), which should have been considered at the same place, since it leads to a noble polyhedron just as the former does (see Figure 21). Brückner's declared intention that the polygons considered as faces of polyhedra be unicursal could have been met by doubling up the vertices as we did in the construction of the third and fourth noble polyhedra described in Figure 21. But Brückner remained inconsistent and imprecise concerning allowed polygons, and polyhedra; neither Brückner nor anyone else explained, at any time, precisely what is to be understood by a "polyhedron" in which some or all faces have vertices at which four edges of one face meet, or have multiple vertices at the same point. Until such a definition is adopted, statements about such "polyhedra" are devoid of meaning. After a definition is adopted (be it the one we used, or some other) it is possible to decide whether a bunch of polygons is, or is not, a polyhedron of a specified kind.

One other shortcoming of [4] is the lack of consideration of regular polygons {n/d}, with n and d not relatively prime. Clearly, one can require from the start that all vertices of a polygon should not only be distinguishable by the role they play in the polygon (that is, by their labels), but be represented by distinct points. With this additional condition, the classical enumeration of regular polyhedra becomes valid, as do various other results. However, it should be realized that this is an arbitrary restriction which should not be imposed for several reasons: (i) a more satisfactory and far richer theory can be developed without the restriction — with the restriction, various polyhedra discussed elsewhere in [4] or [5] are inadmissible; (ii) continuity and consistency are lost in such situations as the isotopies considered in our Section 4, or in considerations of winding numbers and rotation numbers; (iii) some very beautiful results are lost if this restriction is imposed. The validity of (i) is evident even from the modest efforts presented here. Some aspects of (ii) are discussed in [16] and [19], but it is strange that Hess, by failing to adhere consistently to his own definitions, was unable — in an article [23] of well over a hundred pages — to rid himself of the confusion between winding numbers and rotation numbers of polygons. However, (iii) deserves a short explanation.

It is curious fact that a set of beautiful results concerning polygons has been developed over the last 150 or so years, without attracting any attention on the part of most geometers; the author regrets to have been part of that majority till very recently. The results in question deal with various kinds of "smoothing" operations on polygons; some of them replace a given polygon by one in which vertices are weighted averages of sets of vertices of the starting one, or points associated in other ways with the original polygon. Some of the results can be found in [1], [3], [9], [28] and a variety of other books and articles (a survey of the topic, with extensive references, is in preparation); part of the results are often said to be generalizations of "Napoleon's theorem". But the one aspect of all this which is relevant to the present discussion is that these results could not even be formulated, much less established, without using regular polygons in the generality adopted here. The historical account and explanations given in [15] concerning the almost two centuries of narrow interpretation of "regular polygons" in the theory of regular polyhedra remain valid. However, one should add that throughout this period, a group of mathematicians used the more general definition of polygons, without any reservations and, probably, without realizing that their usage is not the one generally adhered to. Still without a satisfactory explanation is the fact that prior to [15] there was no attempt to use regular polygons such as $\{6/2\}$ or $\{8/2\}$ in the formation of regular or other polyhedra.

At this time, there are several open challenges concerning hollow-faced polyhedra. The most obvious one is to find a complete characterization of the regular ones among them, and to determine the corresponding groups of symmetries. This, in fact, consists of several different subproblems, depending on the generality considered (see below). Other related directions of desirable investigation concern polyhedra that are isogonal, or noble, or have some other transitivity properties; also, enumeration of all types of hollow polyhedra with a given (small) number of edges should be interesting.

The family of hollow-faced polyhedra reestablishes in part the correspondence between polyhedra and maps on manifolds that is so useful in the study of convex polyhedra. But it should be noted that in order to get a more satisfactory relationship, a class of polyhedra more general than the hollow-faced ones should be considered. In our presentation, two faces may have several vertices (or edges) in common, while two vertices may not have more than one edge connecting them. A generalization admitting such and other incidences forbidden for hollow-faced polyhedra can be developed, but this has not been done so far. One other benefit of the introduction of such generality would be the unlimited possibility of polarity between polyhedra in question; as is easily seen, the family of hollow-faced polyhedra is not closed under duality.

The noble polyhedron in Figure 20(a) and the polyhedron $\mathcal{P}_2$ of Figure 20(b) were briefly mentioned by Hess [25], and described in detail by Brückner [6]. Both failed to notice the noble polyhedron $\mathcal{P}_1$. Neither Hess nor Brückner appear concerned about the fact that $\mathcal{P}_2$ is not a polyhedron according to the definitions they purport to use. Naturally, they fail to discuss polyhedra such as $\mathcal{P}_3$ or $\mathcal{P}_4$, in which distinct vertices are represented by the same point. The first two noble polyhedra in Figure 21 are also due to Hess and Brückner; these polyhedra are remarkable in that they are noble although their faces and vertex figures are rather irregular. Many other possibilities for noble polyhedra exist (several of which were investigated by Hess and Brückner), but no complete enumeration is available. Particularly interesting would be a complete list of autopolar noble polyhedra.

Also deserving consideration are **compounds** of regular or noble polyhedra, that is, families of separate congruent polyhedra with a large degree of symmetry of the union. In the old literature (and in some more recent works) these are called "discontinuous polyhedra"; however, in modern contexts it appears more reasonable to restrict the concept of "polyhedron" to connected families of polygons, as done in this note.

The topic of polyhedra that are not epipedal is almost completely unexplored. Such polyhedra appear very naturally as Petrie-polyhedra of other (including epipedal) polyhedra, and are probably the most general type of object in 3-space one may wish to call "polyhedron". However, there has been no work done on such polyhedra beyond the initial discussion in [15], the completion of the enumeration of regular ones among these polyhedra by Dress [9], [11], and the papers of Farris [12], [13] on "fully transitive" polyhedra. Moreover, even in these papers the generality is restricted by insisting that the vertices of each skew polygon are represented by distinct points. Thus there is great scope for further work in this direction as well.

8. Comments added in proof (13 December 1993).

Two additional facts concerning the autopolar noble polyhedron in Figure 20(a) have come to the author's attention.

(i) The polyhedron was first described, and shown in a pair of stereoscopic diagrams, by Theodor Hugel, in a booklet "Die regulären und halbregulären Polyeder" (published by Gottschick-Witter, Neustadt a. d. H., 1876). In fact, in Hugel's view this polyhedron should be considered to be as regular as the Kepler-Poinsot polyhedra. Naturally, this depends on the definitions adopted, and the ones proposed by Hugel are not considered appropriate today.

(ii) J. M. Wills observed (in "The combinatorially regular polyhedra of index 2", Aequationes Math. 34(1987), 206 – 220; see p. 212) that the map which corresponds to the polyhedron in Figure 20(a) is, in fact, the regular map W#60.57. No doubt, Hugel would have been pleased!

Not much attention was given in the present paper to non-unicursal polygons and polyhedra, hence a few additional words may be appropriate. A description of the regular polygons of this kind is quite simple. We start from a regular unicursal polygon $\{n/d\}$, with edges e_i, where e_i is adjacent to e_{i+1} for $i = 1, \ldots, n-1$, and e_n is adjacent to e_1. We replace each edge e_i by k edges $e_{i,1}, \ldots, e_{i,k}$, and declare as adjacent edges $e_{i,j}$ and $e_{i+1,j}$ for $i = 1, \ldots, n-1$, as well as $e_{n,j}$ as adjacent to $e_{1,j+1}$. This leads to a regular polygon with n vertices and nk edges, that can be denoted by $\{n_k/d\}$. Examples of isogonal (but not regular) non-unicursal polygons are shown in parts (c) to (h) of Figure 1. The complete characterization of such polygons appears to be not too complicated, but has not been carried out so far.

Regarding polyhedra, one can start from a unicursal polyhedron in which sets of vertices are represented by the same point, and identify all the vertices in some or all such sets; whether or not a polyhedron results from this procedure depends on how the various adjacencies are defined. In case of the regular polyhedra in Figure 12 defining the adjacencies is easy to accomplished in such a way that the resulting polyhedra are again regular. Convenient symbols for these regular polyhedra are $\{3_2/1,3\}$ and $\{3_2/1,4\}$. However, only a few such examples are known, no results of a general nature are available at present.

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