A New Look at Euler's Theorem for Polyhedra

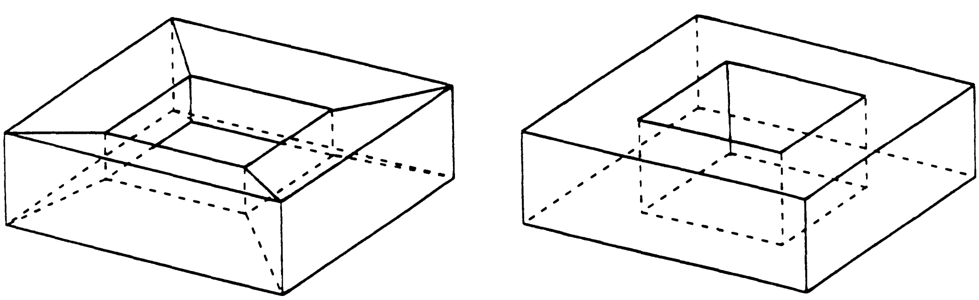
Branko Grünbaum and G. C. Shephard

1. INTRODUCTION. Euler's Theorem for Polyhedra is one of the most beautiful results of elementary geometry. If v, e and f are, respectively, the number of vertices, edges and faces of a polyhedron P, then the relation

$$v - e + f = 2 \tag{1}$$

is true for cubes, pyramids, prisms, octahedra, and many other polyhedra. One might be tempted to think (as Euler himself apparently did) that this equality holds for *all* polyhedra, but it is easily seen that it fails for the *picture frame* of FIGURE 1(a). Here $v = 16$, $e = 32$ and $f = 16$ so $v - e + f = 0$. The discrepancy is usually dealt with by saying that (1) holds only for polyhedra without any "holes", and then rewriting it in the form

$$v - e + f = 2 - 2g \tag{2}$$



(a) (b)

Figure 1. (a) A polyhedron to which Euler's theorem in its elementary form $v - e + f = 2$ does not apply. (b) A polyhedron of genus $g = 1$ to which Euler's theorem in the form $v - e + f = 2 - 2g$ does not apply.

for a polyhedron of "genus" g (that is, "with g holes passing through it", or "with g handles"). Note that here by "polyhedron" we mean the 2-dimensional manifold P which is the boundary of the "solid polyhedron". The quantity on the right-hand side of (2) is usually called the Euler characteristic of that manifold and denoted by $\chi(P)$, so that (2) can be restated as

$$v - e + f = \chi(P). \tag{3}$$

Equations (1), (2) and (3) relate the numbers of vertices, edges and faces of the polyhedron to the topological properties of the polyhedron itself. As a picture frame has just one hole (that is, $g = 1$), relation (2) holds for the polyhedron of Figure 1(a). However, this simple solution is not applicable in all cases. The polyhedron shown in FIGURE 1(b) also has $g = 1$, but $v = 16$, $e = 24$ and $f = 10$, so $v - e + f = 2$; hence relations (2) and (3) are no longer true. Also, how should one deal with a polyhedron like the funnel-shaped one shown in FIGURE 2 (the boundary of a cube with a pyramid attached to its base and with a pyramidal cavity drilled into it until the apex of the cavity just meets the apex of the attached pyramid)? Does this have a "hole" or not? In this case no integer g fits equation (2)! How should the right side of (1) be modified to deal with this situation, or with the polyhedra of Figure 3? What appeared at first sight to be a simple numerical identity is now seen to be hedged with additional conditions or exceptional cases.

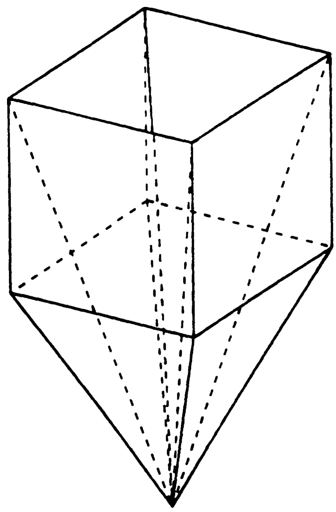


Figure 2. A funnel-shaped polyhedron for which the elementary forms of Euler's theorem are not valid.

The present paper gives the unexpectedly simple answers to these and related questions. We begin by defining a large class of sets called "polyhedral sets" which generalize familiar polyhedra; they may be closed, open, or neither, connected or not, bounded or not, and their parts may have different dimensions. Examples are shown in FIGURES 1, 2, 3 and later diagrams. For each such polyhedral set P we define an integer $\chi(P)$ called the Euler characteristic of P, and show how this is related to the geometric features of P. While this approach to the Euler characteristic is not new (see references given in Section 6), it has the advantage of allowing very easy determination of the Euler characteristic even for complicated and unusual polyhedral sets. However, although—in the unmatchable words of Richard Guy "this is well known to those who well know it"—the circle of those who know it seems to be very small.

To obtain an analogue for polyhedral sets of relation (3) we shall define subsets of P called k-scaffolds (for $k = 0, 1, 2, 3$) and use these to calculate integers V, E, F and C which, in simple cases, correspond to numbers of vertices, edges, faces and cells of P. We then show that

$$V - E + F - C = \chi(P) \tag{4}$$

for *every* polyhedral set P. Clearly, relation (4) is a true generalization of (1), (2) and (3) to polyhedral sets.

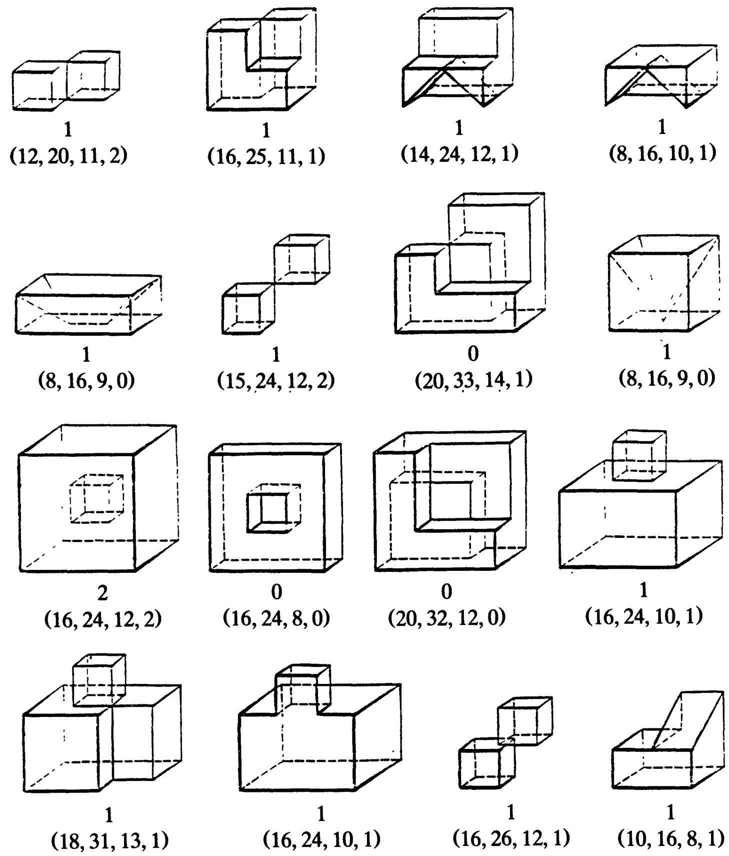


Figure 3. Examples of somewhat unusual polyhedra, after Hajós [**8**]; we interpret all these solids as closed sets. The single integer in the first line of the caption to each part is the Euler characteristic $\chi(P)$ of the solid P; it can be determined by the methods of Section 3. The second line of each caption lists (V, E, F, C); the meaning of these numbers will be explained in Section 4. The Euler characteristic of the boundary of each polyhedron P is $C + \chi(P)$.

The present approach is simple and elementary; for clarity we shall restrict attention to sets in three-dimensional Euclidean space E^3, though generalizations to higher dimensions present no difficulties. As defined below, the Euler characteristic is fully additive, and may be positive, negative or zero. Sums (rather than alternating sums), are used in its definition. Notice also that we regard the faces of polyhedral sets as relatively open, in contrast to the traditional approach in which they are considered to be closed sets.

The paper is organized as follows. In Section 2, after some preliminary definitions which may already be familiar to the reader, and are included here in order to avoid any ambiguities, we define polyhedral sets and their dissections. In Section 3 we define the Euler characteristic and establish its fundamental properties. In Section 4 we define the k-scaffolds, establish the Euler relation in the form (4), and present the definition of j-faces. Extensions of these results to unbounded polyhedral sets are presented in Section 5. Section 6 includes a short synopsis of the historical development of the Euler characteristic and related concepts, as well as references to the literature.

2. DEFINITIONS AND TERMINOLOGY. As usual, we shall use $N(x, \delta)$ for the *δ-neighborhood* of a point x in E^3. A set is *bounded* if it is contained in some neighborhood of a point. The *interior* of a set S is denoted by $\operatorname{int} S$, and the *boundary* of S by $\operatorname{bd} S$. A set S is *closed* if it contains $\operatorname{bd} S$ and is *open* if it contains no point of $\operatorname{bd} S$, that is, if $S = \operatorname{int} S$.

By a *flat*, or *affine subspace*, we mean any translate of a linear subspace; if the dimension k of a flat is specified, we shall say that it is a *k-flat*. For any S, we denote by $\operatorname{aff} S$ the *affine hull* of S, that is, the smallest flat that contains S. A set S is said to be *k-dimensional* if $\operatorname{aff} S$ is k-dimensional. Hence a single point is 0-dimensional, a line segment is 1-dimensional, etc. The dimension of a flat L is denoted by $\dim L$.

The above definitions of interior and boundary apply to sets of any dimension, but for our purposes, *relative* properties are more important. If E is a k-flat and if $N(x, \delta)$ is a neighborhood of a point $x \in E$, then the intersection $N(x, \delta) \cap E$ is called a *k-neighborhood* of x. Thus a 2-neighborhood of a point x is a small open circular disk of radius δ centered at x, and a 1-neighborhood of x is an *open* line segment (that is, a segment without its endpoints) of length 2δ centered at x. The 0-neighborhood of a point x is just the point x itself.

The *relative interior* of S is the set of those points $x \in S$ for which there exists a k-neighborhood of x, that is contained in S, but for no point y is any $(k+1)$-neighborhood of y contained entirely in S; clearly, there is a unique value of k with this property. Sometimes the relative interior of S will be called its *k-interior*. The 3-interior is the same as the interior for any set in E^3. If the k-interior of a set

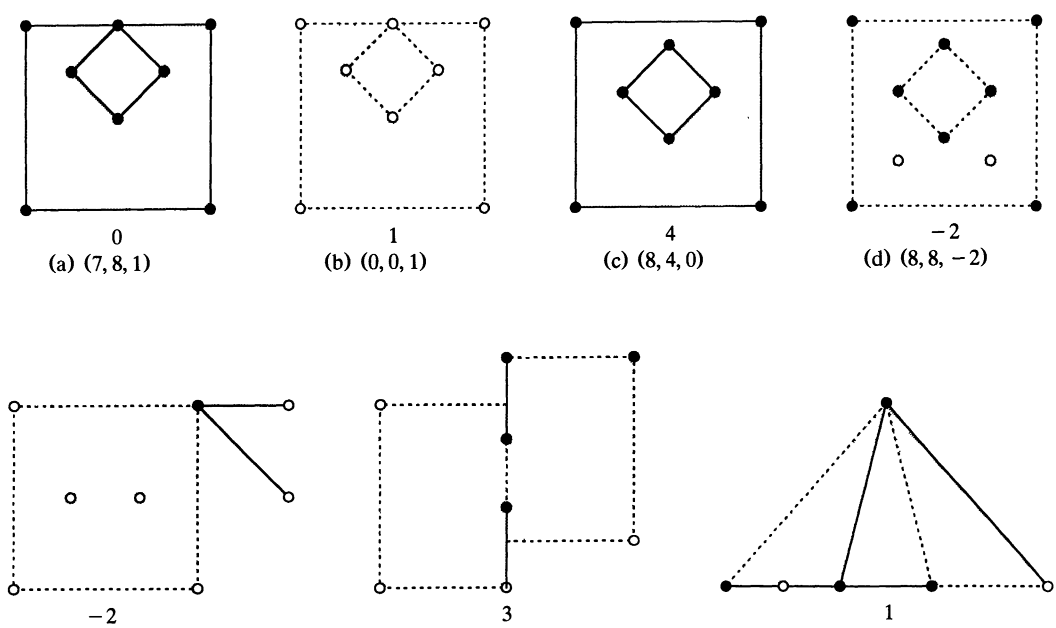


Figure 4. Examples of bounded polyhedral sets P. For simplicity of description and graphical representation, these sets are all in the plane, and all polygonal regions and segments are taken as relatively open. The polygonal regions are shaded. Solid lines and solid dots indicate segments and points included in the set, dashed lines and hollow dots indicate segments and points that are not included. The data in the caption to each part are as in Figure 3, except that C is not listed since it is 0 in all cases. The Euler characteristic of $P \cap \operatorname{relbd} P$ is $\chi(P) - F$.

S is nonempty, we define the *relative boundary* of S as the set of all points of bd S which do not belong to the relative interior of S. The relative interior and relative boundary of S are denoted by relint S and relbd S, respectively. The set S is said to be *relatively open* if $S = \text{relint } S$.

A single point, an open ray or segment, a straight line, a plane, or the whole space E^3 are examples of relatively open sets. If Q is a square region in E^3 (so aff Q is a plane, and Q is 2-dimensional) then relbd Q consists of the union of four open segments (each of which is the relative interior of one of the sides of the square) together with the four vertices of the square. Further, relint Q is the part of aff Q that lies inside relbd Q. If S is the boundary of a cube then S contains no points whose 3-neighborhoods lie in S, hence int S is the empty set. The points whose 2-neighborhoods lie in S are the points in the faces of the cube, that is, the points of S apart from the edges and vertices. This set is therefore the 2-interior of S. Moreover, since int S is empty, the 2-interior of S is, by definition, the relative interior of S. Thus relint S is the union of the relative interiors of the six square

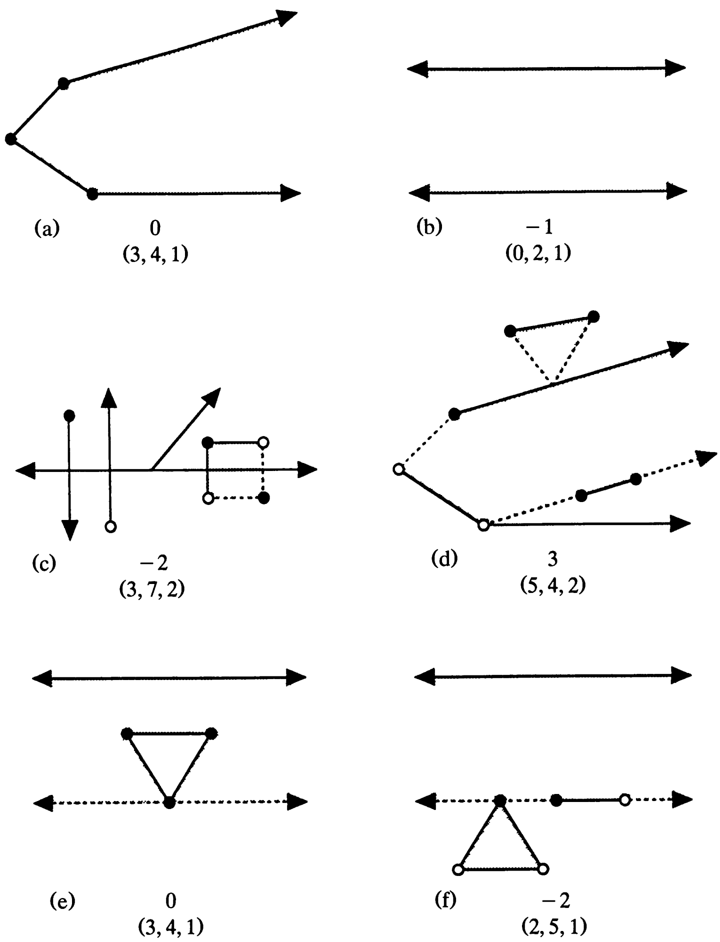
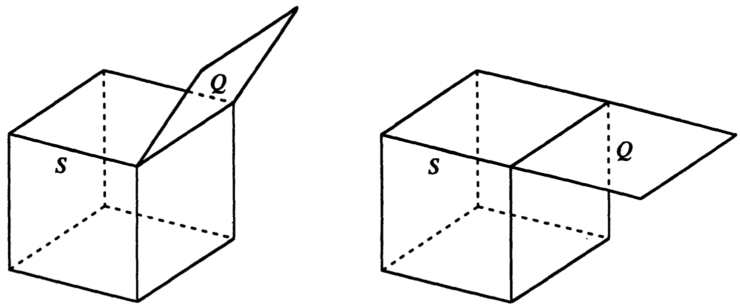


Figure 5. Examples of unbounded polyhedral sets it the plane. The conventions are the same as in Figure 4; in addition, arrowheads are used to distinguish halflines and lines from segments. (a) shows a line-free closed convex set, and the set in (b) is closed and convex, but not line-free (L is a line and $k = 1$ in the notation of Section 5).



(a) (b)

Figure 6. An illustration of the dependence of the relative interior of a set on the mutual position of its parts. In (a), the set $T = S \cup Q$, where S is the boundary of a cube and Q is a closed square not coplanar with any face of the cube. Hence relint T = relint $S \cup$ relint Q. In (b), for the set $U = S \cup Q$, in which Q is coplanar with the top face of the cube, meeting it along the edge E, we have relint U = relint $S \cup$ relint $Q \cup$ relint E. The Euler characteristic is 2 in both cases, as can be verified using the definition given in Section 3. In the notation introduced in Section 4, $\text{scaf}_2 T$ consists of seven open squares, $\text{scaf}_1 T$ consists of 15 open segments, and $\text{scaf}_0 T$ of 10 vertices; hence $V = 10$, $E = 15$, and $F = 7$. On the other hand, $\text{scaf}_2 U$ consists of five open squares and one open rectangle, $\text{scaf}_1 U$ of 12 open segments, and $\text{scaf}_0 U$ of eight points.

faces of the cube. If T consists of the set S just considered together with a square Q which is not coplanar with any face of the cube (see FIGURE 6(a)) attached to S along a common edge E, then relint T = relint $S \cup$ relint Q; but for a set U (see FIGURE 6(b)) in which Q is coplanar with a face of S we have relint U = relint $S \cup$ relint $Q \cup$ relint E. A more complicated example is shown in FIGURE 7 and explained in the caption.

A set S is called *convex* if, given any two distinct points $x, y \in S$, the closed line segment with endpoints x and y lies entirely in S. Thus a line segment is necessarily convex; a single point and the empty set are also convex, since the definition is vacuous in this case. A *closed half-space* is the (unbounded convex) set of points that lie to one side of, or on, a plane in E^3. Any set which is the intersection of a finite number of closed half-spaces is called a *closed convex polyhedron*. Familiar examples of closed convex polyhedra are (closed) cubes, (closed) squares, (closed) line segments, and single points. These are of three, two, one and zero dimensions respectively. But our definition includes also unbounded convex polyhedra, such as the examples in FIGURES 5(a) and 5(b); their edges can be rays (halflines) or straight lines, and faces and cells can be unbounded as well.

Euler's Theorem in its basic form (1) holds for the boundary of 3-dimensional closed and bounded convex polyhedra. Many elementary proofs of this fact are known, see Section 6. The case of unbounded polyhedra will be considered in Section 5.

A *relatively open convex polyhedron* is the relative interior of a closed convex polyhedron. It should be noted that whereas an open convex polyhedron (which is necessarily 3-dimensional) is the intersection of a finite number of *open* half-spaces, the same is *not* true for relatively open convex polyhedra of dimension less than three. On the other hand, if P is a relatively open convex polyhedron lying in a d-flat E = aff P (where $d = 1$ or 2) then P can be written as the intersection of finitely many open half d-flats (each of which is a relatively open set) lying in E. Alternatively, P is the intersection of E with a family of open halfspaces. As we

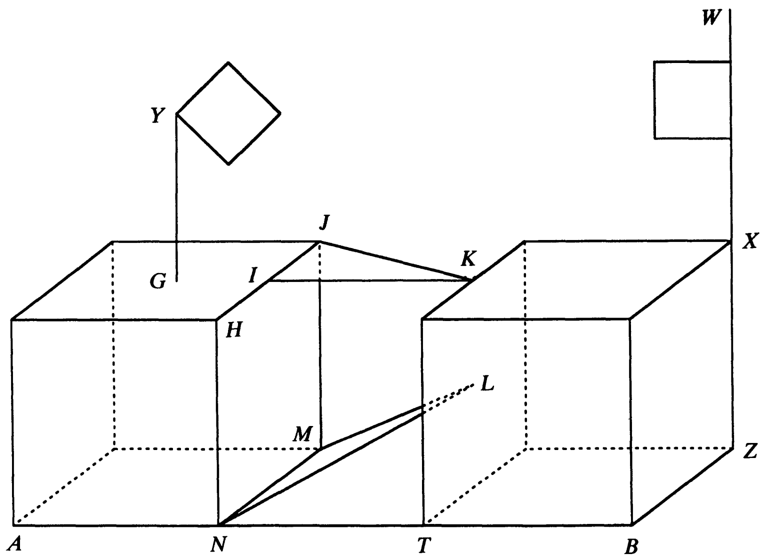


Figure 7. A polyhedral set P (assumed closed, to simplify the discussion) consisting of two solid cubes joined by two triangles and a segment, and of two squares attached to segments which touch the cubes; clearly int $P = \text{scaf}_3 P$ consists of the two open cubes, so $C = 2$. The polyhedral set $Q = \text{bd}\, P$ consists of the 14 squares shown, together with the two triangles and three segments. Its relative interior relint $Q = \text{scaf}_2 Q = \text{scaf}_2 P$ consists of the relative interiors of the squares and the triangles, together with the open segment IJ along which the upper triangle is joined to the cube; hence $F = 15$. The set $R = \text{relbd}\, Q$ consists of the following open segments: all the edges of the two cubes except HJ, the four edges of one of the squares and three of the other, and the segments $GY, HI, IK, JK, LM, LN, NT, WX$; moreover, R also contains all the vertices of the cubes and squares, and the point W (but not the points G and L). It follows that relint $R = \text{scaf}_1 R = \text{scaf}_1 Q = \text{scaf}_1 P$ consists of the following open segments: AB, GY, HI, WZ, twenty edges of the two cubes, seven edges of the two squares, and four edges of the two triangles; hence $E = 35$. Finally, relbd $R = \text{scaf}_0 R = \text{scaf}_0 Q = \text{scaf}_0 P$ consists of all the vertices of the two cubes except N, T, X, as well as of the four vertices of one of the squares and two of the other, and the points I and W; hence $V = 21$. It follows that $\chi(R) = -1$, $\chi(Q) = 1$, and $\chi(P) = -1$. These values can easily be verified by directly using the definition of the Euler characteristic.

shall see, relatively open convex polyhedra play a central rôle in our treatment of the Euler characteristic.

From the definition it is clear that the intersection of any two closed convex polyhedra is a closed convex polyhedron. Analogously, though this requires some additional reasoning, it may be verified that regardless of their dimensions the intersection of any two relatively open convex polyhedra is a relatively open convex polyhedron. Following convention, we shall sometimes refer to a bounded convex polyhedron of two dimensions as a *polygon*, and a bounded convex polyhedron of one dimension as a *line segment* or simply *segment*; in each case we must, of course, specify whether the polyhedron is closed, relatively open, or neither.

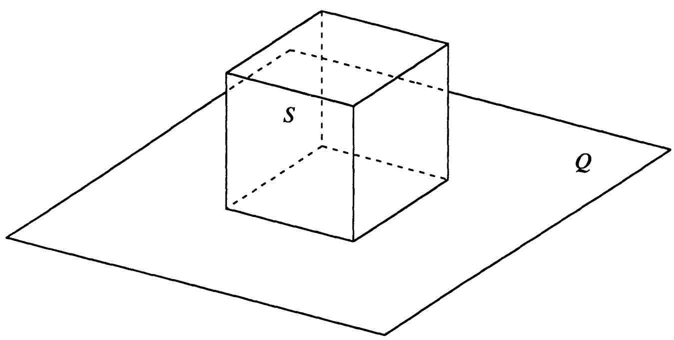
If a set P is the union of members a finite family $\mathscr{C} = \{C_1, C_2, \dots, C_n\}$ of pairwise disjoint sets $C_1, C_2, \dots, C_n$, we say that $\mathscr{C}$ is a *dissection* of P and we write $P = \dot{\bigcup} \mathscr{C}$, where the dot in the union symbol indicates that the sets C_i are pairwise disjoint. If P is a set that admits a dissection $\mathscr{C}$ all elements of which are relatively open convex polyhedra we shall say that P is a *polyhedral set*, and express this by writing

$$P = \dot{\bigcup}_c \mathscr{C} \tag{5}$$

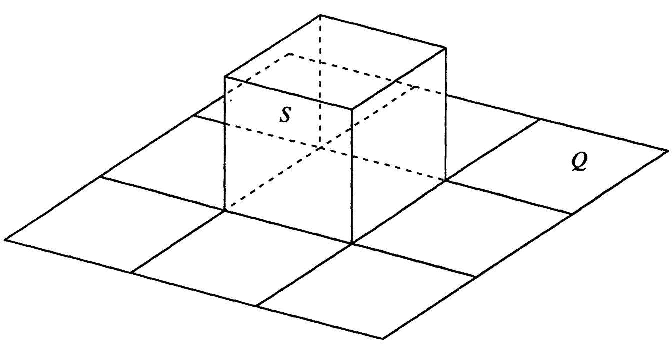
where the subscript c is to remind us that each element of $\mathscr{C}$ is a relatively open convex polyhedron. Such a family $\mathscr{C}$ is called a *relatively open convex dissection* of P and each element of $\mathscr{C}$ is called an *element* of the relatively open convex dissection. All the sets shown in FIGURES 1 to 9 are polyhedral sets in this sense. It will be observed that the definition is very general in that it does not require P to be open or closed, connected, simply connected or even *homogeneous* in the sense that neighborhoods of all points of P are of the same dimension. Nor are there any restrictions on the incidences of the closures of the elements. On the other hand, a bounded and closed polyhedral set is necessarily compact. The collection of all polyhedral sets will be denoted by $\mathbb{P}$. These polyhedral sets are the sets to which relation (4) will apply.

It is clear that except when P consists of a finite number of points, its expression (5) as a disjoint union of relatively open convex polyhedra is not unique. In fact, every relatively open convex polyhedron of dimension greater than 0 admits infinitely many open dissections.

Let $P = \dot{\bigcup}_c \mathscr{C}$ and $P = \dot{\bigcup}_c \mathscr{F}$ be two dissections of P; we shall say that the latter is a *refinement* of the former if each element of $\mathscr{C}$ is a disjoint union of elements of $\mathscr{F}$. If we are given *any* two dissections $P = \dot{\bigcup}_c \mathscr{C}$ and $P = \dot{\bigcup}_c \mathscr{D}$ of P then it is possible to find another dissection $P = \dot{\bigcup}_c \mathscr{F}$ which is a refinement of each; any such $\mathscr{F}$ will be called a *common refinement* of the dissections $\mathscr{C}$ and $\mathscr{D}$. In fact it suffices to take, as elements of $\mathscr{F}$, all non-empty sets of the form $C \cap D$,



(a)



(b)

Figure 8. (a) A polyhedral set P consisting of an open cube S and a relatively open square Q. This dissection of P is not complex-like. (b) A relatively open convex dissection of P which is complex-like.

where $C \in \mathscr{C}$, and $D \in \mathscr{D}$. From the properties of relatively open convex sets it is immediate that if the dissections $\mathscr{C}$ and $\mathscr{D}$ are relatively open convex dissections, then the common refinement $\mathscr{F}$ is another dissection of the same kind. Moreover, among such common refinements of the relatively open convex dissections $\mathscr{C}$ and $\mathscr{D}$ of a polyhedral set P it is possible to find a common refinement $\mathscr{F}$ which is *complex-like*. By this we mean that whenever an element of $\mathscr{F}$ meets the relative boundary of another element of $\mathscr{F}$, it is contained in that boundary, see FIGURE 8.

These simple observations are basic for all that follows.

3. THE EULER CHARACTERISTIC. Throughout this section we shall restrict attention to bounded polyhedral sets; the extension of these results to unbounded sets will be given in Section 5.

For any polyhedral set $P \in \mathbb{P}$ we define an integer $\chi(P)$ in the following way:

(a) $\chi(\varnothing) = 0$.
(b) If P is a relatively open convex polyhedron of dimension d, then $\chi(P) = (-1)^d$.
(c) If $P = \dot{\bigcup}_c \mathscr{C}$ is a relatively open convex dissection of P then $\chi(P) = \Sigma_{C \in \mathscr{C}} \chi(C)$.

This integer $\chi(P)$ is called the *Euler characteristic* of P. Part (c) of the definition appears to imply that the definition of $\chi(P)$ depends on the dissection $\mathscr{C}$ that is used. However, this dependence is only apparent. This, and other important properties of $\chi(P)$ are given by the following theorems:

Theorem 1. *The Euler characteristic $\chi(P)$ is well-defined in the sense that its value, as given by (c) above is independent of the relatively open convex dissection $\mathscr{C}$ of P used in the computation.*

Theorem 2. *If P is a closed and bounded convex polyhedron then $\chi(P) = 1$.*

Theorem 3. *The Euler characteristic $\chi(P)$ satisfies the valuation property: if $P_1 \in \mathbb{P}$ and $P_2 \in \mathbb{P}$, then*

$$\chi(P_1) + \chi(P_2) = \chi(P_1 \cap P_2) + \chi(P_1 \cup P_2). \tag{6}$$

It should be noted that Theorem 2 refers to the Euler characteristic of the (bounded and closed) convex polyhedron itself, and not—as in the discussion in Section 1—to the Euler characteristic of its boundary.

In some treatments of the Euler characteristic Theorems 2 and 3 are used to define $\chi(P)$ for closed convex sets and for those sets which can be obtained from them by taking finite unions. Thus the definition of $\chi(P)$ given here may be regarded as an extension of the traditional approach to sets which need not be closed. On the other hand, our definition is restricted to polyhedral sets, hence not applicable to non-polyhedral sets even if they are convex.

The proofs of Theorems 1, 2 and 3 are omitted since they follow the usual techniques. We note only that for Theorem 1 we rely on the fact mentioned earlier that two convex dissections $\mathscr{C}$ and $\mathscr{D}$ of a polyhedral set P have a complex-like common refinement $\mathscr{F}$.

Immediate consequences of these theorems include the following:

Corollary 1. *If $\mathscr{C}$ is any dissection of the polyhedral set P such that each $C \in \mathscr{C}$ is a polyhedral set, then*

$$\chi(P) = \sum_{C \in \mathscr{C}} \chi(C).$$

Corollary 2. (Inclusion-exclusion property). *If $\mathscr{C}$ is any family of polyhedral sets such that $P = \bigcup \mathscr{C}$, then*

$$\chi(P) = \sum \chi(C) - \sum \chi(C_1 \cap C_2) + \sum \chi(C_1 \cap C_2 \cap C_3) - \cdots$$

where the first sum is over all $C \in \mathscr{C}$, the second sum is over all sets $C_1, C_2 \in \mathscr{C}$ of two distinct elements of $\mathscr{C}$, the third sum is over all sets of three distinct elements of $\mathscr{C}$, etc.

Corollary 3. *For any polyhedral set $P \in \mathbb{P}$,*

$$\chi(P \cap \text{relbd } P) = \chi(P) - \chi(\text{relint } P);$$

hence, if P is closed, $\chi(\text{relbd } P) = \chi(P) - \chi(\text{relint } P)$.

Using the above results and suitable dissections, it is easy to verify that each of the polyhedral sets in FIGURES 3, 4 and 5 has the Euler characteristic indicated.

4. EULER'S THEOREM FOR BOUNDED POLYHEDRAL SETS. We now show how the Euler characteristic, as introduced in Section 3, can be used to derive analogues of the traditional Euler equations (1) and (2) which are valid for bounded but otherwise general polyhedral sets. The basic approach is to express each such polyhedral set $P \in \mathbb{P}$ as the disjoint union of four well-determined sets, called *k-scaffolds* of P and denoted $\text{scaf}_k P$ for $k = 0, 1, 2, 3$.

The 3-scaffold $\text{scaf}_3 P$ is simply int P, the set of all interior points of P. We delete $\text{scaf}_3 P$ from P, leaving $P_2 = P \setminus \text{scaf}_3 P$; clearly, $P_2 \subset \text{bd } P$. The 2-scaffold $\text{scaf}_2 P$ is the set of all 2-interior points of the set P_2. (Equivalently, $\text{scaf}_2 P$ is the relative interior of the set $P \cap \text{bd } P$.) Similarly, $\text{scaf}_1 P$ is the set of all 1-interior points of $P_1 = P_2 \setminus \text{scaf}_2 P$, and $\text{scaf}_0 P$ is the finite set of points $P_0 = P_1 \setminus \text{scaf}_1 P$.

Each k-scaffold of P is a relatively open set; in fact, it is a relatively open polyhedral set, and $S = \{\text{scaf}_0 P, \text{scaf}_1 P, \text{scaf}_2 P, \text{scaf}_3 P\}$ is a relatively open dissection of P into polyhedral sets. To verify this claim, we only have to show that each $\text{scaf}_k P$ is a polyhedral set. This follows at once from the observations:

(i) the definition of $\text{scaf}_k P$ is independent on the relatively open convex dissection $\mathscr{C}$ which establishes that P is a polyhedral set, and

(ii) each relatively open convex dissection of P can be refined to one that is complex-like. For such a refinement it is clear that each element is contained in one and only one scaffold of P, and so defines a relatively open convex dissection of each $\text{scaf}_k P$.

As a consequence we have:

Corollary 4. *If p is a bounded polyhedral set then*

$$\chi(P) = \chi(\text{scaf}_0 P) + \chi(\text{scaf}_1 P) + \chi(\text{scaf}_2 P) + \chi(\text{scaf}_3 P).$$

We note that if P is a convex polyhedron, then $\text{scaf}_2 P$ is the union of the relative interiors of the faces of P, $\text{scaf}_1 P$ is the union of the relative interiors of the edges of P, and $\mathrm{scaf}_0 P$ is the set of vertices of P. These remarks motivate the following notation: $V = \chi(\mathrm{scaf}_0 P)$, $E = -\chi(\mathrm{scaf}_1 P)$, $F = \chi(\mathrm{scaf}_2 P)$, and $C = -\chi(\mathrm{scaf}_3 P)$. Note that we use upper case symbols in order to stress the distinction between the precisely defined entities and the somewhat vague quantities mentioned in the Introduction and Section 1. Corollary 4 implies:

Theorem 4. *If P is a polyhedral set then*

$$V - E + F - C = \chi(P).$$

Thus relation (4) has been established.

Theorem 4 will now be illustrated by means of examples.

Consider first the closed polyhedral set P of FIGURE 9(a); it is to be understood as a solid, the interior of which is $\mathrm{scaf}_3 P$. The 2-scaffold $\mathrm{scaf}_2 P$ consists of 12

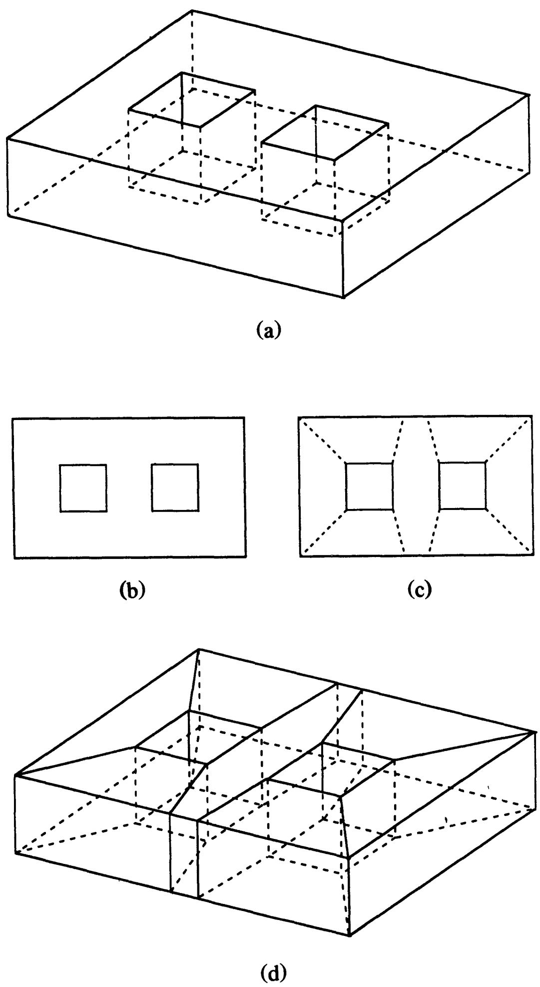


Figure 9. (a) A (closed) polyhedral set; if interpreted as a solid P, it has $V = 24$, $E = 36$, $F = 10$, $C = -1$, and $\chi(P) = -1$. If P is interpreted as a 2-manifold then $\chi(P) = -2$. (b) One of the polygons B which occurs as the upper or lower face of the polyhedral set P shown in (a), and (c) a dissection of B. Interpreting B as a relatively open set, this shows that $\chi(B) = -1$. (d) A dissection of int P, which shows that $\chi(\mathrm{int}\, P) = 1$.

relatively open rectangular faces (each of which has Euler characteristic equal to 1) and two other relatively open polygons, namely the top and bottom of P, each of which is an open rectangle from which two closed squares have been removed. (FIGURE 9(b)). The Euler characteristic of each of these polygons (which we shall denote by B) can be easily determined using the open dissection shown in FIGURE 9(c). This shows B expressed as the union of seven relatively open convex polygons and eight open line segments, so $\chi(B) = 7 \times 1 + 8 \times (-1) = -1$. We deduce that $F = 12 \times 1 + 2 \times (-1) = 10$. As $\text{scaf}_1 P$ consists of 36 open segments, each with Euler characteristic -1, it follows that $E = 36$. The 24 vertices form $\text{scaf}_0 P$, hence $V = 24$. To determine C, consider the open convex dissection of $\text{int}\, P = \text{scaf}_3 P$ indicated in FIGURE 9(d); it consists of seven open 3-dimensional convex polyhedra, and of 8 relatively open convex polygons, hence $C = -\chi(\text{scaf}_3 P) = -(7 \times (-1) + 8 \times 1) = -1$. Therefore

$$\chi(P) = V - E + F - C = 24 - 36 + 10 - (-1) = -1.$$

The value of $\chi(P)$ can be verified by using the inclusion-exclusion principle (Corollary 2). We represent P as the union of seven closed convex sets, all of which are prisms (six on quadrangular bases, and one on an octagonal base—FIGURE 9(d) can also be interpreted as showing this representation). The intersections by pairs are eight closed polygons (rectangles), and every set of three has empty intersection. Hence $\chi(P) = 7 - 8 = -1$, as before.

It should be noted that the flexibility built in into the approach followed here enables one in many cases to avoid subdivisions when calculating $\chi(P)$. For the example in FIGURE 9(a) we could argue that P, together with two open rectangular boxes and four relatively open squares, forms a dissection of a closed rectangular box. Therefore $1 = \chi(P) + 2(-1) + 4$, leading to the same value $\chi(P) = -1$.

The numbers calculated above can also serve to illustrate Corollary 3, since

$$\chi(\text{relbd}\, P) = 24 - 36 + 10 = -2.$$

which coincides with

$$\chi(P) - \chi(\text{relint}\, P) = -1 - 1 = -2,$$

since $\chi(\text{relint}\, P) = -C = 1$.

As a second example, consider the closed polyhedral set P in FIGURE 2. Using a plane through two opposite vertical edges of the cube, partition P into two parts, each of which is a closed polyhedron and has Euler characteristic equal to 1. The intersection of these parts is the union of two closed triangles with a vertex in common, and again this has Euler characteristic 1. Hence, by the inclusion-exclusion principle, $\chi(P) = 2 - 1 = 1$. Considering scaffolds, we can verify $V = 9$, $E = 20$, $F = 12$, and so, from Corollary 4,

$$C = -\chi(\text{scaf}_3 P) = -\chi(\text{int}\, P) = 1 - 12 + 20 + 9 = 0.$$

This can be verified using a dissection of $\text{int}\, P$ analogous to that just used for P. Each half of $\text{int}\, P$ has Euler characteristic -1, but their intersection now has characteristic 2, since it consists of two disjoint, relatively open triangles. Thus $\chi(\text{int}\, P) = 2 \times (-1) + 2 = 0$, as before. For another verification we may consider the dissection consisting of P, an open 4-sided pyramid, and an open square, from which we have $1 = \chi(P) + (-1) + 1$, hence $\chi(P) = 1$.

Additional examples illustrating our theorems and corollaries appear in FIGURE 3. The quantities V, E, F, C are indicated in the form of a vector (V, E, F, C) below each part of the diagram. Each of the polyhedral sets is considered to be solid, that is, to have non-empty interior, and to be equal to the closure of its interior. Other examples are shown in FIGURES 4, 6, 7, 9 and described in the captions.

Finally, we note that the use of scaffolds leads to a definition of j-faces ($j = 0, 1, 2, 3$) which is applicable to *all* polyhedral sets. We define a *j-face* of a polyhedral set P to be a connected component of $\text{scaf}_j P$. For each j, these components are well determined. In the case of convex polyhedra—and many others as well—these j-faces are the relative interiors of faces and edges in the traditional approach to the subject. However, in more complicated situations rather unusual sets can appear as faces. For example, in FIGURE 4, one 1-face of the first polyhedral set in (a) consists of the union of three open segments, namely the upper segment and the two contiguous sides of the deleted square. In FIGURE 4(g), one 1-face is the union of two open segments. In FIGURE 7, one 2-face consists of the open segment IJ and the two open squares and one open triangle that contain IJ in their boundary; another 2-face consists of the open square that contains the point L, and the open triangle with vertex L, while one 1-face consists of the open segment AB together with the five additional open segments that have N or T as an endpoint. It remains to be seen whether this generalization of the usual concept of "face" will lead to interesting mathematics.

5. UNBOUNDED POLYHEDRAL SETS. Now we shall extend the results of the previous sections to unbounded polyhedral sets. As we pointed out in Section 2, our definition of a closed polyhedron applies equally to the unbounded case. We may also allow the elements in the definition of a polyhedral set (5) to be unbounded, and thus extend the family $\mathbb{P}$ to include unbounded sets. From now on, $\mathbb{P}$ will be used in this wider sense. We shall assume that the reader has some familiarity with the structure of unbounded convex sets as explained, for example, in Grünbaum [1967], Section 2.5 and the references in Section 2.7.

For any non-empty convex polyhedron P in the d-dimensional space E^d, let L represent a k-flat, contained in P, and chosen to have maximal possible dimension k. Then P is called *line-free* if and only if $k = 0$. A bounded convex polyhedron is necessarily line-free, as is also a set such as that shown in FIGURE 5(a). For the convex polyhedron in FIGURE 5(b), k takes the value 1. It is known (see Grünbaum [1967], page 24) that if P is a closed, convex polyhedron then it may be written as a "direct product" or "vector sum"

$$P = L \oplus \tilde{P}, \tag{7}$$

where L is the maximal k-flat defined above and $\tilde{P}$ is a line-free polyhedron whose dimension is k less than the dimension of P. The sign $\oplus$ means that every point $x \in P$ can be written uniquely in the form $x = y + z$ (vector addition) where $y \in L$ and $z \in \tilde{P}$. We may take for $\tilde{P}$ any set of the form $P \cap \tilde{L}$, where $\tilde{L}$ is a $(d - k)$-flat orthogonal to L in E^d. We shall refer to (7) as the *standard linear decomposition* of P. In the example of FIGURE 5(b), L is a line and $\tilde{P}$ is a segment.

For the boundary of a 3-dimensional unbounded line-free convex set P a relation analogous to (1) holds, namely

$$v - e + f = 1, \tag{8}$$

where v, e, f are, respectively, the numbers of vertices, edges and faces of P (see Grünbaum [1967], Section 8.5). If P is 2-dimensional, the corresponding result for the relative boundary is

$$v - e = -1. \tag{9}$$

An example is provided by the line-free set in FIGURE 5(a), where $v = 3$ and $e = 4$; two of the edges are line segments, and the other two are rays (unbounded half-lines).

Just as the definition of the family $\mathbb{P}$ in Section 2 was formulated in such a way as to apply also to unbounded sets, so were Theorems 1 and 3, together with the three corollaries in Section 3 phrased in such a way as to apply also in the unbounded case. The result analogous to Theorem 2 is as follows:

Theorem 2*. *Let P be a closed convex polyhedron, and $P = L \oplus \tilde{P}$ be a standard linear decomposition of P with $\dim L = k$, then*

$$\chi(P) = \begin{cases} 0 & \text{if } \tilde{P} \text{ is unbounded} \\ (-1)^k & \text{if } \tilde{P} \text{ is bounded.} \end{cases}$$

Proof of Theorem 2.* The case where P is bounded has been dealt with in Section 3, Theorem 2. Now let P be unbounded but line-free. If P is a closed half-line (ray), then $\chi(P) = 1 - 1 = 0$. If P is 2-dimensional then, by (9),

$$\chi(P) = v - e + 1 = 0,$$

and if P is 3-dimensional then, by (8),

$$\chi(P) = v - e + f - 1 = 0.$$

If P is not line-free, then to each j-dimensional element of P there corresponds a $(j - k)$-dimensional element of $\tilde{P}$ (note that all elements of P have dimensions $\geq k$). Hence $\chi(P) = (-1)^k \chi(\tilde{P})$ and equals $(-1)^k$ or 0 depending on whether $\tilde{P}$ is bounded or not. This completes the proof of the theorem.

The definitions of the scaffolds of a polyhedral set P, together with Theorem 4 and Corollary 4, hold with trivial modifications in the unbounded case. Examples of the application of Theorem 4 to unbounded sets appear in FIGURE 5. Details are given in the caption to the figure.

6. HISTORICAL REMARKS AND COMMENTS. The history of Euler's Theorem and concepts related to it are both interesting and voluminous. It involves many of the ideas that led to modern algebraic topology, and also many of the errors which were committed in that development. At least two books have been devoted to the early history and attempts at clarification of Euler's Theorem (Lakatos [**13**], Federico [**4**]), and countless books and articles contain short accounts. Here only a very brief survey will be given.

Euler first published his theorem in 1750, stating that he had no satisfactory proof but was convinced of its general validity by a wealth of examples. (The frequently encountered assertion that Euler's Theorem was known to Descartes a century before Euler is unsupported by any evidence, and based on erroneous interpretation of some of Descartes' writings. For a discussion of this and other historical errors concerning Euler's Theorem see Malkevitch [**17**] and the references given there, and Federico [**4**].) Euler's formulation was, essentially, that "for every solid bounded by flat surfaces, the number of surfaces increased by the number of vertices exceeds by two the number of edges." Later, Euler presented a proof of the theorem, as did several other mathematicians. Early in the nineteenth century it was observed that the assertion cannot be true in the generality claimed

(L'Huilier [16], Hessel [9]). In a masterpiece of understatement, Hessel remarks:

> Other excellent mathematicians (Legendre, Cauchy, Gergonne, Rothe and Steiner) supplied proofs for the general validity of the theorem. But in fact, it suffers from exceptions.

In order to illustrate such exceptions and the difficulties in removing them, Hessel shows pairs of polyhedra such that "for each shape for which Euler's rule is not valid, a rather similar one can be found for which the rule is valid." Hessel's examples are reproduced in FIGURE 10; we shall return to them shortly.

These developments led to a number of reformulations of the basic version of Euler's Theorem (given by relation (1)) through the introduction of various parameters that replaced the value 2 of the right-hand side. All the discussions were dogged by two difficulties.

On the one hand, no precise definitions were given for the polyhedra under consideration or for their faces, edges and vertices. It was more or less generally assumed that one is dealing with solids and considering features on their surfaces —but how to determine faces (or edges) was illustrated by examples rather than defined by unambiguous rules. A glance at the collection of examples of polyhedra in FIGURE 3, taken from Hajós [8], should convince the reader that the concepts of face, edge and vertex are not very straightforward to define even for those special polyhedra which are compact and such that the boundary of the polyhedron coincides with the boundary of its interior. This observation explains why, in the later decades of the nineteenth century, Euler-type relations with more and more complicated right-hand sides were appearing in the literature.

On the other hand, in its original formulation and in the minds of many early workers, it was the hallmark of Euler's theorem that it involved only the *numbers* of vertices, edges and faces—*without any consideration of the nature of the faces*. Thus, in Hessel's presentation, his even-numbered examples are "good" because for them $v - e + f = 2$, and the odd-numbered polyhedra are "bad" because the numbers to be inserted in the left-hand side do not yield 2. Remarkably, some of this attitude survived even to our days: in Seydel [30, page 322] the same example as in Hessel's diagram labelled 2 (and our FIGURE 1(b)) is given, with the comment that this illustrates the validity of relation (1) for *all* polyhedra! We shall return to the faces of polyhedra later in this discussion.

The second half of the nineteenth century saw the gradual clarification of the difficulties; one step was the insight that relations such as (1) deal with the surface, and not directly with the solid. The concept of genus of those special surfaces which are called orientable manifolds helped to reach the relation (2). In particular, this led to the understanding that the original formulation of Euler's theorem applies to the boundary of 3-dimensional convex polyhedra and, more generally, to maps on the sphere and other closed surfaces.

At the same time, Euler's theorem was extended to higher dimensions as one of the first results in the emerging discipline of algebraic topology. It was established (or, at least, stated!) that the Euler characteristic equals to the alternating sum of so-called "Betti numbers". But again, there were more good intentions than mathematically proved results. In the words of Dieudonné [3]:

> ...the mathematicians of the second half of the nineteenth century which were busy with these questions [of algebraic topology] speak freely of curves, of surfaces, of deformations,..., without ever saying what they mean by these words.

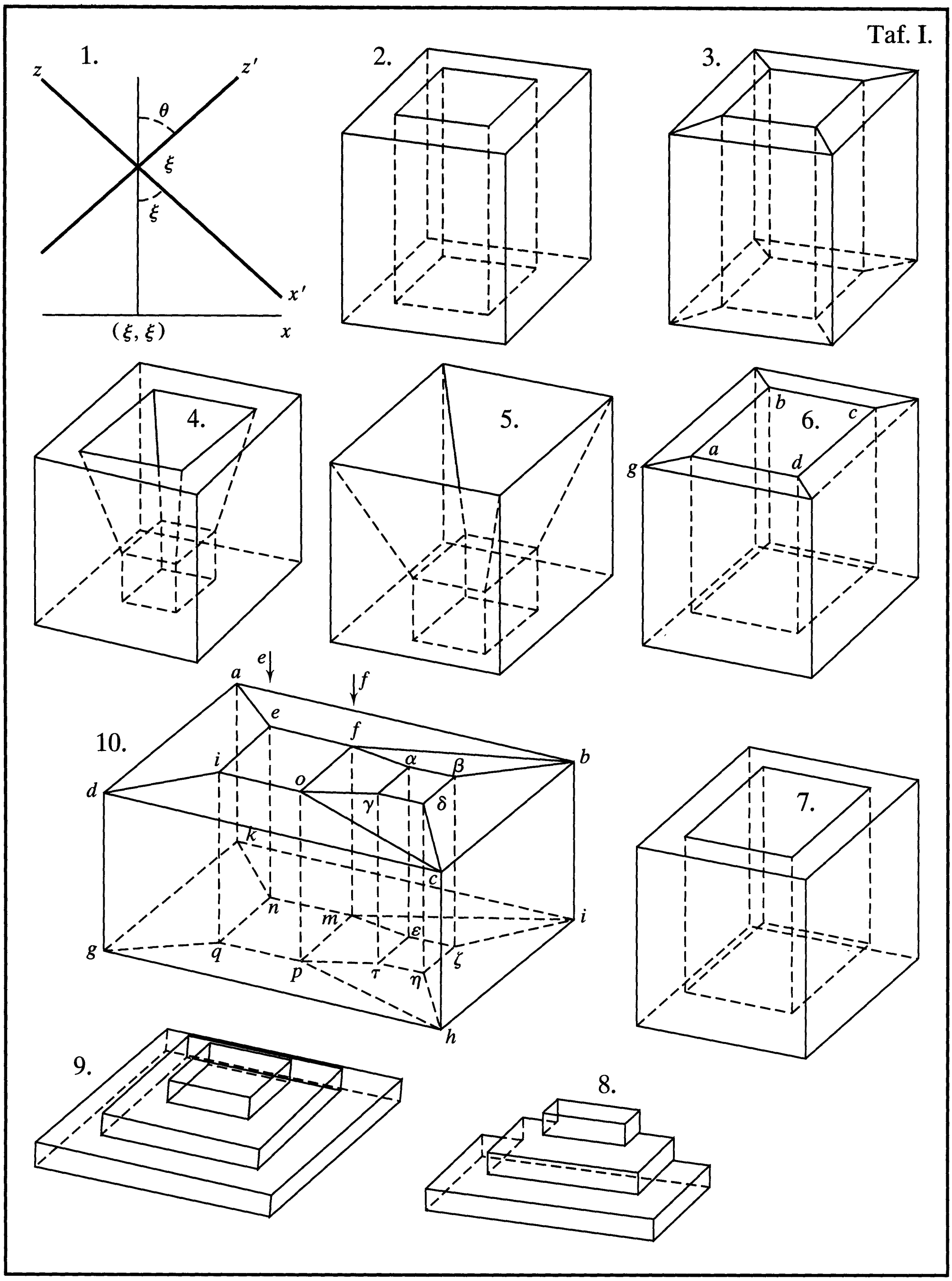


Figure 10. Plate taken from Hessel [9]. These diagrams illustrate the difficulties that early authors had in extending the simple Euler relation (1) to more complicated polyhedra. Parts 2 to 5 show a cube with a hole drilled through it, and parts 6 and 7 show a cube with an indentation. Part 10 shows a block with two prismatic holes through it. In parts 3, 6 and 10 some faces have been subdivided.

Starting with Poincaré [**26**] and completed, simplified and extended by others (in particular, Alexander [**1**]) these concepts involved in the definition of Euler characteristic acquired precise meaning for simplicial complexes and for more general objects; see, for example, Hilton-Wylie [**11**], Singer-Thorpe [**32**], or any other modern text on algebraic topology. However, this development led to a loss of the connection to the origins of Euler's theorem as a relation involving the vertices, edges and faces of a polyhedral object. Apart from these developments, in the comparatively simple case of convex polytopes of arbitrary dimensions, the analogue of Euler's theorem was discovered by Schläfli in 1850 (but not published till 1901), and independently discovered and published in the 1880's by several mathematicians. But all these proofs—including the one in Sommerville [**33**]—were grossly incomplete; as noted in Grünbaum [**5**, p. 141], to validate these proofs one would have to show that the faces of the polytopes in question can be arranged in a special order known as *shelling*. Although it was later shown that every convex polytope can be shelled (Bruggesser-Mani [**2**]), the absence of this concept in the proofs shows that they were invalid in an essential aspect. (For elementary proofs of the Euler theorem for convex polytopes of all dimensions, which avoid the use of shelling, see Grünbaum [**5**, p. 134], McMullen-Shephard [**19**, p. 94].)

Returning to the situation dealing with polyhedra, a new direction opened up with the work of Hadwiger [**6**]. Hadwiger observed that the Euler characteristic can be defined consistently by assigning to each compact convex set C the value $\chi(C) = 1$, and proceeding to extend this to the "Konvexring" (family of sets each of which is a union of finitely many compact convex sets) by using the valuation property. Later, Klee [**12**] simplified and generalized this approach by putting it in a lattice setting, and observing that it also worked for unions of open convex sets (of a fixed dimension) if one starts by assigning to all open convex sets the Euler characteristic 1. Without mentioning Hadwiger or Klee, Shashkin [**31**] gave a detailed elementary exposition of Hadwiger's approach to the Euler characteristic, restricted to closed and bounded polyhedral sets in the plane. However, Shashkin's assertion (on page 82) that certain types of such sets admit a unique decomposition into "components" of a particularly "simple" type is incorrect.

In a later paper, Hadwiger [**7**] considered polyhedral sets, defined as those admitting relatively open convex dissections, but—due to his insistence on defining $\chi(C) = 1$ for every relatively open convex polyhedron—failed to obtain the general version of Theorem 3.

The approach followed here, to assign to a relatively open convex set of dimension d the Euler characteristic $(-1)^d$, is due to Lenz [**15**]. An account of the relevant writings by Lenz, Groemer and others appears in McMullen-Schneider [**18**]. Related developments, and in particular the relations between the Euler characteristics and valuations on appropriate families of sets are presented in Schneider [**28**], with extensive references to earlier literature. It should be stressed that, in all these works, the definition of the Euler characteristic by a relation such as (c) in Section 3 does not lead to any result of the nature of Theorem 4. Although involving the relatively open convex elements of dissections, these formulae apply equally to *all* such dissections of the given set (just as the topological approach applies to all simplicial complexes that represent a given set); as a consequence, they do not reflect in any way the particular facial structure of a polyhedral set. In fact, it seems that objects like the k-scaffolds have not been considered in the literature at all.

Neither McMullen and Schneider [**18**] nor Schneider [**28**] mention the work of Nef [**20–25**], which our note parallels to some extent. Nef's definition of polyhedral sets differs from ours, but is equivalent to it; his approach to the Euler characteristic is the same as ours. However, we believe our definitions lead to simpler proofs, and there is a significant difference between Nef's treatment of the faces of polyhedral sets and that given here. In view of its importance in the history of the subject, it seems appropriate to devote a few lines to a discussion of this topic.

In the case of convex polyhedra it is generally accepted that the "faces" are the intersections of the polyhedra with supporting planes. The only differences between the treatments of various authors is in deciding whether to regard the intersections themselves, or their relative interiors, as "faces", and whether the convex set itself and/or the empty set should also be included. Whatever choices are made, the faces of a convex polyhedron form a well-determined family; there is a bijection between the family of relatively open faces, and the family of closed faces. However, when more general polyhedra are considered there seems to be no agreement at all! In fact, most writers seem content to avoid the topic all together.

One of the few writers devoting some attention to this question is Hajós [**8**]. His "polyhedra" are closed, bounded, polyhedral sets which coincide with the closure of their interiors. To find the "faces" of such a polyhedron P, he proceeds as follows: If L is a plane that meets bd P in a set with non-empty relative interior, then the closure of any connected component of the relative interior of $L \cap \text{bd } P$ is a face of P. A vertex of P is any point that belongs to some three different faces such that their planes do not pass through one line. If the intersection S of two non-coplanar faces of P contains a segment, then S contains two or more vertices of P; segments determined by these vertices, contained in S and not containing any vertices in their relative interior, are called edges of P. Hajós appears not to realize that these definitions lead to such oddities as an edge that is in the relative interior of every face that contains it, or a vertex that is a relatively interior point of every face that contains it (see FIGURE 11(a)). Rather unsurprisingly, Hajós reports no results concerning these definitions, and it is reasonable to expect that there is little hope for establishing any connection between the Euler characteristic of P and the vertices, edges and faces defined in this way. The calculation in the caption to FIGURE 11 bears out this interpretation.

Nef [1978] defines as faces of a polyhedral set P any family of disjoint, relatively open convex sets that is a dissection of P; hence a relation analogous to Theorem 4 holds if V, E, F, C denote the numbers of "faces" of dimension $0, 1, 2, 3$, respectively. However, these "faces" are, in general, not uniquely determined, and have only a limited geometric significance due to the following fact. There exist closed polyhedral sets homeomorphic to a closed ball, for which any convex dissection must use as vertices points that cannot be regarded as vertices of P in any reasonable sense. Such polyhedral sets were first described by Lennes [**14**]; the simplest of these, shown in FIGURE 11(b), is due to Schönhardt [**29**]. For this set P our definitions yield six vertices, twelve edges and eight faces. In any of the dissections of P used by Nef there are at least seven vertices (and correspondingly larger numbers of edges and faces), hence at least one of them is devoid of geometric meaning. The same polyhedron P is also a counterexample to Lemma 2 of Szabó [**34**], which asserts that each polyhedron (according to a definition that includes P) has a simplicial decomposition in which all vertices of the tetrahedra involved are also vertices of the polyhedron. For interesting results concerning polyhedra that lack simplicial decompositions free of additional vertices see Ruppert and Seidel [**27**].

The results concerning polyhedral sets presented here can be extended to more general sets. For example, one could admit as "basic constituents", besides

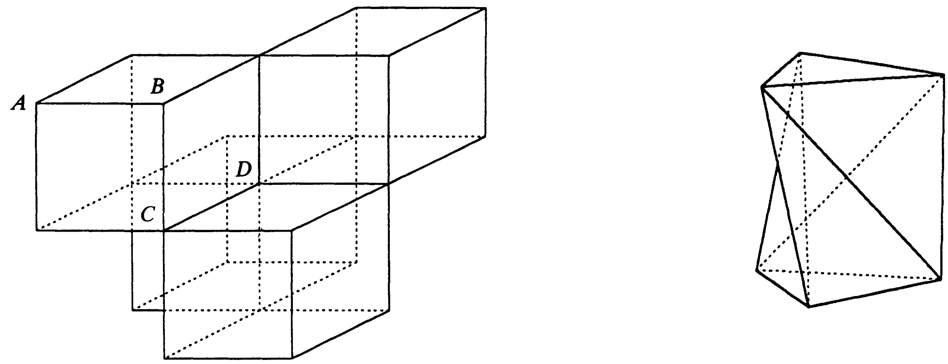


Figure 11. (a) A closed polyhedral set P, which is the union of four cubes. According to the definitions of Hajós [8, p. 37], this P has 23 vertices, 42 edges and 15 faces, hence the Euler characteristic of its boundary would be $23 - 42 + 15 = -4$. According to the definitions adopted here, $C = 4$, $V = 16$ (four vertices such as A, and 12 like B), $E = 30$ (twelve edges like AB, and six edges in the shape of two crossing open segments such as those containing the point C, each of Euler characteristic -3), and $F = 19$ (twelve faces are small squares, and one face, of Euler characteristic 7, consists of the three large squares that contain the point D), hence $\chi(\text{bd } P) = 5$, and $\chi(P) = 1$—in agreement with the topological interpretation of the Euler characteristic. (b) A nonconvex octahedron P, which has the property that in every relatively open convex dissection of P, some vertices of the polyhedra in the dissection are not vertices of P.

relatively open convex polyhedra, also spheres, open balls, closed balls, and the (unbounded) complements of closed balls. Extending property (b) by assigning to these sets the Euler characteristic 2, -1, 1, and -2, respectively, and to circles and open circular disks the values 0 and 1, and considering sets obtainable as finite unions of intersections of "basic constituents", results analogous to Theorems 1 to 4 can be obtained. Various other generalizations are also possible; their investigation is left to the reader.

We hope that the present account will lead to a better understanding of Euler's Theorem, and possibly also to analogous results for other valuations on polyhedral sets.

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Department of Mathematics, GN-50
University of Washington
Seattle, WA 98195
grunbaum@math.washington.edu

School of Mathematics
University of East Anglia
Norwich NR4 7TJ, England
k101@cpc865.east-anglia.ac.uk