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QUADRANGLES, PENTAGONS, AND COMPUTERS

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Already Euclid knew (as do all high-school students — or do they?) that the perpendicular bisectors of the sides of a triangle are concurrent, meeting at the center of the circle circumscribed to the triangle. Strangely enough, the question what do the perpendicular bisectors of the sides of a quadrangle do (see Figure 1) seems to have

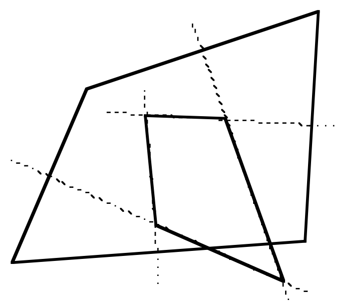

Figure 1.

been asked only relatively recently, in 1953. Problem E 1085, proposed by Josef Langr [3], is the following (reproduced in full from [3]):

The perpendicular bisectors of the sides of a quadrilateral Q form a quadrilateral Q_1, and the perpendicular bisectors of the sides of Q_1 form a quadrilateral Q_2. Show that Q_2 is similar to Q and find the ratio of similitude.

No solution of problem E 1085 was published in the intervening forty years. The next appearance of the topic is in the book [4] by C. Stanley Ogilvy, page 80; the reference to Langr is given on page 177. It is not clear whether Ogilvy had a solution or not; if he did, he kept it to himself. In fact, to judge by the subtitle of the book, he may well not have had it.

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The only later mention of the matter is in a very interesting book [1] of a completely different character; through [1] I first became aware of the question. As its title indicates, Chou's book is devoted to the theory and practice of having theorems in elementary geometry proved by specially developed computer software. As Example 65 in [1] he has his software prove the similarity part of Langr's problem. (In the next Example 66 Chou uses the software to establish that (in Langr's notation) if Q_3 is formed by the perpendicular bisectors of Q_2 then Q_3 and Q_1 bear the same relation as Q_2 and Q. Since this is completely trivial in view of the result of Example 65, I suspect that some misunderstanding occurred here. This opinion is reinforced by Chou's attributing this as a problem to Ogilvy, although Ogilvy mentions it as obvious.)

Intrigued by all this I set up a simple program in Mathematica® software to draw any preassigned quadrangle as Q and to compute and draw the corresponding quadrangles Q_1 and Q_2. The outcomes of several such experiments appear in Figure 2. Numerical data yield evidence that (to at least 12 decimal places) Q_2 is similar to Q in each of the cases. In fact, the experiments show that Q and Q_2 are not only

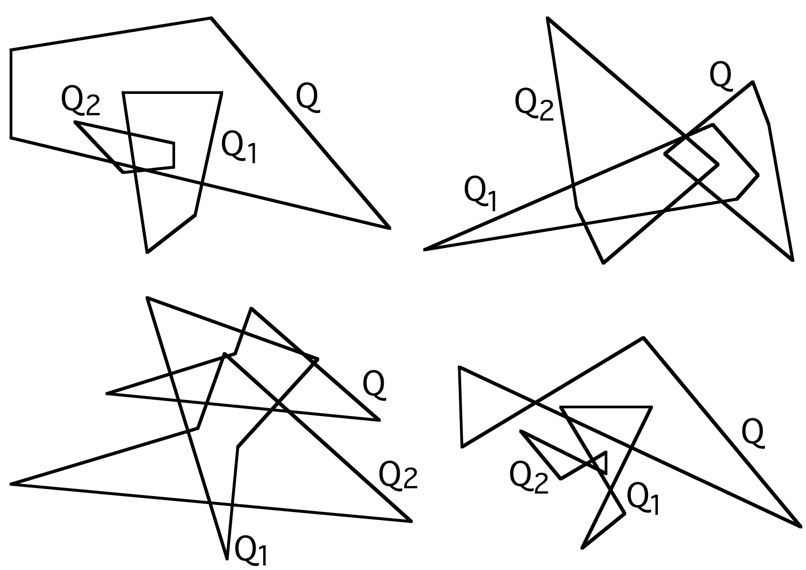

Figure 2.

similar but are homothetic (that is, one is obtained from the other by uniform stretching from a fixed point, without any rotations or reflections). Moreover, for each of the three types of quadrangles (convex, nonconvex simple, and selfintersecting) Q_1, and hence Q_2 as well, is of the same type as Q. One further observation: the ratio of homothety between Q and Q_2 is positive for simple nonconvex Q, and negative for convex and for selfintersecting Q. As shown by the examples in Figure 2, the absolute value of the ratio can be either less or more than 1. However, I am not closer than any of the other writers to guessing what is, for a given quadrangle, the homothety ratio for which Langr asked. This is only one of the many questions that arise in this context; we shall discuss several other aspects — some much more basic — after we briefly look at the case of pentagons.

The idea is clear, even though it does not seem to have been mentioned in the literature. We should ask what happens if, starting from a suitable pentagon P, we use intersection points of perpendicular bisectors of adjacent edges of P as the vertices of a new pentagon P_1, go from this one by the same procedure to P_2, etc.? An easy modification of my earlier program made it possible to conduct experiments without much effort. As is visible from Figure 3, no obvious relationship between P and P_2 emerges. However, with a bit more patience we find that now it is P_1 and P_3 that are homothetic (see Figure 4), even though P and P_2 are not. Naturally, it follows that P_2 and P_4 are homothetic, and so on. The same result, that P_2 and P_4 are homothetic, is obtained starting with any pentagon of whatever shape (as long as no four of its vertices are concyclic).

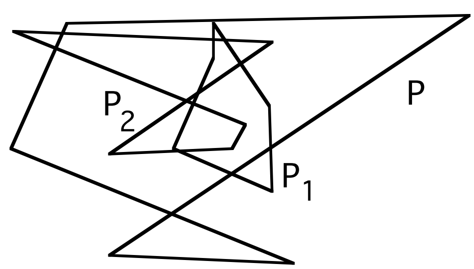

Figure 3.

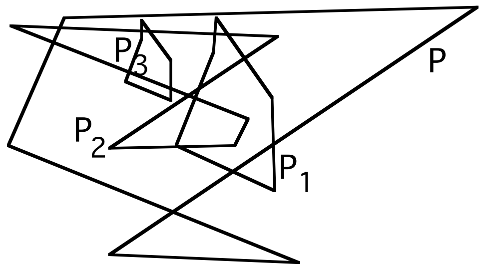

Figure 4.

If the reader now has the same idea as the author had — that for hexagons the similarity will start with the third and fifth iteration — we share in the disappointment. In fact, I have no idea what happens for hexagons or polygons with more sides. The only indisputable fact is that nothing as simple as similarity after a few steps takes place. So let us now list a number of open problems, in what probably is an increasing level of difficulty.

Problem 1. Find a full solution of Langr's problem; that is,

(i) prove directly that Q_2 and Q are homothetic; and

(ii) determine the ratio of homothety in dependence on the shape of the starting quadrangle.

Notice that this would, among other consequences, also find all the quadrangles for which the ratio is 1 or -1.

Problem 2. Characterize those quadrangles Q for which Q is similar to Q_1.

Problem 3. Investigate in detail what happens with pentagons.

Some of the puzzling aspects here are:

(i) While every quadrangle is obtainable as Q_1 for a suitable Q, this is clearly not the case with pentagons; which ones can be obtained ? In particular, it seems that no convex pentagon is P_1 for any P ! For which pentagons is P similar to P_2, or to P_1 ?

(ii) What is the relation between the "types" of P, P_1 and P_2 ? Here by "type" I mean any characteristic such as convexity, or selfintersection, or whatever works.

Problem 4. What is actually happening with hexagons ? In some instances, iterating the construction sufficiently many times seems to lead to stabilization of the shapes, while in other cases nothing like that appeared in my experiments.

Problem 5. The attentive reader may have noticed that I did not use the word "theorem" so far. So the question arises what is the character of the facts I have been discussing ? Do we start trusting numerical evidence (or other evidence produced by computers) as proofs of mathematics theorems ? In some situations this is easily answered in the affirmative. For example, if somebody claims that a certain integer n has been proved by computer to be prime, and if another person independently checks this assertion and comes up with the same result, there seems to be no doubt that n has been proven to be a prime. Probably similar is the situation regarding "mechanical proofs" such as those of Chou [1]. If the same result is "mechanically" verified by somebody else, there will probably be general agreement that the assertion in question is indeed a fact. But, disregarding Chou's "mechanical proof" of the similarity part of Langr's problem, is there any consequence to be drawn from the fact that in example after example numerical evidence establishes the homothety of Q and Q_2 ? Seeing that anybody with even a very superficial knowledge of computers and software can write a program that will check the validity of the claims made above, how much can one doubt that regardless of which pentagon we start with, P_1 and P_3 will be homothetic ? On the other hand, if we have no doubt — do we call it a theorem ? According to my Webster [5] a theorem is "a statement in mathematics that has been proved or whose truth has been conjectured". While I find this to be far too inclusive a definition, I do think that my assertions about quadrangles and pentagons are theorems.

A few remarks may possibly make the reader more receptive to the conclusion that regardless of the pentagons and hexagons discussed here, the mathematical community needs to come to grips with the possibilities of new modes of investigation that have been opened up by computers.

First, it should be noticed that if we try to prove the assertions about pentagons (or quadrangles, or hexagons) using analytic geometry, we wind up with some algebraic relations between the variables specifying the coordinates of the starting points. Thus a validation of the claim is equivalent to the verification that certain (possibly quite complicated) algebraic expressions are identically satisfied, that is, are identities. Now, just as in case of polynomials of a single variable and degree d the identity can be verified by checking numerical coincidence in $d+1$ cases, so for polynomial expressions in several variables identity can be established by a sufficient number of numerical verifications. But in fact, as pointed out long ago by Davis [2], an algebraic identity can be conclusively established by a single numerical check: we only need to use algebraically independent transcendental numbers! It is true that this is not very practical, since our computers do not operate with transcendental numbers. But in a sense, we can achieve a similar result much more simply. Assume that the question whether P_1 is always homothetic to P_3 reduces to the question whether a polynomial $p = p(x_1,...,x_{10})$ of some degree (say 10000) in the ten variables of the coordinates of the vertices of P is identically zero or not. If p is not identically 0 then $p = 0$ determines a set of dimension at most 9 in the 10-dimensional space of the x_i's; hence, if we choose at random a set of ten values for the variables, the probability that they will represent a point of the low-dimensional set is very small. Thus, if experiment after experiment comes up with the result that the value of p for the chosen entries is 0, it would be a very unwise investment to put one's money against the general validity of $p = 0$. (Notice that we are not talking about odds of 1 to a million — but to incomprehensibly large numbers !)

In short, while this is not the right place to enter into deep arguments on one or the other side of the issue, it seems clear that we have here a serious mathematical, philosophical and practical problem that needs to be addressed, and also that the approach followed in the first part of this note can lead in many other cases to new and worthwhile results and insights.

References

[1] S.-C. Chou, Mechanical Geometry Theorem Proving. Reidel, Boston 1988.

[2] P. J. Davis, Proof, completeness, transcendentals, and sampling. J. Assoc. Comp. Machin. 24(1977), 298 - 310.

[3] J. Langr, Problem E 1050. Amer. Math. Monthly 60(1953), 551.

[4] C. S. Ogilvy, Tomorrow's Math. Unsolved Problems for the Amateur. Oxford Univ. Press, New York 1962. 2nd ed. 1972. The page references given are to the second edition.

[5] Webster's Third New International Dictionary of the English Language. Unabridged. Encyclopædia Britannica, Chicago 1966.