Pick's Theorem

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Some years ago, the Northwest Mathematics Conference was held in Eugene, Oregon. To add a bit of local flavor, a forester was included on the program, and those who attended his session were introduced to a variety of nice examples which illustrated the important role that mathematics plays in the forest industry. One of his problems was concerned with the calculation of the area inside a polygonal region drawn to scale from field data obtained for a stand of timber by a timber cruiser. The standard method is to overlay a scale drawing with a transparency on which a square dot pattern is printed. Except for a factor dependent on the relative sizes of the drawing and the square grid, the area inside the polygon is computed by counting all of the dots fully inside the polygon, and then adding half of the number of dots which fall on the bounding edges of the polygon. Although the speaker was not aware that he was essentially using Pick's formula, I was delighted to see that one of my favorite mathematical results was not only beautiful, but even useful. (From DeTemple [1989].)

The discoverer of the theorem in question, Georg Alexander Pick, was born in 1859 in Vienna, and died around 1943 in the Theresienstadt concentration camp. He made significant contributions to analysis and differential geometry. The theorem we are concerned with was first published in 1899 [**15**]. It became widely known through Steinhaus' delightful book [**18**].

Pick's theorem concerns lattice polygons ("geoboard polygons"), that is, polygons with all vertices at points of the square unit lattice L, see Figure 1. The original form of the theorem concerns simple polygons, whose edges do not cross one another. (More formally, a polygon is simple if its edges have no mutual intersections other than those of adjacent edges at the common vertices.) The theorem asserts that the area of a simple lattice polygon P is given by the expression

$$i + b/2 - 1,$$

where i is the number of lattice points in the interior of P, and b is the number of lattice points on the boundary of P, that is, points which are either vertices of P or relatively interior points of edges of P. Many proofs of Pick's Theorem are known, see, for example, [**1**], [**2**], [**3**], [**6**], [**7**], [**10**], [**11**], [**12**], [**14**]; there are various generalizations: to more general polygons [**9**], [**15**], [**19**], to lattices other than the square lattice [**4**], [**5**], and to higher-dimensional polyhedra [**13**], [**16**], [**17**], [**20**].

In this paper we shall extend Pick's theorem to more general lattice polygons, by allowing multiple intersections, and even overlapping, of the edges. We shall

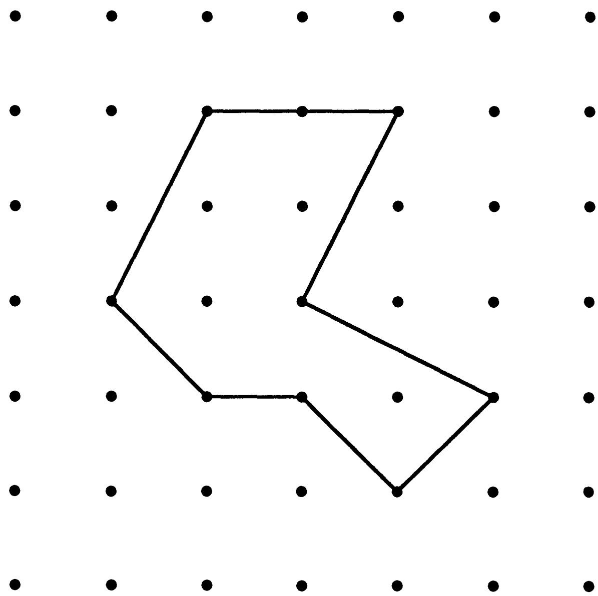


Figure 1. A simple lattice 8-gon P illustrating the classical form of Pick's Theorem. Here $i = 4$ is the number of lattice points in the interior of P, $b = 9$ is the number of lattice points on the boundary of P, and

$$A(P) = i + \frac{b}{2} - 1 = \frac{15}{2}$$

is the area of P.

make use of results on rotation numbers, winding numbers and tangent numbers of such polygons P; a brief account of the necessary definitions and facts concerning these numbers will be given here, but for more details, examples, and proofs of some of our assertions, the reader should consult [**8**].

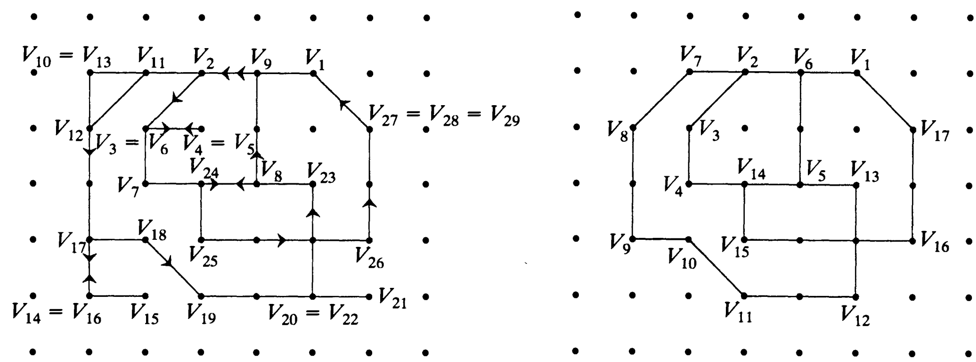
1. BASIC DEFINITIONS. By an *abstract polygon* or *n-gon* Q we mean an ordered sequence $(V_1, \dots, V_n)$ of n distinct symbols $V_1, \dots, V_n$, called the *vertices* of Q. Adjacent pairs $(V_1, V_2), (V_2, V_3), \dots, (V_n, V_1)$ (where the subscripts are taken mod n) are called the *edges* of Q, and two sequences which differ only by a cyclic permutation of the symbols are regarded as identical. Thus Q has a definite *orientation* and the edge (V_i, V_{i+1}) is said to be *oriented* or *directed* from V_i to V_{i+1}.

A *lattice polygon* is any embedding of an abstract polygon and its edges in the plane, such that the following conditions hold:

(i) The image of each V_i is a point of the square unit lattice L. Without confusion we may continue to denote the image of V_i by the same symbol, and to call it a vertex of the polygon.

(ii) Each edge (V_i, V_{i+1}) (mod n) is represented by a straight line segment connecting the image points V_i, V_{i+1} in the plane. In the diagrams it is convenient to denote the direction of the edge by an arrow.

An example of a lattice polygon is shown in Figure 2(a). If we impose two additional restrictions, namely



(a) (b)

Figure 2. (a) An example of a lattice polygon (29-gon) with multiple vertices ($V_3 = V_6$, $V_4 = V_5$, $V_{10} = V_{13}$, $V_{14} = V_{16}$, $V_{20} = V_{22}$, $V_{27} = V_{28} = V_{29}$) and whiskers (V_9, V_{10}, V_{11}; V_{12}, V_{13}, V_{14}; V_{14}, V_{15}, V_{16}; V_3, V_4, V_6 also becomes a whisker after the multiple vertex $V_4 = V_5$ has been removed, and V_{12}, V_{14}, V_{17} becomes a whisker after the removal of whiskers V_{12}, V_{13}, V_{14} and V_{14}, V_{15}, V_{16}). (b) The polygon that results from shaving that shown in (a); it is a 17-gon.

(iii) no two consecutive vertices V_i, V_{i+1} (mod n) map onto the same lattice point, and

(iv) two edges with a common vertex do not overlap (that is, the polygon has no "whisker"),

then we say that the polygon is *shaven*. Examples of shaven polygons appear in Figures 2(b) and 3. (The numbers attached to the lattice points in the latter figure will be explained in the next section.) Figure 3 shows some of the possibilities for multiple intersections and overlaps of edges that are not excluded by conditions (i) to (iv).

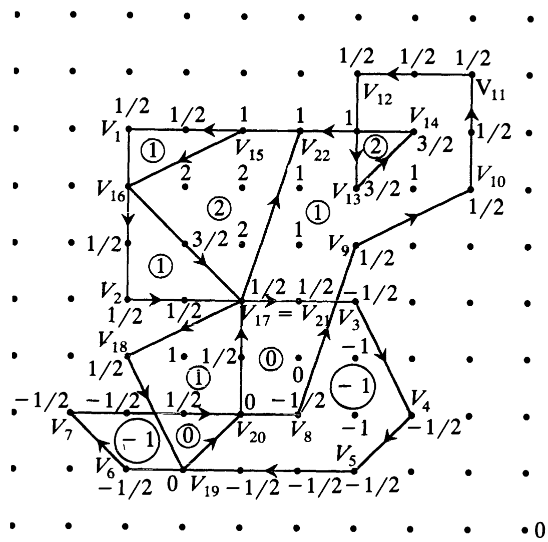


Figure 3. A general lattice polygon P with 22 vertices. The winding numbers of the various cells are shown by circled numbers. The rotation number of P is 2, and the indices of the lattice points are indicated. (The index of every point in the exterior is 0, and is not marked.) The area of P is 18 and the sum of all the indices is $i(P) = 20$, in agreement with the assertion of Theorem 1 that $A(P) = i(P) - r(P)$.

We shall usually denote a lattice polygon (shaven or otherwise) by the letter P. The complement $\mathbb{E}^2 \setminus P$ of P in the plane consists of a finite number of connected open regions called the *cells* of P. All are bounded except for one, which is called the *exterior of* P.

Now let X be a given point of the plane, which does not belong to P, and let $R(X)$ be a ray (a closed half-line) with endpoint X which does not pass through any vertex of P. Then for each edge E_j we define

$$\omega(R(X), E_j) = \begin{cases} 0 & \text{if } R(X) \text{ does not intersect } E_j, \\ 1 & \text{if } E_j \text{ crosses } R(X) \text{ in a counterclockwise} \\ & \quad \text{direction as viewed from } X, \\ -1 & \text{if } E_j \text{ crosses } R(X) \text{ in a clockwise direction.} \end{cases}$$

The *winding number* $w(P, R(X))$ of P with respect to $R(X)$ is defined as $\Sigma_j \omega(R(X), E_j)$ summed over all the edges of P. It can be shown that $w(P, R(X))$ depends *only* on the endpoint X of $R(X)$, and *not* on the particular ray $R(X)$ that was used. In fact, this even applies to rays that pass through vertices of P if the definition is suitably modified. In view of this we may use the notation $w(P, X)$ unambiguously. Further, it can also be shown that if X and Y belong to the same cell of P, then $w(P, X) = w(P, Y)$. Hence we can define the winding number $w(P, C)$ of a cell C with respect to P as the winding number of any point in the cell. In Figure 3 the winding numbers of the cells are indicated. These are the same winding numbers which are well known from calculus for their rôle in defining the area enclosed by curves with selfintersections.

The *area* $A(P)$ of P is defined as $\Sigma_j w(P, C_j)|A(C_j)|$ summed over all the cells C_j of P. Here $A(C_j)$ is the (usual) elementary area of the polygonal region C_j. Figure 3 serves to illustrate the calculation of the area of the polygon P. The area of a polygon can be positive, negative or zero. Since the winding number $w(P, C_j)$ of a cell changes its sign if we reverse the orientation of P (that is, reverse the order of the vertices in the definition of the polygon), the same is true for the area of P.

Next, we need the concept of "rotation number" (sometimes called "tangent winding number") of a polygon. Throughout we shall use the *absolute system* of angle measure, in which a complete counterclockwise turn of 2π radians has value 1. At a vertex V_j of a shaven polygon P let W lie on the extension of (V_{j-1}, V_j) beyond V_j. Then the signed angle $\angle WV_jV_{j+1}$ (which necessarily satisfies $-\frac{1}{2} < \angle WV_jV_{j+1} < \frac{1}{2}$) is called the *deflection* $d(V_j)$ of P at V_j. It is easy to show that $r(P) = \Sigma_j d(V_j)$, with summation over all the vertices of P, is necessarily an integer, called the *rotation number* of P. For the polygon P of Figure 3 we have $r(P) = 2$, as indicated in the caption. It should be observed that the rotation number is only defined for shaven polygons, since the definition of deflection is not applicable at multiple vertices or whiskers.

To facilitate the formulation of the next definition, we note that if a vertex V_j of the lattice polygon lies on a ray $R(X)$ with endpoint X, then the two edges (V_{j-1}, V_j) and (V_j, V_{j+1}) which meet at V_j can lie in six different configurations with respect to the ray, see Figure 4. If the edges lie on different sides of $R(X)$ (cases (a) and (b)) we say that $R(X)$ *cuts* the polygon P at V_j. In the four other cases we say that P is *tangent* to $R(X)$ at V_j. In (c) and (f) we say that the tangency is *concordant* since the directions induced on $R(X)$ by the edges (V_{j-1}, V_j), (V_j, V_{j+1}) are consistent with that on $R(X)$ oriented away from X. In (d) and (e) we say that the tangency is not concordant.

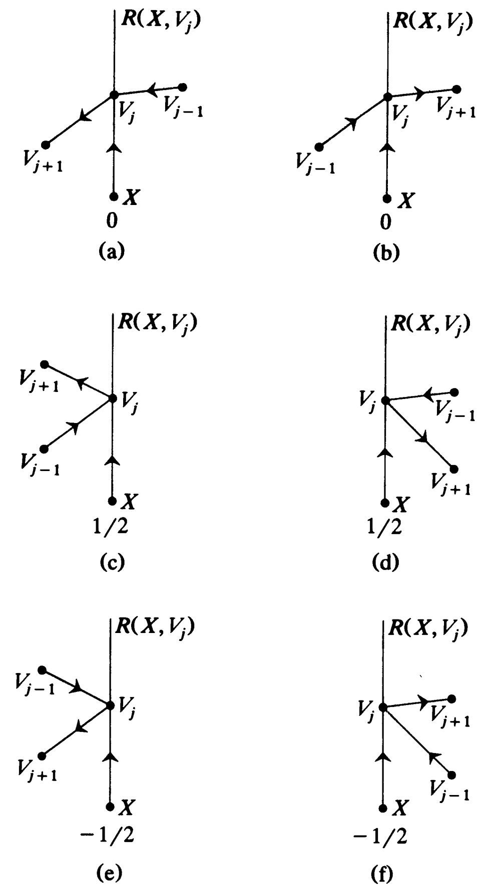


Figure 4. The possible positions of edges (V_{j-1}, V_j) and (V_j, V_{j+1}) relative to ray $R(X)$, and the contributions to the tangent number $t(X, P)$.

Let now X be any point of the plane which does not lie on a line containing an edge of a lattice polygon P, and consider a closed ray $R(X)$ with endpoint X that passes through a vertex V_j of P. Then we define

$$\tau\big(R(X), V_j\big) = \begin{cases} 0 & \text{if } R(X) \text{ cuts } P \text{ at } V_j \text{ (cases (a) and (b) of Figure 4)}, \\ \frac{1}{2} & \text{if the edges } (V_{j-1}, V_j) \text{ and } (V_j, V_{j+1}) \text{ lie to the left} \\ & \quad \text{of } R(X) \text{ and the tangency is concordant (case (c))}, \\ & \quad \text{or they lie to the right of } R(X) \text{ and the tangency is} \\ & \quad \text{not concordant (case (d))}, \\ -\frac{1}{2} & \text{in all other cases ((e) and (f) in Figure 4)}. \end{cases}$$

For fixed X let $t(P, X) = \sum_j \tau(R(X), V_j)$, where the sum is over all the vertices of P. It can be shown that $t(P, X)$ is necessarily an integer; it is known as the *tangent number* of X with respect to P. (This definition differs slightly from the one given in [**8**], but is equivalent to it.) We require the following important property of $t(P, X)$ (see [**8**]):

If X lies in the exterior of P, then $t(P, X) = r(P)$.

For example, in Figure 3 the tangent number $t(P, O)$ of the point O with respect to P is 2, which is the same as the rotation number of P.

2. INDICES OF LATTICE POINTS WITH RESPECT TO THE POLYGON P. An essential ingredient in the main theorem will be the index $i(P, X)$ of a lattice point X with respect to P. In Figure 3 the indices of all the lattice points are indicated; these are zero for all points in the exterior of P.

The definition of $i(P, X)$ is similar in many respects to both that of the winding number and that of the tangent number. However, in this case we do not exclude the possibility that X belongs to P. Let $R(X)$ be a closed ray with endpoint X, which intersects no edge of P in a line segment of positive length. Then for each point $Y \in R(X) \cap P$ we define

$$\iota(R(X), Y) = \begin{cases} 1 & \text{if } Y \neq X \text{ and } P \text{ crosses } R(X) \text{ in a counter-clockwise direction viewed from } X, \\ \frac{1}{2} & \text{if } Y = X \text{ and } P \text{ crosses } R(X) \text{ from right to left,} \\ -1 & \text{if } Y \neq X \text{ and } P \text{ crosses } R(X) \text{ in a clockwise direction,} \\ -\frac{1}{2} & \text{if } Y = X \text{ and } P \text{ crosses } R(X) \text{ from left to right,} \\ 0 & \text{if } Y \neq X \text{ and } P \text{ is tangent to } R(X) \text{ at } Y, \\ \frac{1}{2} & \text{if } Y = X\text{, and either } P \text{ is concordantly tangent to } R(X) \text{ at } Y \text{ and the two edges meeting at the point of tangency lie on the left of } R(X)\text{, or } P \text{ is tangent to } R(X) \text{ at } X \text{ but the tangency is not concordant and the two edges lie to the right of } R(X), \\ -\frac{1}{2} & \text{if } Y = X \text{ and } P \text{ is tangent to } R(X) \text{ in the other two cases, that is, the tangency is concordant and the edges lie on the right, or the tangency is not concordant and the edges lie on the left of } R(X). \end{cases}$$

An example of the calculation of $\iota(R(X), Y)$ is shown in Figure 5. Now let $i(P, X) = \Sigma_Y \iota(R(X), Y)$, where summation is over all the intersections of P with

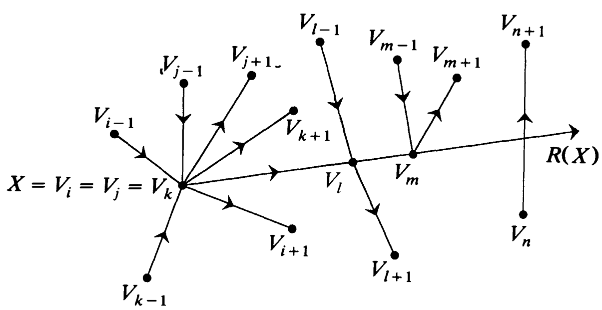


Figure 5. The calculation of $i(X, P)$, where X is a lattice point which coincides with vertices V_i, V_j and V_k. The ray $R(X)$ also intersects the polygon three times (at V_l, V_m and the edge (V_n, V_{n+1})) at points distinct from X. According to the definition the contributions of these six intersections are, in order, $-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$ at X and $-1, 0, 1$ at other points of $R(X)$, so $i(X, P) = \frac{1}{2}$.

$R(X)$. (In the case of multiple intersections, the contributions due to each arc of P are added.) From the definition, it is clear that if $X \notin P$ then $i(P, X) = w(P, x)$.

Lemma. *The value of $i(P, X)$ depends only on the lattice point X and the polygon P and not on the ray $R(X)$ chosen to define it.*

Proof: Consider changes in the value of $i(P, X)$ that occur as $R(X)$ is rotated in a counterclockwise direction about X. Let Z be any point on $R(X)$ such that the open line segment $]X, Z[$ lies entirely in some cell of P. Suppose the initial position of the ray is $R_1(X)$ (see Figure 6) and Z is at Z_1. It is clear that during the rotation, so long as Z does not cross any edge of P, then the fact that $w(P, Z)$ remains constant shows that $i(P, X)$ does so also.

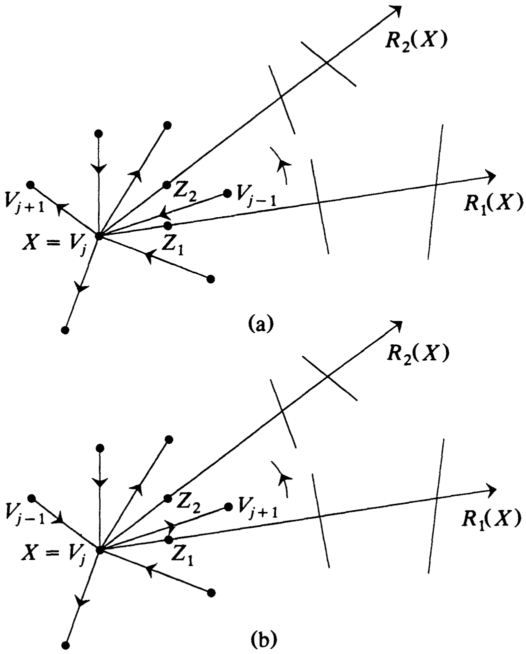


Figure 6. Diagrams illustrating the proof of Lemma. The intersection of P with $R(X)$ at points other than X are shown schematically at right in each of the diagram.

Now let $R(X)$ cross an edge directed towards X (see Figure 6(a)) to position $R_2(X)$ with Z moving to Z_2. The change in $i(X, P)$ will be twofold:

(i) the contribution to $i(P, X)$ from the intersections of P with $R(X)$ at points other than X (which equals the winding number of Z with respect to P) will decrease by 1, and

(ii) the contribution to $i(X, P)$ from the intersections of P with $R(X)$ at X will increase by 1. (In the diagram the edges (V_{j-1}, V_j) and (V_j, V_{j+1}) form a non-concordant tangency to $R_1(X)$ at X and so contribute $-\frac{1}{2}$. However, they cut $R_2(X)$ from right to left and so in this new situation contribute $\frac{1}{2}$. The total change in the contribution is $+1$. It is easy to check that in all other cases the same holds.)

On the other hand, if $R(X)$ crosses an edge directed away from X (see Figure 6(b)) then the corresponding changes are $+1$ and -1 respectively. If pairs of edges coincide, then their contributions are added.

Thus it will be seen that in all cases the rotation of $R(X)$ about X does not alter the value of $i(P, X)$ as we have defined it, and so the lemma is proved.

At first sight the definition of $i(P, X)$ may appear somewhat artificial. In fact, as will be seen from Figure 1, in the case of a simple polygon P oriented in a positive (counterclockwise) direction, the index of each lattice point in the interior of P is 1 and the index of each lattice point on the boundary of P is $\frac{1}{2}$. Theorem 1, stated at the beginning of the next section, reduces immediately to the classical form of Pick's Theorem in this case. The complications in the definition of $i(X, P)$ arise because of the need to deal with lattice points that occur at multiple intersections and overlapping edges which may occur in the general lattice polygons which we are considering here.

Finally, we write $i(P) = \Sigma i(P, X)$, where summation is over all the lattice points X of L. We note that this sum is finite since $i(P, X) = 0$ for all X in the exterior of P.

3. SHAVEN POLYGONS. We begin with the basic theorem from which the more general result (Theorem 2 of the next section) can be derived.

Theorem 1. *Let P be any shaven lattice polygon. Then*

$$A(P) = i(P) - r(P),$$

where $A(P)$ is the area of P, $r(P)$ is the rotation number, and $i(P)$ is the sum of the indices with respect to P of all the lattice points.

Proof: Let O be any point in the plane not belonging to P. We calculate the area of P in the classical way, as follows. Let T_j be the triangle obtained by joining the edge E_j of P to O (that is, T_j is the convex hull of E_j and O with the orientation induced by that of E_j). Then $A(P) = \Sigma_j A(T_j)$, summation being over all the edges of P, and areas being counted with appropriate signs ($A(T)$ is positive if T is oriented in a counterclockwise direction, and negative if T is oriented clockwise.)

In the present context we take O as a lattice point in the exterior of P, and apply the classical form of Pick's Theorem to find the area of each triangle. All that is needed for the proof is an investigation as to how the indices of the lattices points and rotation numbers change when two triangles T_1 and T_2 corresponding to adjacent edges (V_{j-1}, V_j) and (V_j, V_{j+1}) are welded together along their common boundary $[O, V_j]$, see Figure 7. Consider, to begin with, the case where T_1 and T_2 are oriented positively (case (a)). The index of a lattice point in the relative interior of T_1 is 1 and of a point on its boundary is $\frac{1}{2}$. Also the rotation number is 1, and so $A(T_1) = i(T_1) - 1 = i(T_1) - 2i(T_1, O)$. Similarly $A(T_2) = i(T_2) - 2i(T_2, O)$. After welding the triangles together we obtain a quadrilateral Q with vertices V_{j-1}, V_j, V_{j+1}, O and we note that the index of each lattice point with respect to Q is the sum of the indices assigned to the point by consideration of T_1 and T_2 *except for the points V_j and O*. Since $i(T_1, V_j) = i(T_2, V_j) = i(Q, V_j) = \frac{1}{2}$, we must *subtract* $\frac{1}{2}$ from the indices of V_j and O, and then

$$A(Q) = i(Q) - 1 = i(Q) - 2i(Q, O).$$

The other cases (b) to (f) in Figure 7 are dealt with similarly. In case (c), for example, $i(T_1, V_j) = -\frac{1}{2}$, $i(T_2, V_j) = i(Q, V_j) = \frac{1}{2}$ and $i(T_1, O) = -\frac{1}{2}$, $i(T_2, O) = i(Q, O) = \frac{1}{2}$. Hence, after summing the indices we must *add* $\frac{1}{2}$ to those of V_j and

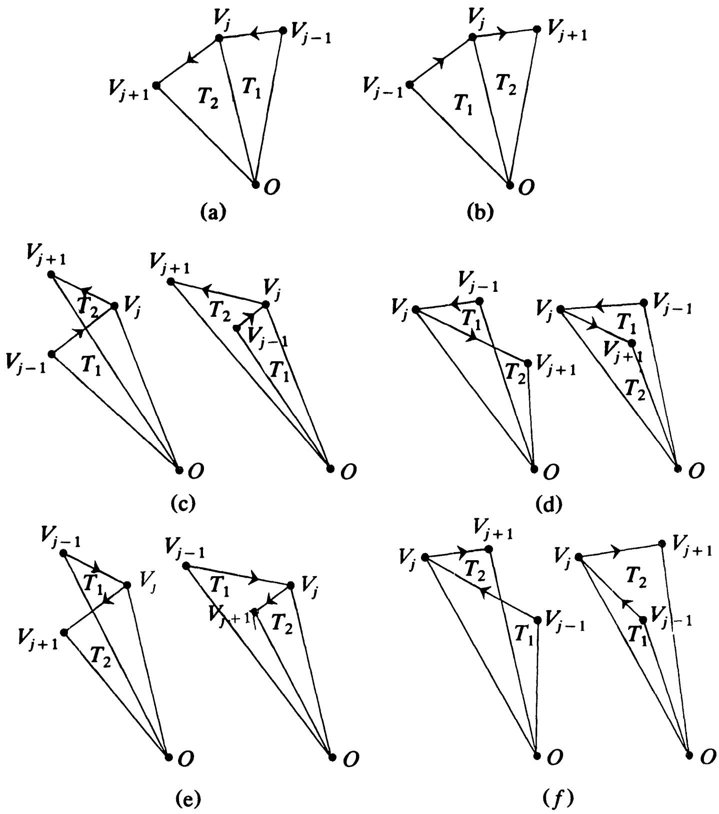


Figure 7. The possible configurations of oriented triangles $OV_{j-1}V_j, OV_jV_{j+1}$ with an edge OV_j in common. These are used in the proof of Theorem 1.

O. However, the relation

$$A(Q) = i(Q) - 1 = i(Q) - 2i(Q,O)$$

continues to hold. It does so also in the other cases: for (b) (c) and (d) we must *add* $\frac{1}{2}$ to the indices of V_j and O after amalgamation of the triangles, and we must *subtract* $\frac{1}{2}$ in cases (a), (e) and (f).

We build up P triangle by triangle, making the necessary modifications to the indices at the ends of the common edges. By natural extension of the above, at each stage X except the final one,

$$A(X) = i(X) - 2i(X,O).$$

When the triangle corresponding to the last edge of P is adjoined, and the modifications to the indices are applied as described above, we obtain the indices $i(P, X)$ of all the lattice points X with respect to P, *except that at O there will be an index i^* which is not equal to $i(P,O)$*. Thus the sum of all the indices will be $i(P) + i^*$, and the area will be given by

$$A(P) = (i(P) + i^*) - 2i^*.$$

To complete the proof we need only evaluate i^*. Let t^+ and t^- be the numbers of positively and negatively oriented triangles which meet at O, and $t_a, t_b, t_c, t_d, t_e, t_f$ be the numbers of vertices V_j of P at which the adjacent edges lie in the configurations (a), (b), (c), (d), (e), (f) of Figure 7, respectively. Then the above construction shows that

$$i^* = \tfrac{1}{2}(t^+ - t^- - t_a + t_b + t_c + t_d - t_e - t_f)$$
$$= \tfrac{1}{2}\big((t^+ - t_a) - (t^- - t_b) + t_c + t_d - t_e - t_f\big).$$

Now $t^+ - t_a$ is the number of connected chains of edges of P which are oriented counterclockwise viewed from O, and $t^- - t_b$ is the number of connected chains of edges of P which are oriented clockwise viewed from O. As O is exterior to P these must be equal, and the first two terms cancel leaving

$$i^* = \tfrac{1}{2}(t_c + t_d - t_e - t_f).$$

Comparing Figures 7 and 4, we see that $i^* = t(P, O) = r(P)$ by the result quoted above. Thus

$$A(P) = i(P) - r(P),$$

and the theorem is proved.

Now let $\mathbb{P} = \{P_1, P_2, \dots, P_n\}$ be a family of shaven polygons, that is to say, a finite set of such polygons. We define

$$r(\mathbb{P}) = \sum_{j=1}^{n} r(P_j), \qquad A(\mathbb{P}) = \sum_{j=1}^{n} A(P_j),$$
$$i(\mathbb{P}, X) = \sum_{j=1}^{n} i(P_j, X), \qquad i(\mathbb{P}) = \sum_{j=1}^{n} i(P_j).$$

Then, by additivity, Theorem 1 immediately implies the following:

Corollary. *If $\mathbb{P}$ is any family of shaven lattice polygons then*

$$A(\mathbb{P}) = i(\mathbb{P}) - r(\mathbb{P}).$$

4. GENERAL LATTICE POLYGONS. We began, in Section 1, by defining a general lattice polygon, but our main result (Theorem 1) applies only to shaven polygons. It is of some interest to consider whether this restriction is necessary. For *any* lattice polygon P the indices of the lattice points are uniquely defined as in Section 2, hence $i(P)$ is well defined. The area $A(P)$ is also uniquely determined, as already explained. However, the rotation number $r(P)$ is indeterminate if P has whiskers, or pairs of adjacent vertices which coincide; therefore to obtain a result analogous to Theorem 1 which applies to general polygons we need to obtain some number which plays a rôle analogous to $r(P)$.

To do this, we obtain a shaven polygon P' from the general polygon P by

(a) removing each whisker. More precisely, if V_{j-1}, V_j, V_{j+1} is a whisker we delete the vertex V_j and replace the edges (V_{j-1}, V_j) and (V_j, V_{j+1}) by either the edge (V_{j-1}, V_{j+1}) or by the two coinciding adjacent vertices $V_{j-1} = V_{j+1}$.

(b) removing coincident adjacent vertices. Thus if $V_j = V_{j+1} = \cdots = V_{j+s}$ we remove $V_{j+1}, \dots, V_{j+s}$ and replace the edge (V_{j+s}, V_{j+s+1}) by (V_j, V_{j+s+1}).

Either operation can produce further whiskers, and operation (a) can produce multiple vertices. However, it can be shown easily that repeated applications of the operations lead eventually to a shaven polygon P'. Even though the polygon P' is in some cases not uniquely determined by P, the rotation number $r(P')$ is independent of the particular P' obtained, and we may therefore define $r(P)$ to have the value $r(P')$. With this convention we obtain the following form of Pick's theorem.

Theorem 2. *Let P be any lattice polygon, $A(P)$ its area, $i(P)$ the sum of the indices at the lattice points, and $r(P)$ the rotation number as defined above. Then*

$$A(P) = i(P) - r(P).$$

Proof: This is immediate from the observation that $A(P) = A(P')$, $i(P) = i(P')$, $r(P) = r(P')$, and from Theorem 1 applied to the shaven polygon P'.

5. FINAL REMARKS. A major difference between our treatment of Pick's Theorem and that of earlier papers is that *oriented* polygons are used here. If suitable orientations are introduced our main theorem implies all known variants of Pick's Theorem as it relates to plane lattice polygons. It does not, of course, include the three-dimensional version of [**13**], [**16**], [**17**], nor the results of [**4**], [**5**] concerning the "hexagonal lattice" (which is not a lattice in the usual terminology) or Archimedean tilings.

Since the index of a lattice point, the rotation number of a polygon, and the area of a polygon are invariant under affine transformations of determinant 1, it follows that our theorems are invariant under such transformations. Hence it includes results relating to polygons whose vertices lie at lattice points of the equilateral triangular lattice, see [**5**]. For "polygons of higher genus" such as those of [**14**, Figures 5, 6, 7], after suitable orientations of the edges are introduced, the results follow from our Theorem 1 and its corollary.

Many generalizations of Pick's Theorem introduce, as one of the variables in the formula for the area, the Euler characteristic of the polygon (or the polygonal region). Since the area depends on the orientation of the polygon, whereas the Euler characteristic does not, it is not an appropriate variable to use for polygons or families of polygons that may have regions with winding numbers other than 0 or 1. If the only winding numbers are 1 or 0, then it is trivial to orient the polygons and deduce the results in the literature from ours.

The same remark applies to the setting in [**9**]. Here a further simplification is possible. Each of the "zweiseitige Randstrecken" ("two-sided boundary edges") corresponds to two overlapping edges with opposite orientations. Even if these are not whiskers they may be removed in a manner exactly analogous to that described in (a) above. This reduces the problem of determining the area, in the examples given, to an application of the corollary to Theorem 1.

It was pointed out to us by Prof. Rolf Schneider that our results could be reformulated in the following way: For each lattice point X in the plane assign an index $j(X, P)$ that equals the sum of the angular lengths of arcs of a small circle centered at X that are contained in P, with the appropriate signs and multiplicities. Then $A(P) = \Sigma j(X, P)$, where the summation is over all lattice points in the plane.

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REFERENCES

1. H. S. M. Coxeter, *Introduction to Geometry*, Wiley, New York 1969.
2. D. DeTemple, Pick's formula: A retrospective, *Mathematics Notes from Washington State University*, Vol. 32, Nos. 3–4 (November 1989).
3. D. DeTemple and J. M. Robertson, The equivalence of Euler's and Pick's Theorems, *Math. Teacher* 67 (1974), 222–226.

4. R. Ding, K. Kolodziejczyk and J. R. Reay, A new Pick-type theorem on the hexagonal lattice, *Discrete Math.* 68 (1988), 171–177.
5. R. Ding and J. R. Reay, The boundary characteristic and Pick's theorem in the Archimedean planar tilings, *J. Combinat. Theory* A44 (1987), 110–119.
6. W. W. Funkenbusch, From Euler's formula to Pick's formula using an edge theorem, *Amer. Math. Monthly* 81 (1974), 647–648.
7. R. W. Gaskell, M. S. Klamkin and P. Watson, Triangulations and Pick's theorem, *Math. Mag.* 49 (1976), 35–37; comments by J. Staib and R. A. Gibbs, *ibid.* pp. 104–105, 158.
8. B. Grünbaum and G. C. Shephard, Rotation and winding numbers for polygons and curves, *Trans. Amer. Math. Soc.* 322 (1990), 169–187.
9. H. Hadwiger and J. M. Wills, Neuere Studien über Gitterpolygone, *J. reine angew. Math.* 280 (1975), 61–69.
10. G. Haig, A 'natural' approach to Pick's theorem, *Math. Gazette* 64 (1980), 173–177.
11. R. Honsberger, *Ingenuity in Mathematics*, New Mathematical Library, volume 23, Math. Association of America, Washington, D.C. 1970, pp. 27–31.
12. A. C. F. Liu, Lattice points and Pick's theorem, *Math. Mag.* 52 (1979), 232–235.
13. I. G. MacDonald, The volume of a lattice polyhedron, *Proc. Cambridge Philos. Soc.* 59 (1963), 719–726.
14. I. Niven and H. S. Zuckerman, Lattice points and polygonal area, *Amer. Math. Monthly* 74 (1967), pp. 1195–1200. Reprinted in *Selected Papers in Geometry*, A. K. Stehney et al., eds., Math. Assoc. of America, 1979, pp. 149–153.
15. G. Pick, Geometrisches zur Zahlenlehre, *Sitzungber. Lotos* (Prague) 19 (1899), 311–319.
16. J. E. Reeve, On the volume of lattice polyhedra, *Proc. London Math. Soc.* (3) 7 (1957), 378–395.
17. J. E. Reeve, A further note on the volume of lattice polyhedra, *J. London Math. Soc.* 34 (1959), 57–62.
18. H. Steinhaus, *Mathematical Snapshots*. Oxford Univ. Press, New York 1969.
19. D. E. Varberg, Pick's theorem revisited, *Amer. Math. Monthly* 92 (1985), 584–587.
20. J. M. Wills, Kugellagerungen und Konvexgeometrie, *Jahresber. Deutsch. Math.-Verein*, 92 (1990), 21–46.

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John Wiley & Sons recently published a volume entitled "Theory and applications of finite groups," consisting of three parts. Part I, written by Professor G. A. Miller, consists of 192 pages and is entitled "Substitution and abstract groups"; Part II, written by Professor H. F. Blichfeldt, consists of 86 pages and is entitled "Finite groups of linear homogeneous transformations"; Part III, written by Professor L. E. Dickson, consists of 103 pages and is entitled "Applications of finite groups." The work is dedicated to Camille Jordan, and is the first treatise on group theory written by American mathematicians.

—*American Mathematical Monthly* 23, (1916) p. 317.