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INFINITE UNIFORM POLYHEDRA

*by Branko Grünbaum**)
University of Washington, GN-50
Seattle, WA-98195

Infinite polyhedra are an interesting but poorly explored family of surfaces. By *infinite polyhedron* P we mean an infinite collection of planar convex polygons, such that each side of each polygon is also a side of precisely one other polygon. The polygons are called *faces* of P, the common sides are the *edges* of P, their endpoints –– vertices of P. Without additional mention, we will assume that the following conditions are satisfied by all polyhedra considered:

(i) The intersection of any two faces of P is either empty, or a vertex of P, or an edge of each of P.

(ii) All faces that contain a given vertex form a simple circuit.

(iii) From any face one can get to any other face by moving from face to face across common edges.

(iv) Each bounded region of the space meets only a finite number of faces.

Clearly, except for the assumption that the number of faces is infinite, these conditions are the same as one imposes on polyhedra in the usual sense. (Note that by "polyhedron" we mean "polyhedral surface", and not "solid".).

It is easy to verify that any planar tiling is an "infinite polyhedron", but there are many others that are of interest. In fact, we shall be mainly interested in infinite analogues of the "uniform" or "Archimedean" polyhedra. However, the following definitions apply equally to finite and to infinite polyhedra.

A polyhedron P is *vertex-transitive* (resp. *edge-transitive*, *face-transitive*, *flag-transitive*) if the group of isometric symmetries of P acts transitively on the vertices (resp. edges, faces, flags) of P. (A *flag* of a polyhedron P is a triplet consisting of a face of P, and edge of that face, and a vertex which is an endpoint of the

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edge.) A polyhedron P is *uniform* if it is vertex transitive and all the faces are regular polygons; P is *regular* if it is flag-transitive. It is *Archimedean* if all the faces are regular polygons and for any two vertices the figure formed by the faces containing the vertex are congruent. Thus uniform polyhedra are Archimedean, but Archimedean polyhedra are not necessarily uniform. The "pseudorhombicuboctahedron" discovered by J. C. P. Miller (see Ball and Coxeter [4, page 138]) and by Ashkinuze [3] is the only such example among finite Archimedean polyhedra; see Figure 1.

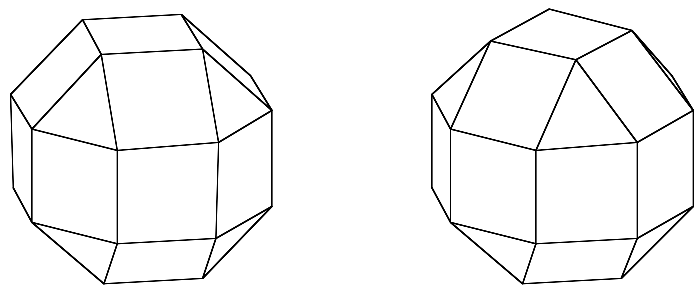


Figure 1. The uniform rhombicuboctahedron and the Archimedean (but not uniform) pseudorhombicuboctahedron.

An infinite polyhedron that is Archimedean but not uniform can be built from the (infinitely extended) "modules" shown in Figure 2; it is shown on page 60 of Wachman et al. [7]. The vertical modules are formed by squares, the horizontal ones by squares and equilateral triangles; both kinds have square holes in front and back. Infinitely many modules of the two kinds should be so interwoven (in alternate layers) that the square holes in each each are matched by square holes in the other kind. It is easily seen that since the horizontal modules admit no reflective symmetries with vertical mirrors, uncountably many infinite Archimedean polyhedra can be constructed in this way.

From now on we shall discuss only polyhedra that are infinitely extended in three independent directions; in fact, we shall assume that they are periodic in three such directions.

Coxeter [5] proved that there are precisely three regular infinite polyhedra of this type. Denoting any Archimedean polyhedron by the cyclic sequence of the numbers of sides of the polygons that meet at each vertex, Coxeter's regular infinite polyhedra are $(4{\cdot}4{\cdot}4{\cdot}4{\cdot}4{\cdot}4) = (4^6)$ with six squares at each vertex, $(6{\cdot}6{\cdot}6{\cdot}6) = (6^4)$, and $(6{\cdot}6{\cdot}6{\cdot}6{\cdot}6{\cdot}6) = (6^6)$. (We use exponents in an obvious way in order to shorten the symbols.) Making models of (finite parts of) these polyhedra is a very interesting project. In the hope that the reader will be challenged to construct such models, we provide here no diagrams.

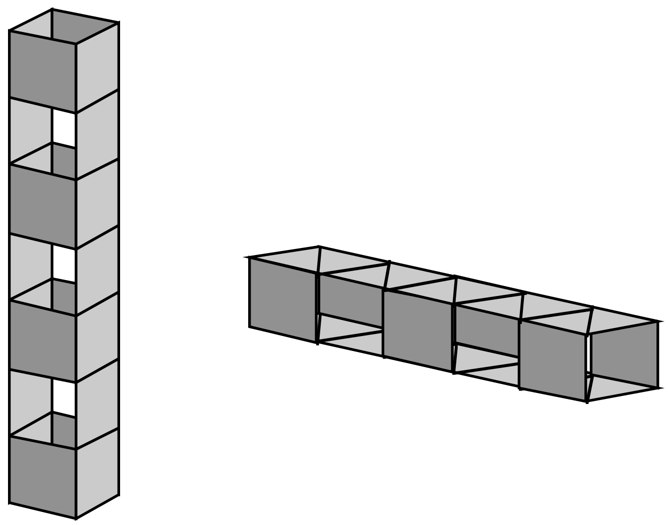


Figure 2. Two "modules" which can be used to construct infinitely many distinct infinite Archimedean polyhedra that are not uniform.

Many non-regular uniform infinite polyhedra are known; ApSimon [2] presents three examples, and Gott [6] shows several. Wachman et al. [7] describe and illustrate a very large number of such polyhedra; however, they fail to notice that many of the types lead to infinitely many pairwise incongruent polyhedra, as explained below. Theoretical chemists have long been interested in infinite uniform polyhedra (Wells [8]), and recently it seems their investigations acquired practical importance (Alper [1]).

An infinite uniform polyhedron (3^{12}), at each vertex of which meet 12 equilateral triangles, has been described by many authors. It is even face-transitive. Other infinite uniform polyhedra with only equilateral triangles as faces are also known (see, in particular, Wachman et al. [7]): one (3^7), three distinct (3^8), five distinct (3^9), four distinct (3^{10}).

Conjecture 1. There are no uniform polyhedra with equilateral triangles as faces other than the 14 mentioned in the preceding paragraph.

The most interesting part of the above is the question whether any uniform polyhedron can have more than 12 triangles meeting at each vertex. We venture even the stronger

Conjecture 2. There are no uniform polyhedra such that each vertex belongs to more than 12 faces.

There are at least eleven different uniform polyhedra with eight faces meeting at each vertex; three of these have only triangles as faces. The remaining eight contain triangles as well as faces with more sides.

Conjecture 3. If each vertex of a uniform polyhedron is incident with more than eight faces then all the faces are triangles.

Wachman et al. [7] described a uniform polyhedron ($6^2.8^2$).

Conjecture 4. No uniform polyhedron exists in which all faces have more than six sides.

Although there are many types of infinite Archimedean polyhedra that are not uniform, it seems likely that all these conjectures hold for Archimedean polyhedra as well.

There are many uniform polyhedra in which all faces are squares, meeting by fives at each vertex, or else meeting by sixes. One uniform polyhedron (4^5) which seems not to appear in the published literature, can be desribed as follows. Consider infinite solid prisms with unit square cross-sections, and with axes parallel to the x-axis, spaced one unit apart, in horizontal layers at heights 0, ±4, ±8, ±12, ..., while analogous sets of prisms parallel to the y-axis form layers at heights ±2, ±6, ±10, ... At all odd heights, these layers of prisms are connected by solid unit cubes. The union of the prisms and cubes (which occupies only 3/8 of the space) has as its boundary the unifom polyhedron (4^5) we wanted to describe. An exploded view of the three kinds of modules used in the construction is shown in Figure 3; the dotted line indicates how the modules fit together.

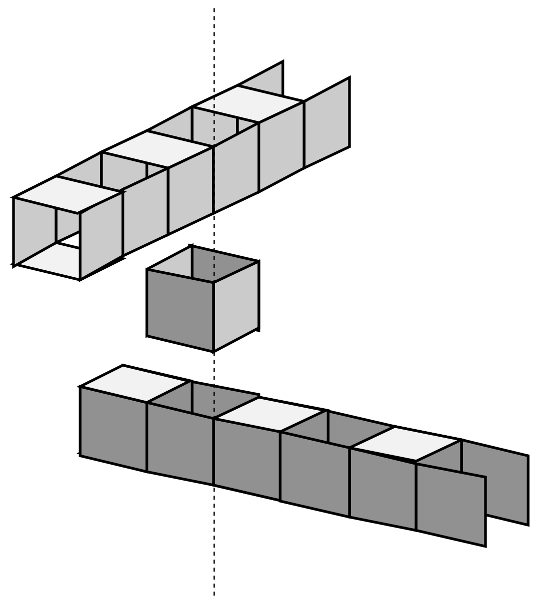
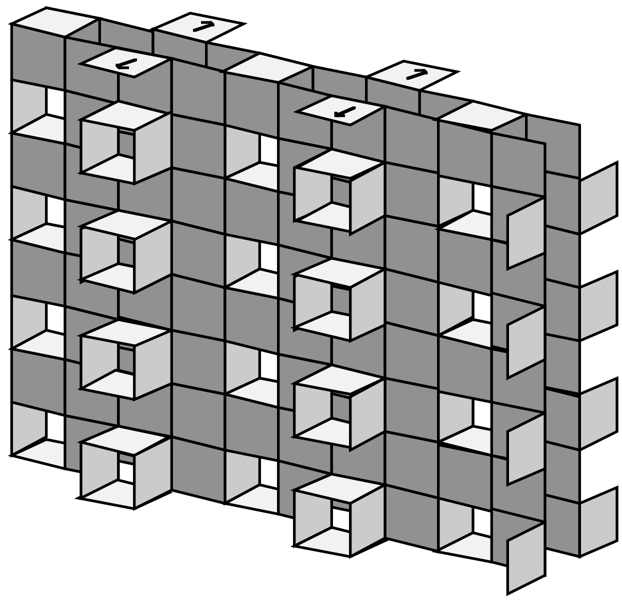


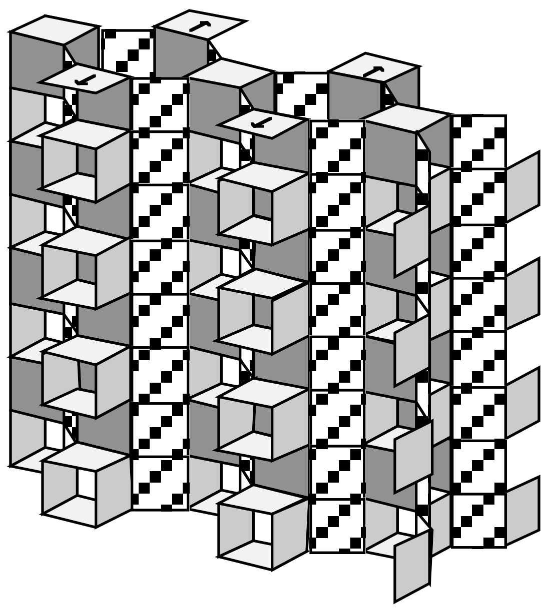
Figure 3. The modules that can be put together as explained in the text, to obtain a uniform polyhedron (4^5).

Two representatives of a family of uniform polyhedra (4^5) that depends on a real-valued parameter are shown in Figure 4. Cardboard models of these polyhedra can be deformed into each other by continuous motions. The polyhedra are obtained by stacking copies of the "slabs" shown. The polyhedron in part (a) is shown on page 50 of Wachman et al., while the one in part (b) appears to be new, as is the observation that they are both part of a continuous family. Another movable family of uniform polyhedra (4^5) results from the one in Figure 4 by shifting one half of the "connecting passages" one square up (for example, the columns marked by arrows in Figure 4 may be shifted one square up to obtain representatives of that family). Polyhedra in the second family corresponding to those in Figure 4 are shown in Wachman et al. on pages 16 and 20, but with no indication ot the relation between them. A third family of movable uniform polyhedra consists of polyhedra of type (4^6). One of its members is the regular infinite polyhedron (4^6) mentioned above. As noted by Ball and Coxeter [3, p.153] this polyhedron is not only movable but may be collapsed to a plane.

Conjecture 5. There are no movable families of uniform polyhedra besides the three described above.



(a)



(b)

Figure 4. Two polyhedra in a continuous family of infinite uniform polyhedra (4^5). The various representatives can be obtained by moving the "columns" marked by arrows in their top squares in the direction of the arrows (or in the opposite direction). Another continuous family can be obtained by shifting the same "columns" one square up.

We conclude by mentioning the probably most surprising of all infinite uniform polyhedra, the polyhedron (5^5) discovered by Gott [6] (and missed by all other investigators of infinite uniform polyhedra). We shall leave to the reader the pleasure of constructing a model of this polyhedron; as a helpful hint we mention that it is convenient to start with a generous supply of strips like the one shown in Figure 5.

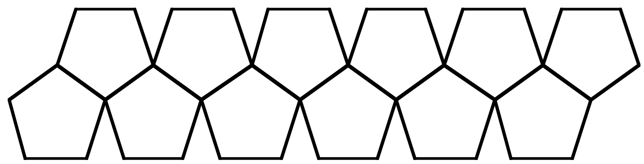


Figure 5. Strips that are convenient to use in the construction of the uniform infinite polyhedron (5^5).

Clearly, all the problems about infinite uniform polyhedra raised above can be subsumed in the question of a complete enumeration of such polyhedra. This seems to be a hard -- but not hopeless -- problem. More generally, it would seem both desirable and interesting to study all the vertex-transitive infinite polyhedra periodic in three independent dimensions, or the analogous edge-transitive, or face-transitive polyhedra. Concerning the latter two, the literature seems to contain no information whatsoever.

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