Geombinatorics 1(1992), No.4, pp. 5 - 12.

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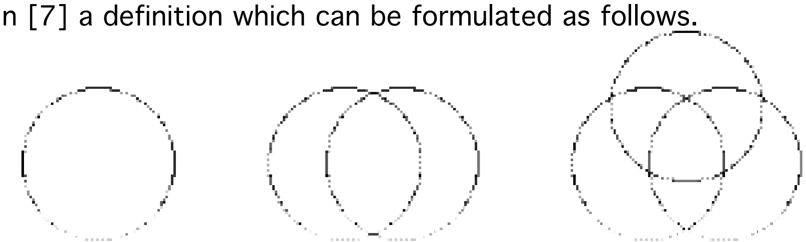
VENN DIAGRAMS

Almost every book on discrete mathematics shows some or all of the *Venn diagrams* of one, two, or three circles, shown in Figure 1. Named after the logician John Venn, these diagrams are meant to help in discussing details of logical possibilities. Modifying work by various predecessors (including Euler), Venn proposed in [6] and popularized in [7] a definition which can be formulated as follows.



(a) (b) (c)

Figure 1.

A family $\mathbf{C} = \{C_1, C_2,...,C_n\}$ of simple (Jordan) curves in the plane is a Venn diagram provided each of the 2^n sets

$$X_1 \leftrightarrow X_2 \leftrightarrow ... \leftrightarrow X_n,$$

where each X_j is one of the two connected components of the complement of C_j (that is, each X_j is either the bounded interior or the unbounded exterior of C_j), is nonempty and connected. The last two attributes are the determining ones. Clearly, the families of circles in Figure 1 are Venn diagrams, while the families of curves in Figure 2 are not. It is to be regretted that no book on discrete mathematics pursues the topic much further –– it leads to fascinating mathematics of an elementary character. In this note and in a sequel we shall discuss various results and open questions concerning Venn diagrams.

Already Venn observed that one cannot form Venn diagrams with n circles if $n \geq 4$. This is easily seen by experimentation, and can be confirmed by an elegant argument using Euler's theorem for planar graphs and the fact that two (distinct) circles cannot have more than two points in common. (The reader is invited to find that argument, and to verify the various assertions made below.) On the other hand, it is easy to extend the diagram of three circles to a Venn diagram with four or five curves, as

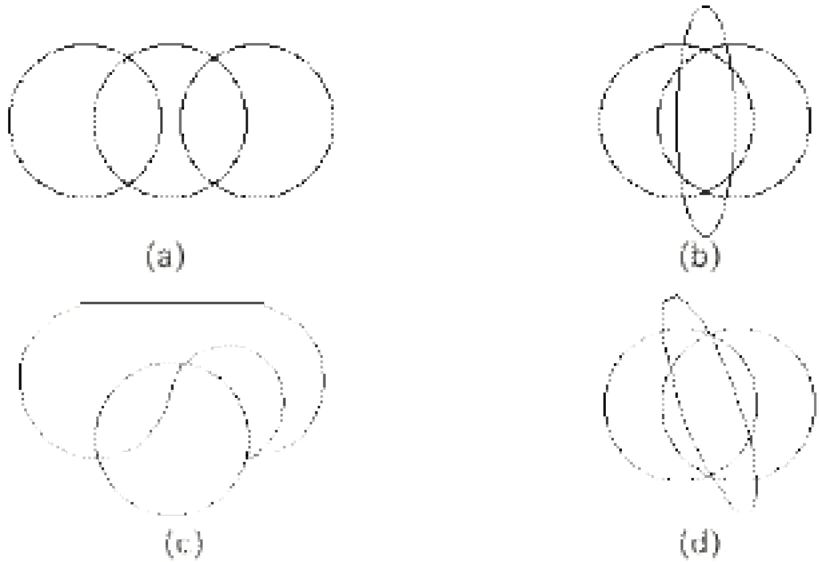


Figure 2.

in Figure 3; as noted by Venn, the continuing possibility of extension to greater numbers is obvious. In this context it is worth repeating a slight generalization of a still open conjecture of Winkler [8]:

Conjecture 1. Every Venn diagram of n curves can be extended to a Venn diagram of $n+1$ curves by the addition of a suitable curve.

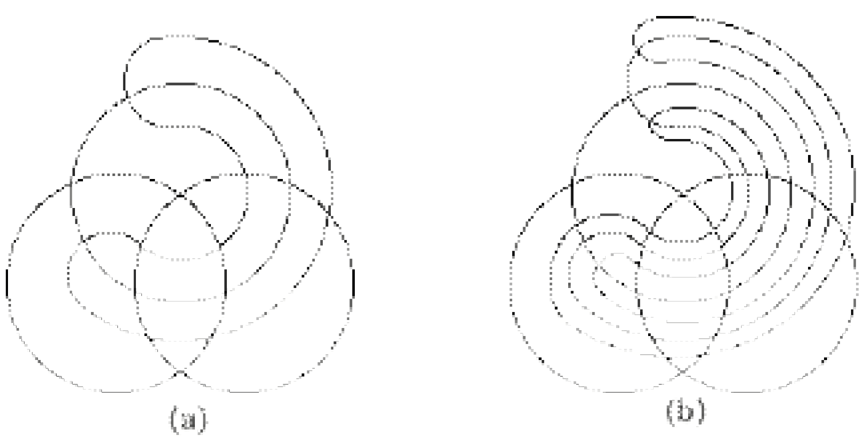


Figure 3.

In Figure 4 are shown three Venn diagrams of four curves. Calling a Venn diagram *simple* if no point lies on three or more of the curves, we see that the first two diagrams in Figure 4 are simple, but the last one is not; all the diagrams in Figures 1 and 3 are simple as well. Some authors include the requirement of simplicity in the definition of Venn diagrams, but this appear to be a needless narrowing of the geometric context.

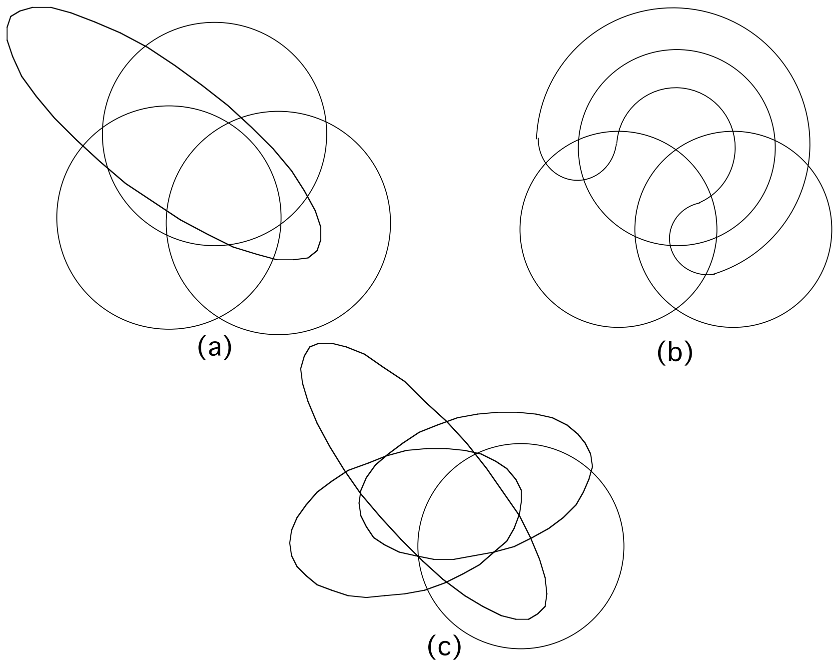


Figure 4.

Two Venn diagrams are called *isomorphic* (or *of the same type*) if by a continuous deformation of the plane one of them may be changed into the other or a mirror image of the other. Thus, for example, the diagram in Figure 4(a) is isomorphic to the one in Figure 3(a), but not to the diagrams in Figures 4(b) or 4(c). Four non-simple diagrams of three curves are shown in Figure 5; they are of different types and, naturally, are not isomorphic to the simple diagram of three circles shown in Figure 1(c).

It is not hard to verify that there is a unique type of Venn diagram with two curves (the type shown in Figure 1(b)), only one type of simple Venn diagrams with three curves (Figure 1(c)), and just two types of simple diagrams with four curves (the types illustrated in Figures 4(a) and 4(b)). In contrast, it is not known how many types (simple or not) are there of all diagrams with three curves (or any larger number), nor is it known how many types of simple diagrams with five (or more) curves exist.

One of the difficulties of such enumerations of types is caused by the fact that not all Venn diagrams with $n+1$ curves are obtainable from Venn diagrams with n curves by the addition of a suitable curve. Specific examples of such "hard-to-get" diagrams will be mentioned below; the interested reader can find additional information on the construction of Venn diagrams from smaller ones in [2] and the references given there, as well as in [3], [4] and [8].

A Venn diagram is said to be *exposed* if each of its curves has an arc on the boundary of the unbounded region. The diagrams in Figures 1, 3, 4(a) and 4(c) are exposed, while those in Figures 4(b) and 5(a) are not exposed. A Venn diagram is said to be *convex* if it is isomorphic to a Venn diagram formed by convex curves. (A curve is convex if it is the boundary of a convex set.) It is well known (see [2]) that there exist convex diagrams with arbitrarily many curves. It is not hard to prove that if a Venn diagram is convex then it is exposed. However, the conjecture that every exposed and simple diagram is convex, is disproved by the example in Figure 6. (In this example, the interiors of two of the curves intersect in a set with two connected components, hence no isomorphism to a diagram with convex curves is possible.) Without assuming simplicity, a negative answer follows already from the diagrams in Figures 5(b) and 5(d).

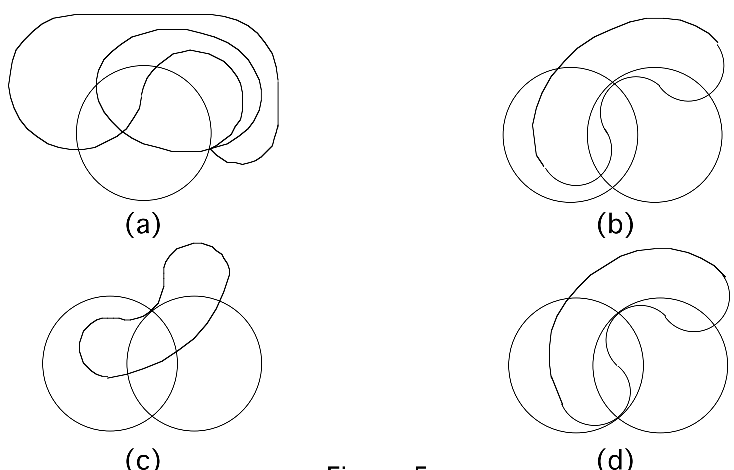


Figure 5.

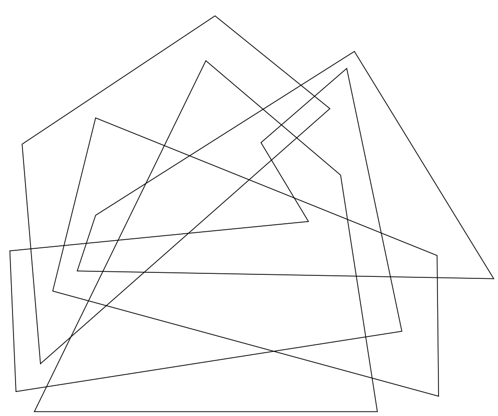


Figure 6.

Conjecture 2. If a simple and exposed Venn diagram has the property that the intersection of the interiors of any subfamily of curves is connected, then it is isomorphic to a convex diagram.

In [6], Venn gave examples of Venn diagrams with four ellipses. However, he mistakenly stated that no five ellipses can form a Venn diagram; indeed, it takes only a little patience to verify that the five congruent ellipses in each part of Figure 7 form a Venn diagram. Venn's erroneous assertion was repeated -- unchecked and unchallenged -- by several authors for almost a century; the first Venn diagram of five ellipses (shown in Figure 7(a)) was published only in 1975 [2]; a non-simple example, which will be shown in the sequel to this note, was found by Schwenk [5]. Using Euler's theorem and the fact that two ellipses can intersect in no more than four points, it follows easily (by an argument similar to the one concerning circles) that there can be no Venn diagrams with six or more ellipses. One possible explanation for Venn's error is that he may have believed that all Venn diagrams can be constructed following a sort of "greedy algorithm" as described above: to get a diagram with n curves first make a diagram with $n - 1$ curves and then add the last one. However, it is easy to verify that none of the Venn diagrams in Figure 7 (nor any other simple Venn diagram of five ellipses) can be obtained by adding a fifth ellipse to a Venn diagram of four ellipses. Probably the same is true without the assumption of simplicity.

The diagrams in Figure 7 are of distinct types. In one sense, they are very special, since each consists of congruent ellipses. (Using ellipses of different sizes it is possible to obtain diagrams in which no region is very small.)

Conjecture 3. Every noncircular ellipse E can be used to obtain Venn diagrams isomorphic to all those in Figure 7, and consisting of ellipses congruent to E.

Conjecture 4. Every simple and convex Venn diagram with five ellipses is isomorphic to one of the diagrams in Figure 7.

Conjecture 5. Every convex Venn diagram with five curves, such that any two of the curves intersect in at most four points, is isomorphic to one with five congruent ellipses.

Any Venn diagram can be transferred from the plane to the sphere by a so-called stereographic projection (see, for example, Coxeter [1], Section 6.9). Briefly

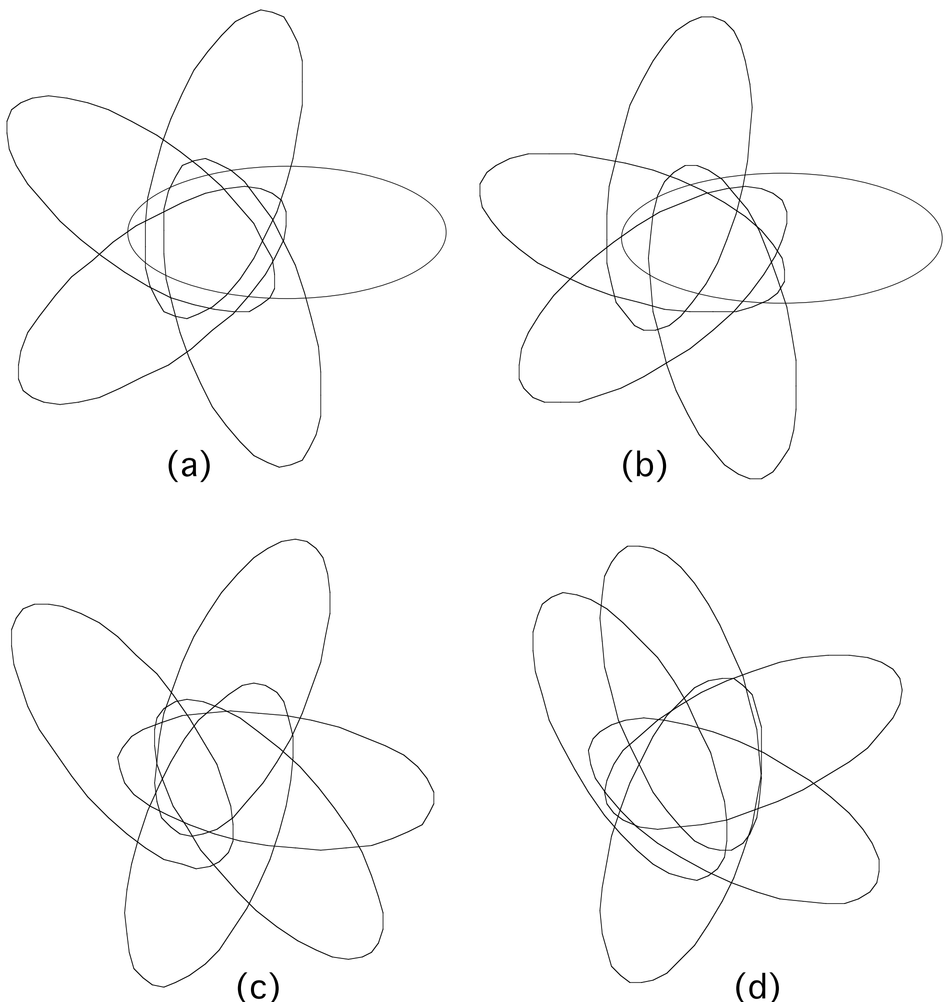


Figure 7.

explained, for a stereographic projection one places the sphere to rest on the plane (touching the plane at the sphere's south pole), and assigns to each point of the plane that point on the sphere at which the ray from the north pole to the point in the plane intersects the sphere. By projecting a Venn diagram to the sphere we obtain a *spherical Venn diagram*, which we shall say is *associated* with the given planar one. In a certain sense, spherical Venn diagrams are more natural than the usual ones, since the two connected components of the complement of a simple closed curve play equivalent roles. This leads to a smaller number of types of spherical Venn diagrams compared with those in the plane. For example, it is easy to check that there is a unique type of simple spherical Venn diagrams with four curves.

Spherical Venn diagrams may be used to define a classification of planar Venn diagrams that is coarser than the one by type. We shall say that two plane diagrams are of the *same class* if the spherical diagrams associated with them are isomorphic. For example, it is not hard to check that the diagrams in Figures 4(a), (b) are of the same class.

Problem 6. Is every simple Venn diagram of the same class as some exposed diagram, or even some convex diagram ?

The diagram in Figure 6 does not provide a counterexample to the stronger version of Problem 6, since it is is of the same class as one of the diagrams in Figure 7. (Which one ?)

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