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NETS OF POLYHEDRA II

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In order to avoid repetition, we use the terminology of the first part (*Geombinatorics*, Vol. 1, No. 2, pp. 5 - 9).

The lead-off question (A), first posed by Shephard [1975], is still open. In the wake of many failed attempts to find a counterexample, we venture:

Conjecture 1. Every convex polyhedron has a net.

If this conjecture should turn out to be wrong, or too hard to decide — here is a weaker version which would affirmatively answer question (D):

Conjecture 1*. Every convex polyhedron is combinatorially equivalent to a polyhedron that has a net.

In contrast, the answer to question (B) (whether every spanning tree of an arbitrary polyhedron leads to a net) is easily seen to be negative. Even among the widely known Archimedean polyhedra (see, for example, Cundy and Rollett [1961], Wenninger [1971], Ball and Coxeter [1974]) there are many examples in which a spanning tree of the polyhedron does not lead to a net; to simplify the language we shall say that such a tree leads to a **would-be net**. In Figure 1 we show (parts of) would-be nets of two Archimedean polyhedra that clearly show overlaps. It is interesting to note that the same phenomenon can occur even with such simple polyhedra as regular prisms. In Figure 2 we show a would-be net for the regular 28-sided prism. The smallest regular prism with a would-be net seems to be the 26-sided one; it is not shown since for it the overlap is so small that the diagram is not very convincing.

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If one considers prisms that are not necessarily regular, much smaller examples can be found which have would-be nets. The reader may find it amusing to show that this happens even with 4-sided prisms.

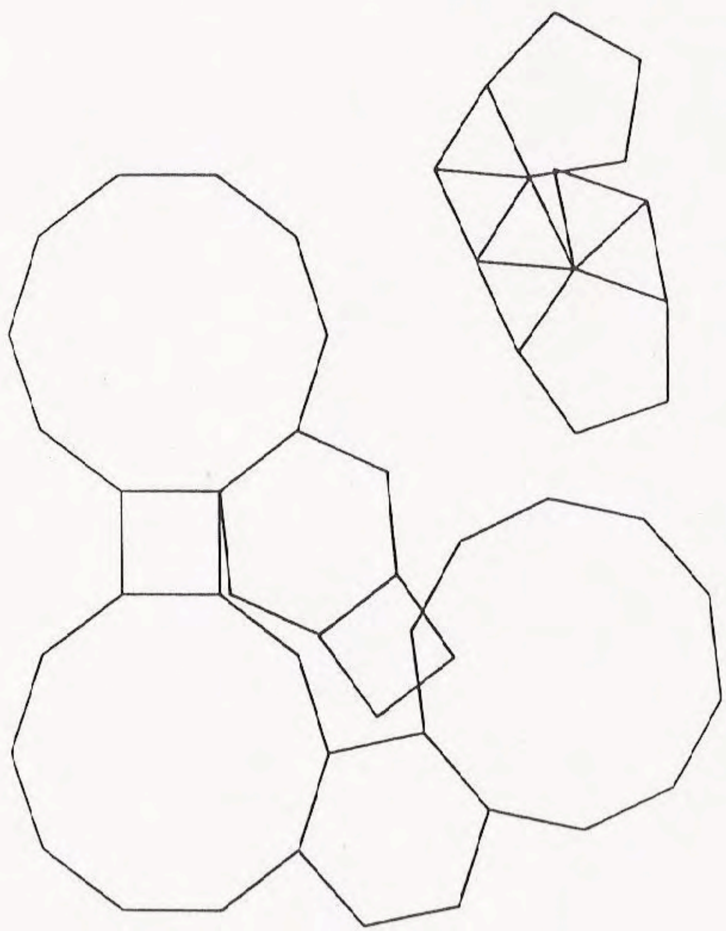

Figure 1. The partial would-be nets of the Archimedean polyhedra (4.6.10) and (3.3.3.3.4)

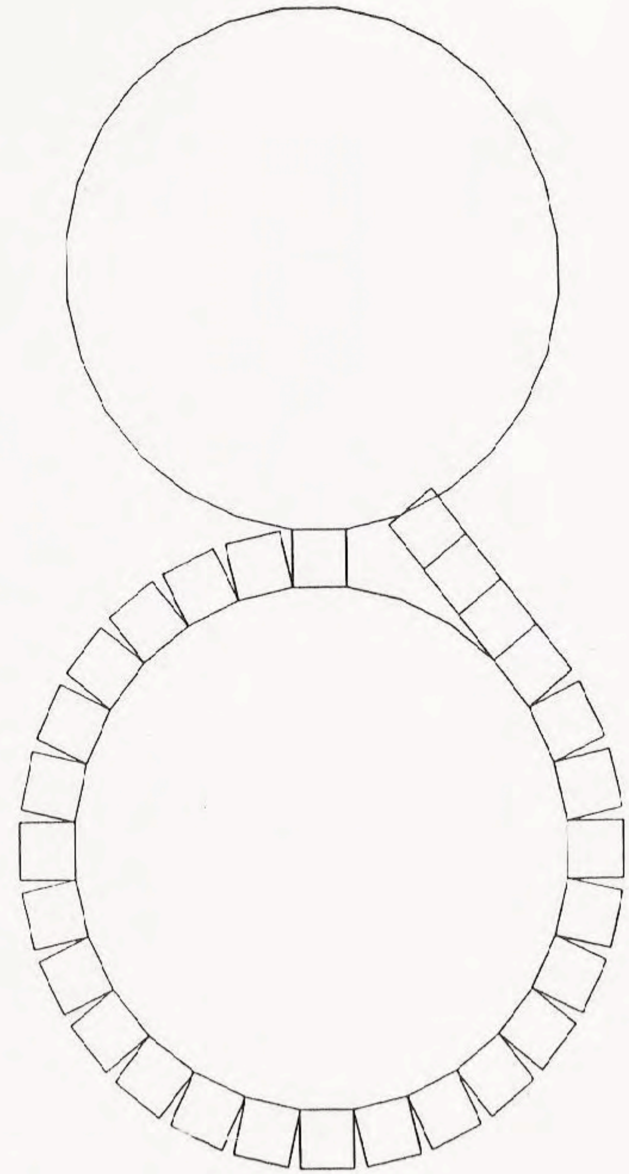

Figure 2. A would-be net of a regular 28-sided prism.

More interesting is the observation that every polyhedron combinatorially equivalent to the 3-sided prism has no would-be nets; the tetrahedron has the same property. The determination of all combinatorial types of polyhedra such that every spanning tree in every polyhedron of that combinatorial type leads to a net seems to be quite hard, and even a conjecture regarding their characterization appears elusive.

The answer to question (E) is probably negative. We believe that every polyhedron combinatorially equivalent to the n-sided prism, where n is sufficiently large (say $n \geq 100$) has some would-be nets. However, there seems to be no easy way to establish this.

On the other hand it is easy to see that question (C) has a negative answer: several well known Archimedean polyhedra (among them the truncated cube and the cuboctahedron, see Figure 3) fail to have single-chain nets. Moreover, all polyhedra combinatorially equivalent to these have the same property. The reason lies not in geometry but in combinatorics: it is not possible to arrange the faces of these polyhedra in a single-chain sequence. Indeed, since in these polyhedra no two triangular faces have a common edge, in any such sequence between two triangular faces there would have to be a non-triangular face; but since there are eight triangular faces and only six other faces, this is clearly impossible.

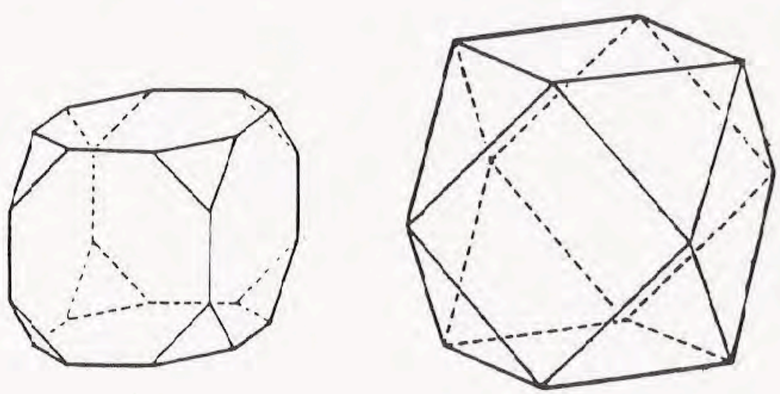

Figure 3. The truncated cube (3.8.8) and the cuboctahedron (3.4.3.4) have no single-chain nets.

We may note that the five Platonic solids (regular polyhedra) not only do admit single-chain nets, but even have the property that every chain of faces of the polyhedron which includes all the faces (but without repeated use of any face, of course) can be flattened to yield a net (it is clear that such chains of faces correspond to Hamiltonian paths in the dual polyhedra). This results from the recent enumeration of all such chains by Reggini [1991]. His enumeration showed that (not counting mirror images as distinct) there are the following numbers of distinct chains of this type:

1	for the tetrahedron
4	for the cube
3	for the octahedron
340	for the dodecahedron
18	for the icosahedron

Moreover, Reggini has sketches of all the corresponding single-chain nets, thus establishing their existence (for the dodecahedron, only 30 of the nets are shown in Reggini [1991]).

We conclude by bringing up several other problems. If P is a convex polyhedron, and if T is a spanning tree of P which determines a net N of P, then in any polyhedron P^* dual to P the net N corresponds to a tree N^* of P^* which determines a net or would-be net T^* of P^*. To simplify the terminology, we shall say that N and T^* are dual to each other. (For explanations of "dual" and "polar" see, for example, Grünbaum [1967], Section 3.4.)

Problem 1. If P has a net N, and if P^* is a polar of P, is T^* necessarily a net or can it happen to be a would-be net?

Barnette [1966] proved that every convex polyhedron has a spanning tree of maximal valence 3. (A tree is said to have maximal valence k if each vertex has at most k neighbors.) This leads to a question analogous to (C) from the first part:

Problem 2. Does every polyhedron P have a net in which each face has at most three neighbors?

Finally, an old question (Grünbaum [1970], p. 1148; Rosenfeld [1990]), an affirmative answer to which would strengthen Barnette's theorem:

Conjecture 2. Does every polyhedron P have a spanning tree T of maximal valence 3 such that the tree N^* of the dual polyhedron also has maximal valence 3?

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