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NETS OF POLYHEDRA

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The idea of the net of a polyhedron is familiar to anyone who has tried to make a model of a cube or a more complicated polyhedron out of paper or thin cardboard. Less well known is the history of that idea. In 1525, the famous painter Albrecht Dürer published an important text on geometry for practical use (Dürer [1525]); it contained many innovations compared to the traditional presentations of Euclidean geometry that were available at that time, and was highly influential and widely circulated. One of Dürer's innovations was the concept of a net for a polyhedron. Figure 1 shows one of the examples given by

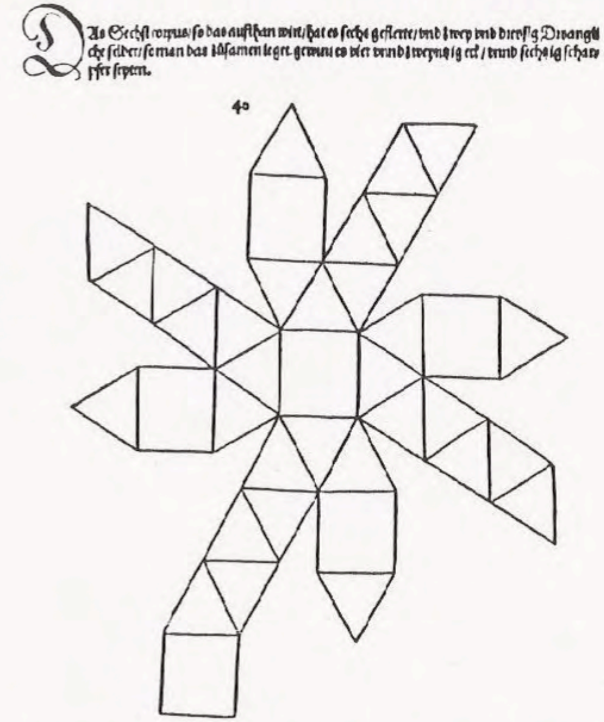

Figure 1. The net of a snub cube, from "The Painter's Manual" by Albrecht Dürer.

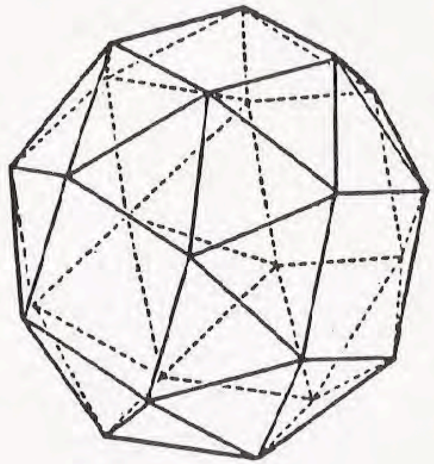

Figure 2. A snub cube.

Dürer, a net for the Archimedean "snub cube" shown in Figure 2. To describe nets, we first note that we are concerned only with convex polyhedra. A cardboard model of a polyhedron represents the boundary of the polyhedron; the polyhedron (or "solid") itself consists of that boundary together with the part of the 3-dimensional space which is enclosed by the boundary. The net of a polyhedron consists of a connected collection of polygons, congruent to the faces of the polyhedron, arranged in one plane (the sheet of cardboard) without overlaps, in such a way that some of the polygons share edges in pairs, and that, if the net is cut out and folded along the shared edges, and the remaining edges joined in pairs in a suitable manner, a model of the polyhedron is obtained. This is illustrated in Figure 3, in which several different nets of the cube are shown, and the paired edges of the resulting cube are indicated in each case by heavy lines. The edges of the polyhedron that are obtained by this pairing of edges of a net form a tree (graph with no circuits) in the graph of the polyhedron (that is, the graph formed by the vertices and edges of the polyhedron). Conversely, starting from a cardboard model of the cube, for example, one can obtain a net by slitting the model along edges which form a tree that spans (that is, includes all vertices of) the graph of the polyhedron.

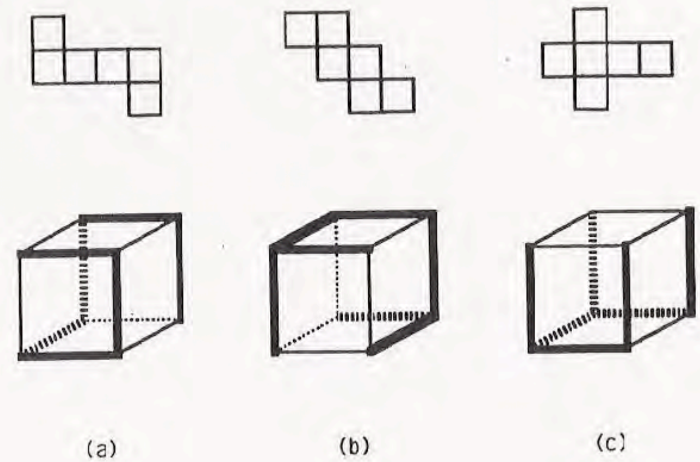

Figure 3. Several nets for the cube, and diagrams of the cubes obtained from these nets.

Countless books show nets that can be used to make models of the more commonly encountered polyhedra, such as the "regular polyhedra" (or "Platonic solids"), the "uniform" (or "Archimedean") solids, etc. However, the following simple question seems to have been explicitly formulated first by Shephard [1975]:

(A) Does every convex polyhedron have a net? In other words, is it always possible to choose such a spanning tree in the edge graph of a convex polyhedron that it leads to a net in the above manner?

Several related questions can be asked; for example:

(B) Does every spanning tree of an arbitrary polyhedron lead to a net of the polyhedron?

(C) Does every polyhedron have a single-chain net?

Here a net is said to be "single-chain" if each of the polygons in the net shares an edge with just one or two other polygons. The nets (a) and (b) of the cube (Figure 3) are single-chain, the more familiar net (c) is not, and neither is Dürer's net in Figure 1.

Convex polyhedra P and Q are called "combinatorially equivalent" (or "isomorphic") if their vertices can be put in a one-to-one correspondence such that a set of vertices of P belongs to a face of P if and only if the corresponding vertices of Q belong to a face of Q.

(D) Is it true that given any polyhedron P, there exists a polyhedron Q combinatorially equivalent to P, which has a net?

(E) Is it true that given any polyhedron P, there exists a polyhedron Q combinatorially equivalent to P, in which every spanning tree leads to a net?

The readers are urged to try their hand at these questions in several ways.

First, before even trying to answer any of the questions, determine the logical dependence among them. In other words, if question (A) were, for example, answered affirmatively, would that imply anything regarding questions (B), (C), ...? What if (A) were decided in the negative? Can you formulate some additional questions of the same character, that are reasonable to ask?

Second, get down to specifics. Take a simple solid (such as the cube, or some other Platonic solid) and try to answer the questions for it. (For example, find all the distinct nets for the cube. This naturally leads to the question when are two nets distinct — and you may enjoy giving several reasonable definitions and seeing how the answer varies from one to the other.) Then be more enterprising, and try some more complicated solids (such as the snub cube; can you find a single-chain net? If you think you found one, make a model and see whether it really works! You may even wish to check out Dürer's net experimentally.

Finally, try answering all questions in general.

Next time I will tell you what others have found out about these problems — in particular, which have been solved and which are still open.

References:

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[Shephard, 1975] G. C. Shephard, Convex polytopes with convex nets. *Math. Proc. Cambridge Philos. Soc.*, Vol. 78 (1975), pp. 389-403.