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Jonathan Ashley*), Branko Grünbaum**), G. C. Shephard and Walter Stromquist:

Selfduality groups and ranks of selfdualities

Abstract

In many contexts, selfdual objects (which are both isomorphic and dual to themselves) are of special interest. The concept of **rank** of a selfdual object has recently been introduced, and the question was raised whether there exist selfdual objects of rank greater than 2. Extending results obtained by other workers we show that the answer is affirmative for convex polyhedra, for planar tilings, and for configurations of points and lines in the projective plane. However, the detailed answers are different in the three cases, and also depend on various natural conditions that may be imposed.

1. Introduction. The concept of **duality** occurs in almost every branch of mathematics. While **isomorphism** preserves certain relations between the mathematical entities under consideration, duality reverses them. In this paper we shall be concerned with duality in three particular contexts:

(i) tilings of the Euclidean plane;

(ii) polyhedra in the Euclidean space of three dimensions; and

(iii) configurations of points and lines in the projective plane.

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Other areas in which duality occurs abound -- for example, vector spaces and topological graph theory -- but these will not be discussed here.

In every situation it is interesting to look for those special objects in which isomorphism and duality occur simultaneously. These are said to be **selfdual**. Again it is possible to find many familiar examples with this property: the regular tiling of the Euclidean plane by squares; three-dimensional convex pyramids; the configurations of Pappus and Desargues in projective geometry. Also, Euclidean spaces and, more generally, all complete normed inner-product spaces are selfdual.

In spite of the fact that selfduality is so familiar and has been investigated for well over a century, one aspect of it escaped attention until very recently. In a short note published in 1988 (Grünbaum & Shephard [1988]) the **rank** of a selfduality was defined, and some questions about its possible values were raised. (The rank, which will be formally defined in the next section, is simply the order of the selfduality regarded as a mapping.) This note spurred interest in the area and led to a number of new results. The only publication so far is Jendrol [1989], which described a polyhedron with 14 vertices and rank 4. This example is shown in Figure 25, and its significance will be explained below. Other examples and related results have been circulated informally, or are in the process of being published (see Archdeacon & Richter [1989], McCanna [1989], McKee [1989], Servatius *et al.* [1989]). It is the aim of this note to present some new results and to point out some of the intriguing problems that are still open.

The paper is organized as follows. In Section 2 we introduce the necessary terminology in a general context, illustrated by reference to a familiar polyhedron. In particular, we introduce the concept of the selfduality group of a given geometric structure; this is the group whose elements are the automorphisms and selfdualities admitted by the structure being investigated. Section 3 is devoted to selfdual tilings, and in Section 4 we are concerned with selfdual polyhedra and show how selfdual tilings may be used to construct them. In each case the possible symmetry groups will be investigated. In Section 5 we discuss selfdual configurations in the projective plane, while the final section is devoted to remarks and open problems. Throughout we find it interesting to consider not only the full selfduality groups, but also the subgroups that arise by limiting attention to isomorphisms and selfdualities that are induced by geometrically interesting mappings.

2. Terminology. Although our interest is in geometric applications we begin by presenting the fundamental definitions in the more abstract context of partially ordered sets. If F and F' are partially ordered sets, then an **isomorphism** $\alpha : F \to F'$ is a bijection which preserves the ordering; that is, $x < y$ (in F) if and only if $\alpha(x) < \alpha(y)$ (in F'). A **duality** $\delta : F \to F'$ is a bijection which reverses the ordering; that is, $x < y$ if and only if $\delta(x) > \delta(y)$. A duality or isomorphism from F to F is called a **selfduality** or **automorphism** of F, respectively. If δ is a selfduality of F then $\delta^2 = \delta \circ \delta$ is an automorphism.

The **rank** of a selfduality δ, denoted by $r(\delta)$, is its order or period -- in other words, $r(\delta)$ is the least positive integer r for which δ^r is the identity; if there is no such integer then we define $r(\delta) = \infty$. We say that F is **selfdual** if F admits at least one selfduality, and in that case we define the **rank** of F, denoted by $r(F)$, as the smallest r such that F has a selfduality of rank r. If $r(F)$ is finite then it must be a power of 2; for if F admits a selfduality δ of rank $2^k n$ for some odd $n > 1$, then δ^n is a selfduality of smaller rank 2^k.

The automorphisms of F form a group called the **automorphism group**, A(F). The selfdualities and automorphisms of F together form a group which we call the **selfduality group** of F and denote by D(F). If F is not selfdual then D(F) = A(F), but if F is selfdual then A(F) is a subgroup of index 2 in D(F). For selfdual F, the rank r(F) equals the minimal order of the elements of D(F) that are not in A(F). It is easily verified that D(F) is isomorphic to a well-determined subgroup of $A(F \oplus F^*)$, where $F \oplus F^*$ is the disjoint unordered union of F and a partially ordered set F^*, which is isomorphic to the set obtained from F by reversing the order. We may think of F and of F^* as of two different "colors" -- then elements of A(F) preserve the colors, whereas the elements of D(F) not in A(F) reverse them. In this way D(F) can be regarded as a "2-color group". In the case of the Euclidean plane, the discrete 2-color groups are well known (see Grünbaum & Shephard [1987], Chapter 8), and it would be interesting to determine which of them can actually occur as representing selfduality groups.

In a geometric context, let us first consider convex polyhedra in ordinary Euclidean space $\mathbb{E}^3$. If P is such a polyhedron, then its proper faces (of dimension 0, 1, or 2) form a set which is partially ordered by inclusion; we call it the **face lattice** of P, and denote it by F_P. All of the definitions given above for partially ordered sets can be applied to F_P, and we may, without possibility of confusion, use the same terms for the polyhedra themselves. For example, an isomorphism from F_P to F_Q can also be called an isomorphism (or "combinatorial isomorphism", or "combinatorial equivalence") from P to Q . A duality δ of P maps 0-dimensional faces (that is, vertices) to 2-dimensional faces, and vice versa. (The action of δ on the 1-dimensional faces (edges) is determined by its action on the 0- and 2-dimensional faces, so we do not need to specify it separately.) It will be convenient and should cause no confusion if the automorphism and selfduality groups $A(F_P)$ and $D(F_P)$ of F_P are denoted $A(P)$ and $D(P)$, respectively.

As an illustration of these concepts, consider the six-sided pyramid P with apex A , base B , base-vertices $F_1, F_3, ..., F_{11}$, and mantle-faces $F_2, F_4, ..., F_{12}$, shown in Figure 1. A selfduality δ of P is defined by $\delta(A) = B$, $\delta(B) = A$, $\delta(F_j) = F_{j+1}$ (for j = 1,..., 12). It is clear that $r(\delta) = 12$ and that δ^3 is also a selfduality of P, with $r(\delta^3) = 4$. We note that P has other selfdualities; for example, the selfduality ε defined by $\varepsilon(A) = B$, $\varepsilon(B) = A$, $\varepsilon(F_i) = F_j$ if and only if $i + j = 13$ has rank 2, hence $r(P) = 2$. Some additional checking shows that δ and ε generate the selfduality group D(P) of P, which is (isomorphic to) the dihedral group $\mathbf{D_{12}}$ of order 24. Also, A(P) is (isomorphic to) the dihedral group $\mathbf{D_6}$ of order 12. More generally, if P_n denotes the n-sided pyramid then $A(P_n)$ is (isomorphic to) the dihedral group $\mathbf{D_n}$ and $D(P_n)$ to $\mathbf{D_{2n}}$. (In Section 4 we shall give a more instructive description of $D(P_n)$.) It can be verified easily that every pyramid has rank 2; in fact, **all** selfdual polyhedra described in the literature prior to 1988 are of rank 2.

If a polyhedron P has a selfduality of rank 2, then a simple way of indicating this selfduality is by **balanced labelling**, that is by assigning the same symbol to any vertex and to the face corresponding to it by the selfduality. (Similar considerations apply to other partially ordered sets.) For the selfduality ε of the six-sided pyramid P discussed above a balanced labelling is indicated in Figure 2. It is remarkable -- from mathematical as well as from psychological and sociological points of view -- that due to a confusion of concepts, many of the authors who defined selfduality in the same way as it is defined here, nevertheless treated it as if selfduality were equivalent to the existence a balanced labelling (see Grünbaum & Shephard [1988] for references to instances of this phenomenon); hence they could not even consider the possibility that selfdualities of rank higher than 2 might exist. It is true that some writers did not make this logical error; instead, they **defined** selfduality as meaning the existence of a balanced labelling.

Besides being inappropriate terminology, such an approach only avoids the difficulties and does not lead to any insights into the nature of the problem.

In the sequel, we shall adopt the very convenient balanced labelling whenever it is appropriate, that is, for indicating a selfduality of rank 2. For selfdualities δ of rank n it is convenient to use a letter with subscripts for elements which arise as images under δ and its powers, thus

$$\rightarrow A_{n-1} \rightarrow A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow \ldots$$

where the odd subscripts relate to points, and the even subscripts to other elements (tiles, faces of a polyhedron, lines in the projective plane, and so on). In this notation, if m is any odd integer then

$$\delta^m: A_j \rightarrow A_{j+m} \pmod n$$

is also a selfduality, but -- as can be seen from the example of the hexagonal pyramid given above -- in general not all selfdualities can be expressed in this way.

To avoid confusion we should stress that when treating polyhedra, tilings and other objects of our consideration as combinatorial entities, then the various groups defined above will be regarded as abstract groups. In constrast, when considering the geometry of these objects, the groups will be groups of isometries or other geometric transformations. This will be reflected in the terminology as well as in the notation.

3. Selfdual tilings. Throughout, and without any further explicit statement of the fact, we shall limit attention to tilings $\mathcal{T}$ of the plane that are well-behaved in the sense that they are **normal**. This means that each tile is a closed topological disk, the diameters of the tiles are uniformly bounded from above, their inradii are uniformly bounded from below, and the intersection of any two tiles is either empty, or a single point, or a single arc (see Grünbaum & Shephard [1987], Chapters 3 and 4, for a discussion of these requirements and their consequences; throughout, we shall use the terminology of this book).

Since the vertices, edges and tiles of a tiling $\mathcal{T}$ form a set $F_{\mathcal{T}}$ partially ordered by inclusion, the definitions of automorphisms, selfduality, and the other concepts introduced above apply to $F_{\mathcal{T}}$. However, the geometric context suggests that besides the groups $A(\mathcal{T})$ and $D(\mathcal{T})$, various subgroups are of interest. This leads to other meanings of "selfduality", for which we have to develop an appropriate terminology. To begin with, if $F_{\mathcal{T}}$ is selfdual as defined above, then we shall say that $\mathcal{T}$ is **combinatori-**

ally selfdual. Using well-known techniques of map-extension (see Sections 4.1 and 4.2 of Grünbaum & Shephard [1987]), it can be shown that $\mathcal{T}$ is combinatorially selfdual if and only if there exists a tiling $\mathcal{T}^*$ of the plane such that $\mathcal{T}^*$ is a homeomorphic image of $\mathcal{T}$, and each tile of $\mathcal{T}$ or $\mathcal{T}^*$ contains precisely one vertex of the other tiling in its interior, and that each edge of each of the tilings crosses, at a single point, precisely one edge of the other tiling. Two tilings $\mathcal{T}$ and $\mathcal{T}^*$ related in this way are said to be **dually situated** or **in dual position**, and we may say that $\mathcal{T}$ and $\mathcal{T}^*$ are **topologically dual**. (The tilings in Figures 3 and 4 can serve as illustrations.) However, as the concepts of selfduality in the discrete (combinatorial) sense and in the continuous (topological) sense coincide, we do not need new terminology for this situation. On the other hand, given a selfdual tiling $\mathcal{T}$, it is not always possible to find an isometric image $\mathcal{T}^*$ of $\mathcal{T}$ such that $\mathcal{T}$ and $\mathcal{T}^*$ are in dual position (see, for example, Figure 5). If this can be done (as it can, for example, in the case of the regular tiling by squares, or the tilings in Figures 3 and 4) then we say that $\mathcal{T}$ and $\mathcal{T}^*$ are **metrically selfdual**.

Since tilings have infinite numbers of faces and vertices, the rank of a selfduality of a tiling is not necessarily finite; in fact, the rank $r(\mathcal{T})$ of a tiling can be infinite. As an example, consider the periodic tiling $\mathcal{T}$ shown in Figure 3a, obtained by a systematic modification of the square tiling. It is easy to verify that $\mathcal{T}$ admits a metric selfduality (illustrated by the tilings $\mathcal{T}$ and $\mathcal{T}^*$ in dual position shown in Figure 3b); moreover, $\mathcal{T}$ has the property that if δ is any combinatorial selfduality of $\mathcal{T}$ then δ can be extended to a metric selfduality of $\mathcal{T}$, such that δ^2 is a non-trivial translational symmetry of $\mathcal{T}$. Hence $r(\mathcal{T}) = \infty$. As demonstrated by its labelling, the tiling $\mathcal{T}$ shown in Figure 4a has a combinatorial selfduality of rank 4; in fact, as shown by Figure 4b this is a metric selfduality of rank 4. Moreover, it is easy to show that every combinatorial selfduality of $\mathcal{T}$ is of rank at least 4, and hence $r(\mathcal{T}) = 4$.

Theorem 1. The rank of any combinatorially selfdual tiling of the plane is either 2, or 4, or ∞. For each of these ranks there exist tilings which are metrically selfdual.

Proof. The tilings shown in Figures 3 and 4, and the square tiling are metrically selfdual of rank ∞, 4 and 2, respectively, and hence establish the second part of the theorem. For the first assertion we assume that the rank $r = r(\delta)$ of a combinatorial selfduality δ of a tiling $\mathcal{T}$ is finite. We shall then show that r must be either 2 or 4.

To see this, let $\mathcal{T}$ and $\mathcal{T}^*$ be in dual position, and $\mathcal{T}$ be the tiling formed by superposition of $\mathcal{T}$ and $\mathcal{T}^*$ (this is illustrated in Figures 3b and 4b). This means that each tile of $\mathcal{T}$ is a quadrilateral (that is, has four adjacent tiles), its edges are the "halfedges" of $\mathcal{T}$ and the "halfedges" of $\mathcal{T}^*$, and its vertices comprise the vertices of $\mathcal{T}$, the vertices of $\mathcal{T}^*$, and the intersection-points of pairs of edges, one from $\mathcal{T}$, the other from $\mathcal{T}^*$. We shall refer to $\mathcal{T}$ as the **dual superposition tiling** of $\mathcal{T}$ (or of $\mathcal{T}^*$).

For each selfduality δ of $\mathcal{T}$ we shall define a homeomorphism ϕ of the plane which maps the tiling $\mathcal{D}$ onto itself (see Figure 6(a)). Let P be any vertex of $\mathcal{T}$. Then $\delta(P)$ is a tile of $\mathcal{T}$ which contains in its interior a unique vertex P^* of $\mathcal{T}^*$, so we define $\phi(P) = P^*$. A vertex Q^* of $\mathcal{T}^*$ is contained in a unique tile of $\mathcal{T}$ which, by δ , maps into a vertex Q of $\mathcal{T}$, so we define $\phi(Q^*) = Q$. If P and Q^* are "opposite" vertices of a tile F_1 of $\mathcal{D}$ then P^* and Q are "opposite" vertices of a tile F_2 of $\mathcal{T}$. Since δ maps the edges of $\mathcal{T}$ incident with P onto the edges of the tile of $\mathcal{T}$ containing P^*, and these correspond to the edges of $\mathcal{T}^*$ incident with P^*, we may define the image under ϕ of an edge PR of F_1 (a "halfedge" of $\mathcal{T}$) to be the corresponding edge P^*R^* of F_2. In a similar way, each of the other edges RQ^*, Q^*S and SP of F_1 are mapped by ϕ onto the edges R^*Q, QS^* and S^*P^* of F_2. By map-extensions (Grünbaum & Shephard [1987], Sections 4.1, 4.2) ϕ can be extended to a homeomorphism of the whole plane onto itself, and by construction $\phi(\mathcal{D}) = \mathcal{D}$. Moreover, it is clear that ϕ^r is the identity if and only if δ^r is the identity -- they are necessarily maps of the same order.

For any tile F of $\mathcal{D}$ consider the union of the images of F under ϕ and its powers. If δ is of finite order r, then so is ϕ and this set may be written $G = \bigcup_{0 \le n < r} \phi^n(F)$. If C is any finite union of tiles of $\mathcal{D}$ which is connected and contains G, then $H = \bigcup_{0 \le n < r} \phi^n(C)$ is also a finite connected set of tiles of $\mathcal{D}$, which is invariant under ϕ. It is possible that H is not simply connected, in which case we augment H by adjoining a finite number of tiles (to fill up the "holes") to form a set J which is connected and simply connected. Clearly J is also invariant under ϕ. By the Brouwer Fixed Point Theorem (see, for example, Alexandroff-Hopf [1935], Satz IIa on page 480, or Eilenberg-Steenrod [1952], Theorem 3.3 on page 301), J contains a point X which is fixed under ϕ. The remaining part of the proof is divided into two cases, depending on whether X belongs to the interior or to the boundary of a tile F of $\mathcal{D}$. Let F have vertices PRP^*S in order, where P and P^* are vertices of $\mathcal{T}$ and $\mathcal{T}^*$ respectively, and each of R, S is a point of intersection of an edge of $\mathcal{T}$ with an edge of $\mathcal{T}^*$.

Case 1. X is an interior point of the tile F of $\mathcal{D}$ (see Figure 6(b)). Since ϕ must map F onto itself, it follows that $\phi(P) = P^*$ and $\phi(P^*) = P$. Also, ϕ either interchanges R and S, or else leaves them invariant. In either case, $\phi^2(R) = R$ and $\phi^2(S) = S$. Therefore ϕ^2, and hence also δ^2, is the identity on $\mathcal{D}$, and hence also on $\mathcal{T}$; thus, in Case 1 we have $r(\mathcal{T}) = r(\delta) = 2$.

Case 2. X lies on the boundary point of the tile F of $\mathcal{D}$. Since ϕ maps PR onto P*R and PS onto P*S, it is clear that X cannot be an interior point of an edge of F; hence let us suppose it coincides with S (see Figure 6(c)) and so is the intersection of the edge PQ of $\mathcal{T}$ with the edge P*Q* of $\mathcal{T}^*$. Since the edges PS, P*S, QS, Q*S of $\mathcal{D}$ (halfedges of $\mathcal{T}$) must be permuted by ϕ and since $\phi(P) = P^*$, there are only two possibilities: either $\phi(P^*) = P$, $\phi(Q) = Q^*$ and $\phi(Q^*) = Q$, or else $\phi(P^*) = Q$, $\phi(Q) = Q^*$ and $\phi(Q^*) = P$. In the first case ϕ^2 is the identity, and therefore so is δ^2, hence $r(\delta) = 2$ and $r(\mathcal{T}) = 2$. In the second case ϕ^4 is the identity, hence so is δ^4, and $r(\delta) = r(\mathcal{T}) = 4$.

Since these cases cover all possibilities we deduce that $r(\mathcal{T})$ is 2 or 4, and so Theorem 1 is proved. □

A tiling is called **periodic** if it admits as symmetries translations in two independent directions. Although Theorem 1 applies to any selfdual normal tiling of the plane, the examples cited to establish the second assertion of the theorem show that the three possible ranks (2, 4, ∞) can each be realized even with periodic tilings.

It is well known that the symmetry groups of periodic tilings fall into 17 classes. These are often called "wallpaper groups" (see, for example, Grünbaum & Shephard [1987], Section 1.4) and it is convenient to denote them by their "international crystallographic symbols". In the following, some familiarity with these groups and their notation is assumed.

Our next objective is to determine the possible symmetry groups of selfdual tilings.

Theorem 2. A periodic selfdual tiling $\mathcal{T}$ can have symmetry group $S(\mathcal{T})$ of one of the classes **p1**, **p2**, **p4**, **pm**, **pg**, **cm**, **pmm**, **pmg**, **pgg**, **cmm**, **p4m** or **p4g**. It cannot have a symmetry group of any of the remaining classes **p3**, **p31m**, **p3m1**, **p6**, or **p6m**.

Proof. The first part of the theorem is established by the existence of the tilings in Figure 7(a) to 7(l). Each of these is obtained by modifying the square tiling (which has symmetry group **p4m**) using the construction described (for polyhedra) by Kirkman [1857] (see also Brückner [1900], p. 93). This states that if a selfdual object is changed in a way that is compatible with any particular duality (that is, by making dual changes at dually corresponding elements), then the resulting object will also be selfdual.

For the second part of the proof we need to deal with each of the five classes of groups separately. We give full details in two cases.

Class p6. Here the centers of 6-fold rotational symmetry (or 6-centers, as we shall call them) are arranged in the form of an "equilateral triangle lattice" (or "dot pattern of type DPP51" in the notation of Grünbaum & Shephard [1987], Section 5.3). See Figure 8(a) where these centers are denoted by small hexagons. Construct the regular hexagonal tiling $\mathcal{H}$ (which is a Dirichlet tiling for the 6-centers) as shown in the diagram; the 6-centers are the centers of the hexagonal tiles. Now suppose $\mathcal{T}$ is a selfdual tiling whose symmetry group of class **p6** has this set of 6-centers in its group diagram (see Grünbaum & Shephard [1987], Section 1.4); we shall show that this assumption leads to a contradiction. Since the 6-centers are all of "one kind", that is, they form one transitivity class under the group, there are two possibilities:

(i) each 6-center is a vertex of $\mathcal{T}$ of valence $6n$ ($n \geq 1$); or

(ii) each 6-center is the center of a $6m$-gonal tile of $\mathcal{T}$ (with $m > 1$).

For case (i), the selfduality δ of $\mathcal{T}$ whose existence is assumed must map the vertex V at a 6-center into a $6n$-gon Q, and we consider the possible positions of Q relative to a hexagon H of $\mathcal{H}$. If Q is interior to H (as Q_1 in Figure 8(a)) then the operations of the group **p6** will map this into five other tiles equivalent to Q; hence H will contain at least six copies of tiles equivalent to Q. If Q straddles an edge of H (as in Q_2, say) then the operations of **p6** (the 6-fold rotations about the center of H, and the halfturns about the midpoints of the edges of H) will map the parts of Q_2 in H and outside H into parts of other tiles which together amount to a total of either three or six tiles of the transitivity class of Q contained in H. The former case arises if Q contains the midpoint of the edge of H, the latter if it does not contain the midpoint. If Q contains a vertex of H (as Q_3, say), then H contains one-third of Q and, by symmetry, five other one-third parts of tiles equivalent to Q, making a total of two copies of Q (actually six one-third copies) in H. Now δ maps each vertex V into just **one** tile Q, and the hexagons are in one-to-one correspondence with the vertices. It follows that each hexagon must contain precisely one copy of Q. As we have seen, this does not happen; whatever the position of Q at least two copies lie in H. This contradiction shows that case (i) cannot occur.

Case (ii) is dealt with similarly. We consider the possible positions of the 6m-valent vertex which is the image of the 6m-gon Q under δ. According to the position of that vertex relative to H there are either six, three or two copies of it in H. Again a contradiction is reached, since an argument analogous to the one given above shows that each H must contain precisely one such vertex.

Class p6m. The argument here is exactly similar except that in this case the number of elements (vertices of valence 6n or 6m-gons) in H is either 2, 3, 6 or 12.

Class p3. The centers of 3-fold rotational symmetry (the 3-centers) form a triangular lattice; see Figure 8(b) where they are indicated by triangles. Suppose $\mathcal{T}$ is a selfdual tiling whose symmetry group has this group diagram; we shall show this assumption leads to a contradiction. The 3-centers are of three kinds (belong to three transitivity classes under the group) as indicated by the symbols in the diagram. For the centers indicated by solid triangles we construct the hexagonal tiling $\mathcal{H}$ whose vertices are the 3-centers of the other two kinds. This tiling is indicated by the dotted lines in the diagram. Consider the possibilities:

(i) each 3-center indicated by a solid triangle is a vertex of $\mathcal{T}$ of valence 3n (n≥1); or

(ii) each 3-center indicated by a solid triangle is the center of a 3m-gonal tile of $\mathcal{T}$ (where m≥1).

For case (i) we proceed as described above for case (i) of the group **p6**. Let V be a vertex as described, with δ(V) = Q, and consider possible positions of Q relative to a tile H of $\mathcal{H}$. If Q is interior to H (at Q_1, say) then the operations of the group lead to three copies of Q in H. If Q straddles an edge of H (at Q_2, say), the the operations of the group lead to three half-copies of Q in H. Both cases lead to a contradiction as before. The only remaining possibility is that Q is at a vertex of H (at Q_3, say) represented by the open (hollow) triangles. In this case one copy (actually, three one-third copies) of Q lies in H, and there is no immediate contradiction. However, we now proceed with an exactly similar argument for the remaining 3-centers. Let us denote such a 3-center by V'. If V' is a 3p-valent vertex, then we count the number of 3p-gonal tiles Q' in a hexagon H'. Again there are three copies (if Q' lies interior to H') or three half-copies (if Q' straddles an edge of H'); however, the exceptional case of three one-third copies cannot arise now, since the vertices of H' are already allocated to vertices or tiles. Hence a contradiction is reached. An exactly similar argument can be used in case (ii), and so group p3 cannot occur.

Classes p31m and p3m1. The argument for p3m1 is identical to that for p3. For p31m we employ an argument similar to that for p6. There are two sorts of 3-centers: each center of the first sort has three lines of reflection passing through it, and all such centers belong to one transitivity class under the group. Take the corresponding Dirichlet tiling and consider how many 3-centers of the second sort (which do not lie on any lines of reflection) are contained in each tile. A contradiction is reached as before, showing that a group of this class can not arise as a symmetry group of a selfdual tiling.

This completes the proof of the theorem. □

We conclude this section with some remarks about Theorem 2. Let us say that a tiling $\mathcal{T}$ is **harmonious** if every automorphism of $\mathcal{T}$ is induced by an (isometric) symmetry of the tiling. In a similar way, we may say that $\mathcal{T}$ is **harmoniously selfdual** if $\mathcal{T}$ is harmonious and there exists an isometric copy $\mathcal{T}^*$ of $\mathcal{T}$ which establishes that $\mathcal{T}$ is metrically selfdual, and, moreover, every symmetry of $\mathcal{T}$ is also a symmetry of $\mathcal{T}^*$. If $\mathcal{T}$ is harmoniously selfdual, then we refer to this special superposition of $\mathcal{T}$ and $\mathcal{T}^*$ as a **harmoniously dual superposition**, and denote it by $\mathcal{T}^H$. Clearly, for harmoniously selfdual tilings $\mathcal{T}$ the selfduality group $D(\mathcal{T})$ is isomorphic to the symmetry group $S(\mathcal{T}^H)$, and the latter, as we have remarked, may be regarded as a 2-color group.

These concepts may appear very special. However, no periodic tiling is known which is not homeomorphic to a harmonious tiling, and no selfdual tiling is known which is not homeomorphic to a harmoniously selfdual tiling. We conjecture that, in fact, no exceptions exist, and all periodic tilings are homeomorphic to harmonious ones, and the same is true for selfdual tilings.

The theorem shows that a group of the class p4g can occur as the symmetry group of a selfdual tiling. However, it is of interest to note that this is not so if we restrict attention to harmoniously selfdual tilings. (It will be observed that the tiling in

Figure 7(l) with this group is **not** harmoniously selfdual.) To see this, suppose $\mathcal{T}$ is harmonious selfdual, with symmetry group of class **p4g**. As in the proof of the second part of Theorem 2, each 4-center must be either a vertex (of valence 4n, n ≥ 1) or the center of a tile (with 4m edges, m ≥ 1). Since $\mathcal{T}$ is harmoniously selfdual, centers of both kinds (with n = m) must exist, and so the set of 4-centers of the first kind must be congruent to the set of 4-centers of the second kind. But this is impossible since, in group **p4g**, all 4-centers are of the same kind, that is, belong to the same transitivity class (see, for example, Figure 1.4.2 of Grünbaum & Shephard [1987]). This is a contradiction showing that such a tiling cannot exist.

Another remark concerns the relationship between the symmetry group of a selfdual tiling, and its rank. This has not been fully investigated, and in Table 1 we show the information available to us. A plus sign (+) indicates that there exists a tiling of the rank in question (2, 4 or ∞); a cross (x) indicates that no such tiling exists; and a minus sign (-) indicates that no tiling is known, although we are not sure that such a tiling cannot exist.

Table 1.

Group / Rank	p1	pg	pm	cm	p2	pgg	pmg	pmm	cmm	p4	p4g	p4m
2	+	+	+	+	+	+	+	+	+	+	+	+
4	x	x	x	x	+	-	-	-	+	-	-	-
∞	+	+	+	-	+	-	-	-	-	-	-	-

In Figures 7, 9 and 10 we show examples of selfdual tilings with all combinations of symmetry group and rank for which we assert existence in Table 1.

We remarked in Section 2 that selfduality groups are best described in terms of 2-color groups. A 2-color group is the group of automorphisms of a 2-colored tiling (that is, a tiling in which each tile has one of two given colors), where the allowable transformations are those that map the tiling isometrically onto itself and either preserve the colors of all tiles, or else reverse them all. It is well known that there are 46 classes of such groups of periodic tilings of the plane, and they are particularly suitable for specifying the selfduality groups of harmonious tilings. (For details of the theory of color groups see Grünbaum & Shephard [1987], Chapter 8, or Schwarzenberger [1984], where references to the ample literature can be found as well.) There exist no generally accepted symbols for these groups, but the symbols recently proposed by Coxeter [1986] are very convenient and may well become standard. (For a comparison of the different systems of notation for 2-color groups see Washburn & Crowe [1988].) The Coxeter notation for the 2-color symmetry group of a 2-colored periodic figure F (or, more generally, for any figure with a binary property) is a symbol of the type G_1/G_2, in which G_1 is the symmetry group of F without regard to the color, and G_2 is the subgroup (of index 2) consisting of those isometries that map F onto itself while conserving the colors. (It should be stressed that the symbol G_1/G_2 is meant to indicate that the 2-color group is related to two other groups which express symmetry properties of a 2-colored figure; it is not intended as an identification of the 2-color group with a quotient group.) In the case of 2-color symmetry groups of tilings, one symbol G_1/G_2 fails to determine the class of the 2-color group uniquely, and an addition to the notation is necessary to distinguish between the two possibilities. Using the notation of 2-color groups, in Table 2 we list the selfduality groups of the harmonious tilings shown in the diagrams of this paper; for convenience of reference, we give also the symbols for 2-color groups used in Grünbaum & Shephard [1987]. It is clear that these examples do not exhaust the possibilities; in fact, we know of several other selfduality groups, which are listed below. It would be desirable for a complete list of selfduality groups of harmonious tilings to be worked out. Since the rank of a harmonious selfdual tiling $\mathcal{T}$ equals the minimum of the orders of the "color-reversing" symmetries of $\mathcal{T}^H$, this would lead to an independent proof of Theorem 1 in the harmonious case. We must stress that for nonharmonious tilings there is no such connection between the color symmetries of $\mathcal{T}^H$ and the selfduality group of $\mathcal{T}$. This can be easily seen from the example of the tiling in Figure 7(l), which has symmetry group p4g, automorphisms group p4m, selfduality group $p4m[2]_5$ = p4m/p4m, and for which $\mathcal{T}^H$ has color symmetry group $pmg[2]_5$ = pmg/p2.

In addition to the listings in Table 2, we have examples of harmonious tilings of rank 2 with selfduality groups $cm[2]_1$ = cm/p1, $pm[2]_2$ = pm/cm, $pm[2]_5$ = pm/pm(m'), $pmg[2]_3$ = pmg/pgg, $pmg[2]_1$ = pmg/pmg, $pmm[2]_4$ = pmm/pmg, $pmm[2]_1$ = pmm/pmm, $pgg[2]_1$ = pgg/pg, $cmm[2]_4$ = cmm/p2, $p4[2]_1$ = p4/p4, $p4g[2]_3$ = p4g/pgg, and a harmonious tiling of infinite rank with selfduality group $pg[2]_1$ = pg/p1.

Table 2.

Tiling	Selfduality group	Tiling	Selfduality group
Fig. 3	$pgg[2]_2$ = pgg/p2	Fig. 4	$p4[2]_2$ = p4/p2
Fig. 7 (a)	$p2[2]_1$ = p2/p1	Fig. 7 (b)	$p2[2]_2$ = p2/p2
(c)	$p4g[2]_1$ = p4g/p4	(d)	$pmg[2]_4$ = pmg/pm
(e)	$pm[2]_1$ = pm/pg	(f)	$cmm[2]_2$ = cmm/cm
(g)	$cmm[2]_5$ = cmm/pmm	(h)	$pmg[2]_1$ = pmg/pmg
(i)	$cmm[2]_1$ = cmm/pgg	(j)	$pmm[2]_3$ = pmm/cmm
(k)	$p4m[2]_5$ = p4m/p4m		
Fig. 8 (a)	$p4[2]_2$ = p4/p2	Fig. 8 (b)	$p4g[2]_2$ = p4g/cmm
Fig. 9 (a)	p1[2] = p1/p1	Fig. 9 (b)	$cm[2]_3$ = cm/pm
(c)	$cm[2]_2$ = cm/pg	(d)	$pgg[2]_2$ = pgg/p2
Fig. 11	$pg[2]_1$ = pg/p1.		

4. Selfdual polyhedra. Although the theory of selfduality for polyhedra differs in many respects from that of tilings, the latter provide a very convenient approach to the construction of selfdual polyhedra. (For other constructions see Archdeacon & Richter [1989]. The related topic of selfdual graphs is discussed by Servatius *et al.* [1989] and McKee [1989].) In particular, we shall show how to construct polyhedra with rank 2^k for any $k \geq 1$ (Corollary to Theorem 3). If, however, we also impose the condition of central symmetry, the situation changes drastically and only $k = 1$ and $k = 2$ are possible (Theorem 5). As for tilings so for polyhedra, the automorphism group A(P) limits the possibilities, but does not in any sense determine, the selfduality group D(P); the possible pairs of groups {A(P), D(P)} have yet to be determined. We explained in the previous section that D(P) is best regarded as a 2-color group; in an obvious modification of the Coxeter notation, we use for 2-color groups on the sphere symbols of the type G_1/G_2, where G_1 is a group and G_2 is a suitable subgroup of index 2. We use v(P) to denote the number of vertices of the polyhedron P, and C_k for the cyclic group of order k.

Theorem 3. For every positive integer n, there exists a convex polyhedron P_n such that $A(P_n) = \mathbf{C}_n$ and $D(P_n) = \mathbf{C}_{2n}/\mathbf{C}_n$. Moreover, these polyhedra can be chosen so that $v(P_1) = 7$, $v(P_2) = 13$, and $v(P_n) = 5n+1$ for $n \geq 3$.

Proof of Theorem 3. The polyhedra P_1 and P_2 in Figures 11 and 12 establish the theorem for $n = 1$ and $n = 2$.

For $n \geq 3$, we consider the periodic tiling in Figure 13(a), which is harmoniously selfdual and of infinite rank, as illustrated in Figure 13(b). We take a "strip" consisting of $n \geq 3$ rows (in Figure 13(a) such a strip with $n = 8$ is shaded), identify the top and bottom ends of the strip, and identify as one vertex V all the vertices along the right edge (see Figure 13(c)). This leads to a 3-connected planar graph G_n with $5n+1$ vertices, whose unbounded region we denote by W (see also Figure 13(d)). By Steinitz's Theorem (see Grünbaum [1967], Chapter 13) this implies that there exists a convex polyhedron P_n, of a unique combinatorial equivalence class, such that the graph of vertices and edges of P_n is isomorphic with G_n; in fact, Figure 13(d) can serve as a Schlegel diagram of P_n. The selfduality group of P_n is generated by the selfduality δ of rank $r(\delta) = 2n$ which is defined by $\delta(V) = W$, $\delta(W) = V$, and $\delta(X_j) = X_{j+1}$ (subscripts mod $2n$) where X represents any of the letters A, B, C, D, E used for faces and vertices in the figure. The odd powers of δ are the only selfdualities of P_n, and the even powers are the only automorphisms. It follows that P_n satisfies the requirements of Theorem 3. In particular, Figure 13(d) illustrates P_8, which is of rank 16. □

Corollary. For every positive integer k, there exists a convex polyhedron Q_k such that $r(Q_k) = 2^k$. Moreover, Q_k can be chosen so that $v(Q_1) = 4$, $v(Q_2) = 13$, and $v(Q_k) = 5 \cdot 2^{k-1}+1$ for $k \geq 3$.

For $k \geq 2$, the corollary follows from the theorem by taking $Q_k = P_n$ for $n = 2^{k-1}$. In general, the rank of P_n is the highest power of 2 that divides $2n$. For $k = 1$ the situation is special, since the regular tetrahedron satisfies the requirements of the corollary for $k = 1$, although it has too many automorphisms to play the role of P_1 in Theorem 3.

To formulate our next result, we need the concept of polarity for convex polyhedra in $\mathbb{E}^3$. If P is a convex polyhedron P then its **polar** polyhedron P^* is defined by

$$P^* = \{ Y \in \mathbb{E}^3 : \langle X,Y \rangle \leq 1 \text{ for all } X \in P \}.$$

This relationship is reciprocal; that is, P^{**} and P are the same subset of $\mathbb{E}^3$ (see, for example, Grünbaum [1967], Section 3.4). There is a natural duality (in the sense defined in Section 1) δ from P to P^*, which maps each face F of P to the face F^* of P^* defined by

$$\delta(F) = F^* = \{ Y \in P^* : \langle X,Y \rangle \le 1 \text{ for all } X \in F \}.$$

(Another way of describing the polarity between convex polyhedra is to say that P^* is obtained from P by **reciprocation** in a unit sphere U.) For convex polyhedra P and Q, we define a **metric isomorphism** from P to Q as an isometry of $\mathbb{E}^3$ which maps P onto Q, and a **metric duality** from P to Q as an isometry of $\mathbb{E}^3$ which maps P onto its polar Q^*. Similarly, we can define the **metric automorphism group** $S(P)$ and the **metric selfduality group** of P. It is clear that every metric automorphism or selfduality of P induces a combinatorial automorphism or selfduality (so that the groups just defined can be considered as subgroups of $A(P)$ and $D(P)$, respectively), but the reverse is generally not the case. A polyhedron P is **harmonious** if $S(P)$ is isomorphic with $A(P)$. A polyhedron P is **harmoniously selfdual** if it is harmonious and there is a unit sphere U such that the polyhedron P^* obtained from P by reciprocation in U is congruent with P and so situated that every symmetry of P is also a symmetry of $P^H = P \cup P^*$. Then the selfdualities of P can be realized by isometries of $\mathbb{E}^3$ which map P onto P^* and vice versa.

It is well known (Mani [1971]) that each convex polyhedron P in $\mathbb{E}^3$ there exists an isomorphic polyhedron Q for which each automorphism is induced by a symmetry. However, it is an open question whether each selfdual convex polyhedron in $\mathbb{E}^3$ is isomorphic to a harmoniously selfdual polyhedron. The next theorem shows that the polyhedra in Theorem 3 are isomorphic to such polyhedra, with the consequence that Theorem 3 is true in a metric as well as in a combinatorial sense. In Theorem 4 we use the usual notation for symmetry groups of a 2-sphere; see, for example, Coxeter & Moser [1980] or Grünbaum & Shephard [1981]. We note that the selfduality group of a harmoniously selfdual polyhedron P, that is, the symmetry group of P^H, can be described by a 2-color group.

Theorem 4. For all positive integers n and k, the polyhedra P_n and Q_k described in the proofs of Theorem 3 and the Corollary can be chosen to be harmoniously selfdual; that is, their automorphisms and selfdualities are all induced by isometries of

$\mathbb{E}^3$ and reciprocations. In particular, the choice can be made so that $A(P_n) = S(P_n) = [n]^+$ and $D(P_n) = S(P_n^H) = [2^+, 2n^+]/[n]^+$ for all $n \geq 1$.

Proof of Theorem 4. In the proofs of Theorem 3 and its corollary, the polyhedra P_n and Q_k were defined only up to isomorphism. To complete the proof of Theorem 4, we need to specify points in $\mathbb{E}^3$ for the vertices of each P_n and Q_k.

In the case of $P_1 = Q_2$ we locate the vertices as follows (see Figure 11(b)):

$V_1 = (\sqrt{2}, 1, 0)$ $\qquad$ $V_2 = (0, -1, 0)$

$V_3 = (\frac{w}{\sqrt{2}}, w - 1, 2w)$ $\qquad$ $V_4 = (\sqrt{2}, 1, 1)$

$V_5 = (-\sqrt{2}, 1, 0)$ $\qquad$ $V_6 = (-\sqrt{2}, 1, -\sqrt{2})$

$V_7 = (0, 1, -\sqrt{2})$

where $w = \frac{2}{5}(\sqrt{6} - 1)$. Now the linear transformation $(x, y, z) \rightarrow (-x, -y, z)$ maps P_1 onto its dual, and so induces the unique selfduality of P_1.

For $P_2 = Q_2$ we choose the vertices as follows (see Figure 12(b)):

$V = (0, 0, +1)$

$A_1 = (+1, 0, -1)$ $\qquad$ $A_3 = (-1, 0\ -1)$

$B_1 = (0, +1, -1)$ $\qquad$ $B_3 = (0, -1, -1)$

$C_1 = (+\frac{1}{2}, +\frac{3}{2}, -\frac{1}{2})$ $\qquad$ $C_3 = (-\frac{1}{2}, -\frac{3}{2}, -\frac{1}{2})$

$E_1 = (+\frac{1}{3}, -\frac{2}{3}, +\frac{1}{3})$ $\qquad$ $E_3 = (-\frac{1}{3}, +\frac{2}{3}, +\frac{1}{3})$

$F_1 = (+\frac{1}{3}, -\frac{1}{3}, +\frac{2}{3})$ $\qquad$ $F_3 = (-\frac{1}{3}\ +\frac{1}{3}, +\frac{2}{3})$

$G_1 = (+1, +1, 0)$ $\qquad$ $G_3 = (-1, -1, 0)$.

Regarding the points of $\mathbb{E}^3$ as column vectors, let δ be the isometry of $\mathbb{E}^3$ defined by left multiplication by the matrix

$$\begin{vmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix} .$$

Then δ maps Q_2 onto Q_2^* , and thus induces a selfduality of Q_2. The isometry, and hence the selfduality, has rank 4, and it is easy to see on combinatorial grounds that no selfduality of Q_2 has smaller rank. It is also easy to check that Q_2 is harmoniously selfdual and has the stated groups of automorphisms and selfdualities. (To verify that δ maps Q_2 onto Q_2^* it is only necessary to show that $\langle\delta(X), Y\rangle \leq 1$ for any pair X, Y of vertices of Q_2, with equality in the appropriate cases. We omit the details.)

There remains the case of P_n with $n \geq 3$, which is somewhat more difficult to calculate. We begin by constructing a simpler polyhedron T_n , whose underlying graph is illustrated in Figure 14(a) (for the case n = 8). Then T_n satisfies $A(T_n) = \mathbf{D_n}$ and $D(T_n) = \mathbf{D_{2n}}$. To realize T_n as a polyhedron in $\mathbb{E}^3$ we locate the vertices as follows:

$$V = (\ 0, 0, 1\ )$$

$$C_j = \left(\frac{1}{b}\cdot\cos\frac{\pi(j+1)}{n},\ \frac{1}{b}\cdot\sin\frac{\pi(j+1)}{n},\ 0\right)$$

$$A_j = \left(b\cdot\cos\frac{\pi j}{n},\ b\cdot\sin\frac{\pi j}{n},\ -1\right)$$

$$F_j = \left(b\cdot\cos\frac{\pi j}{n}\ \ b\cdot\sin\frac{\pi j}{n},\ 1-b^4\right)$$

for j = 1, 3, ... , 2n-1, where $b = \sqrt{\cos\frac{\pi}{n}}$. It can be verified that $S(T_n) = [n]$ and $S(T_n^H) = [2+,2n]/[n]$. (We note that a right n-sided pyramid with a regular basis has the same groups as T_n; this is the description of the selfduality group of pyramids alluded to in Section 2.)

Now, again understanding the vertices as column vectors, let the isometry δ be defined by left multiplication by the matrix

$$\begin{vmatrix} +\cos\frac{\pi}{n} & -\sin\frac{\pi}{n} & 0 \\ +\sin\frac{\pi}{n} & \cos\frac{\pi}{n} & 0 \\ 0 & 0 & -1 \end{vmatrix}$$

which represents a rotary reflection about the Z-axis through $\frac{1}{2n}$ of the full turn. Then it can be verified (we omit the calculations) that δ maps T_n onto T_n^* and that δ has order 2n. To construct P_n , we must modify T_n in such a way as to destroy the symmetries which are not induced by powers of δ . We do this by cutting off each of the vertices F_j (j = 1, 3, ...), adding in its place a new face B_{i-1} which passes through the vertex C_j and which intersects the edges through F_j at points we call D_j and E_j . To preserve the selfduality of the polyhedron under δ, we need also to add a vertex B_j (j = 1, 3, ...) which extends the face C_{j+1} and to replace the face F_{j+1} by new faces D_{j+1} and E_{j+1}, as shown in Figure 14(b); the position of B_j is determined by polarity from the position of the plane through C_j , D_j and E_j . (To avoid cluttering up the diagram, the modifications are shown for one value of j only; they need to be carried out at all vertices F_j .) The calculations of the coordinates of the vertices of the modified polyhedron are essentially elementary, and we do not give details here. It is clear that duality of the new polyhedron P_n and its image under δ is maintained, that the polyhedron obtained is isomorphic to P_n in Theorem 3, and that the groups $S(P_n)$ and $S(P_n^H)$ are as claimed. □

We remark that a slight modification of the above construction leads to another family of harmoniously selfdual polyhedra: convex polyhedra R_n such that $A(R_n) = S(R_n) = [n]^+$ and $D(R_n) = S(R_n^H) = [2,n]^+/[n]^+$, for all $n \geq 2$. For R_2 the polyhedron of Figure 15 can be chosen, and for $n \geq 3$ a typical polyhedron is shown in Figure 16 (where we have taken n = 8). We note that these polyhedra have rank 2; in fact, all their selfdualities have rank 2.

Concerning the possibilities present for centrally symmetric polyhedra we have:

Theorem 5. If P is a combinatorially selfdual polyhedron with a center of symmetry then its rank r(P) is either 2 or 4. Moreover, there exist centrally symmetric convex polyhedra C_2 and C_4, with center at the origin, which are harmoniously selfdual and such that $r(C_2) = 2$, $S(C_2) = [2,2]$, $S(C_2^H) = [2,4]/[2,2]$, and $r(C_4) = 4$, $S(C_4) = [2,2^+]$, $S(C_4^H) = [2,4^+]/[2,2^+]$.

Proof. The fact that 2 and 4 are the only possible values for the rank of a combinatorially selfdual, centrally symmetric polyhedron follows almost word by word as in the analogous part of the proof of Theorem 1, with three points of difference: (i) Instead of working with the polyhedra themselves, we consider their projections onto a fixed sphere on which they induce tilings; thus we discuss tilings of the sphere instead of the plane. (ii) Since the mapping ϕ may be assumed to commute with the antipodal map of the sphere, it has a fixed point (see, for example, Alexandroff-Hopf [1935], Theorem IV on page 481). (iii) Since the number of faces of a polyhedron is finite, the rank of a polyhedron cannot be infinite. Thus we only need to prove the existential part of the theorem.

The smallest centrally symmetric selfdual convex polyhedron that can be taken as C_2 has 8 vertices; a Schlegel diagram of C_2 is shown in Figure 17(a). This polyhedron was first mentioned as being (combinatorially) selfdual by Kirkman [1857], and later also by Hermes [1900] and Brückner [1900]; however, none of these authors noticed that C_2 could be realized as a centrally symmetric polyhedron. This fact was first stated by Jucovic [1970], while the realization of C_2 as a harmoniously selfpolar polyhedron (as shown in Figure 17b) is due to Leichtweiss [1978]. The construction of C_4, which appears to be new, is indicated in Figure 18. In Figure 18(a) a Schlegel diagram of C_4 is given, while a harmoniously selfdual realization is shown in Figure 18(b); the coordinates of the vertices of the polyhedra in Figures 17(b) and 18(b) are listed in the captions. The verification of the claims concerning groups of symmetries and the harmonious selfduality of these polyhedra is routine, and is omitted. □

We conjecture that each of the polyhedra described in this section has the smallest possible numbers of vertices among all selfdual polyhedra with the properties stated in each case.

5. Selfdual configurations in the projective plane. For our third, and final, example of selfdual objects whose rank exceeds 2 we turn to configurations of points and lines in the real projective plane. We shall be exclusively concerned with (n_m)-configurations, that is, sets of n points P_i and n lines L_i, $i = 1,2,\ldots,n$, such that every P_i lies on exactly m of the lines L_j, and conversely. Duality is defined as in Section 1, and it is worth remarking that many of the configurations that occur in elementary projective geometry are selfdual -- for example, the configurations of Pappus and Desargues are selfdual (9_3)- and (10_3)-configurations.

As with tilings, several distinct meanings can be attached to selfduality of configurations. The first, alluded to in the preceding paragraph, is the **combinatorial** sense, with isomorphisms and dualities being understood in the sense of the incidence lattice. Of independent interest are **projective** selfdualities, in which isomorphisms are induced by collineations, and dualities by correlations, of the projective plane in which the configuration has been embedded. (In fact, an even more detailed classification arises from the possibility of using projective planes over various fields; here, however, we shall restrict attention to the **real** projective plane.) For a selfdual configuration C embedded in the projective plane we shall use A(C) and D(C) to denote the automorphism and selfduality groups, while $A_L(C)$ and $D_L(C)$ will indicate the groups of collineations, or of collineations and correlations, of C, respectively.

As an illustration we consider the (21_4)-configuration C shown in Figure 19. This selfdual configuration has been extensively studied by many authors, but only recently has it been observed that it admits an embedding in the *real* Euclidean plane $\mathbb{E}^2$ (embeddings in the complex plane and in finite planes were found long ago); see Grünbaum & Rigby [1990] for the properties of this configuration, and for references to earlier works. It can be shown that A(C) = PGL(2,7) of order 336, and that D(C) has order 672; in contrast, $A_L(C) = \mathbf{D}_7$, $D_L(C) = \mathbf{D}_7 \times \mathbf{C}_2$, and no embedding of C in $\mathbb{E}^2$ has a larger group of collineations and correlations.

From now on we shall consider only (n_3)-configurations. For some special cases, including the configurations of Pappus and Desargues, the various groups have been studied; see, for example, Kagno [1947], Coxeter [1975], [1977]. However, the question of rank of the configurations has not been considered before. (We may note that Coxeter [1955, p. 67] mentions that a "polarity" is an involutory correlation; he thus comes close to the question of rank of a selfduality -- but he does not pursue the topic.) The following is analogous to the result on polyhedra:

Theorem 6. There exist selfdual (n_3)-configurations C_k in the projective plane $(k \geq 1)$ with $r(C_k) = 2^k$. We may choose C_1 to be a (9_3)-configuration, and C_k, for $k \geq 2$, to be a $(64q_3)$-configuration with $A(C_k) = A_L(C_k) = \mathbf{C}_q$, and $D(C_k) = D_L(C_k) = \mathbf{C}_{2q}$, where $q = 2^{k-1}$.

Proof. For C_1 we may take the Pappus configuration. For $k \geq 2$ we construct C_k using the fragments **F** and **G** shown in Figure 20. Fragment **F** is obtained from a configuration usually denoted $(10_3)_5$ (see, for example, Schroeter [1889], Dorwart [1967], Bokowski & Sturmfels [1989, p. 44]), by breaking one of its incidences. Fragment **G** is dual to **F**, and the points and lines of both fragments have been labelled following the convention made in Section 2. Using three copies of **F** and three copies of **G**, together with additional points and lines labelled by W and X (with appropriate primes and/or subscripts), we construct a larger fragment **H**, which is schematically indicated in Figure 21. Using $q = 2^{k-1}$ copies of **H**, placed in a necklace like circuit, yields the configuration C_k. A selfduality δ of this configuration maps A_1,

B_1, C_1, ... onto A_2, B_2, C_2, ... ; these map onto A_3, B_3, C_3, ... in the next fragment, and so on. (Subscripts are taken modulo 2q.) To see that C_k satisfies the requirements of Theorem 6, it is convenient to consider its Levi-diagram (Figure 23; see Coxeter [1950] for more detailed explanations of Levi graphs); this is a graph in which the points of the configuration are represented by black vertices (solid dots), the lines of the configuration by white vertices (hollow dots), and vertices corresponding to incident points and lines determine edges of the diagram. The parts of the Levi-diagram corresponding to F and G are shown in Figure 22; from these it is clear that G is dual to F, and from the different roles A and V play in the diagrams it is obvious that the only way a duality can operate in C_k is by mapping a copy of F onto a copy of G. The points and lines added in the formation of H also map appropriately under dualities that carry copies of F onto copies of G (and vice versa), as is visible from the Levi-diagrams of two consecutive copies of H. It follows therefore that all possible dualities are odd powers of the duality δ described above. Each of these is of rank $2q = 2^k$, which completes the proof of Theorem 6.

We do not assert that C_k is the smallest configuration of rank 2^k, but we believe that the numbers of points and lines cannot be decreased greatly unless a radically different method of construction is used. Also, it would be of interest to decide whether the configurations we described have embeddings for which $A_L(C_k) = C_q$, and $D_L(C_k) = C_{2q}$.

It seems likely that there exist selfdual (n_m)-configurations with $m > 3$ and of arbitrarily large rank; however, this has not been established so far.

6. Remarks and problems.

(1) Plane selfdual tilings of infinite rank can be used to construct selfdual tilings of the torus of arbitrary rank $r = 2^k$. The method is illustrated in Figure 24 for $r = 8$. No results seem to be available concerning the possible automorphism and selfduality groups of such tilings. Also, it appears that selfdual tilings on manifolds of higher genus have not been investigated so far.

(2) The construction of selfdual polyhedra or tilings by performing "dual changes" on "dually corresponding" elements has been used to obtain almost all the examples in this paper, even if the presentation does not indicate this. Although this technique is as old as the investigation of selfdual polyhedra (Kirkman [1857]), it applies

here with an essential modification: if the selfduality is of rank r greater than 2, then the changes must be made at all r elements involved. Thus it is possible that, from a selfdual polyhedron P of rank 2, the use of a selfduality of P with rank $r > 2$ can lead to a selfdual polyhedron of rank r. This happens, for example, in the construction of the Jendrol polyhedron (see Figure 25). Similarly, the use of selfdualities of infinite rank of the square tiling (which is of rank 2) leads to tilings with infinite rank, such as those in Figures 3, 5, 10.

(3) In the hyperbolic plane there exist infinitely many selfdual tilings with a high degree of regularity. However, no characterization of the possible automorphism groups or selfduality groups is known.

(4) As observed by Leichtweiss [1978] and Sztencel & Zaremba [1981], the polyhedron C_2 described in Section 4 can serve as the unit sphere of a 3-dimensional normed space X with the property the dual (adjoint) space X^* is isometric to X (thus X is selfdual) but only the second dual X^{**} is canonically isometric to X. Analogously, the polyhedron C_4 is the unit sphere of a 3-dimensional normed space Y which is isometric to its dual Y^*, but the fourth dual Y^{****} is the first which is canonically isometric to Y. Moreover, Theorem 6 implies that for no 3-dimensional normed space isometric to its dual does one have to go beyond the fourth dual to find a canonical isometry. It is known there exist selfdual centrally symmetric polytopes of all dimensions and of rank 2; these are of interest as unit balls of selfdual normed spaces without inner product norms (see Partingon [1986]). However, it is not known whether ranks greater than 2 are possible for centrally symmetric convex polytopes of dimension exceeding 3. If central symmetry is not required, the polyhedra constructed in Section 4 can be used to obtain selfdual convex polytopes of dimension 4 or higher, with arbitrarily high rank.

(5) S. Jendrol [1990] has determined all symmetry groups that are possible for 3-dimensional selfdual convex polyhedra.

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University of Washington, Seattle, WA 98195, USA

University of Washington, Seattle, WA 98195, USA

University of East Anglia, Norwich, England

Daniel H. Wagner Associates, Paoli, PA 19301

Figures.

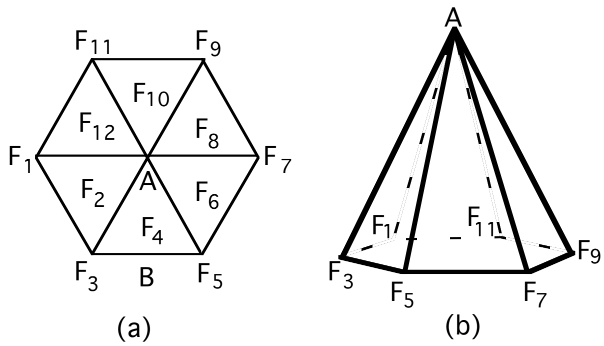


Figure 1. A six-sided pyramid P, illustrating the concept of selfduality. (a) A Schlegel diagram of P, with vertices and 2-dimensional faces labelled so as to indicate the selfduality δ of P discussed in the text. (b) A perspective view of the same pyramid P. To simplify the diagram only the vertices are labelled.

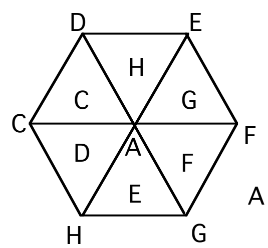


Figure 2. A balanced labelling of the six-sided pyramid P, which shows a selfduality of P of rank 2.

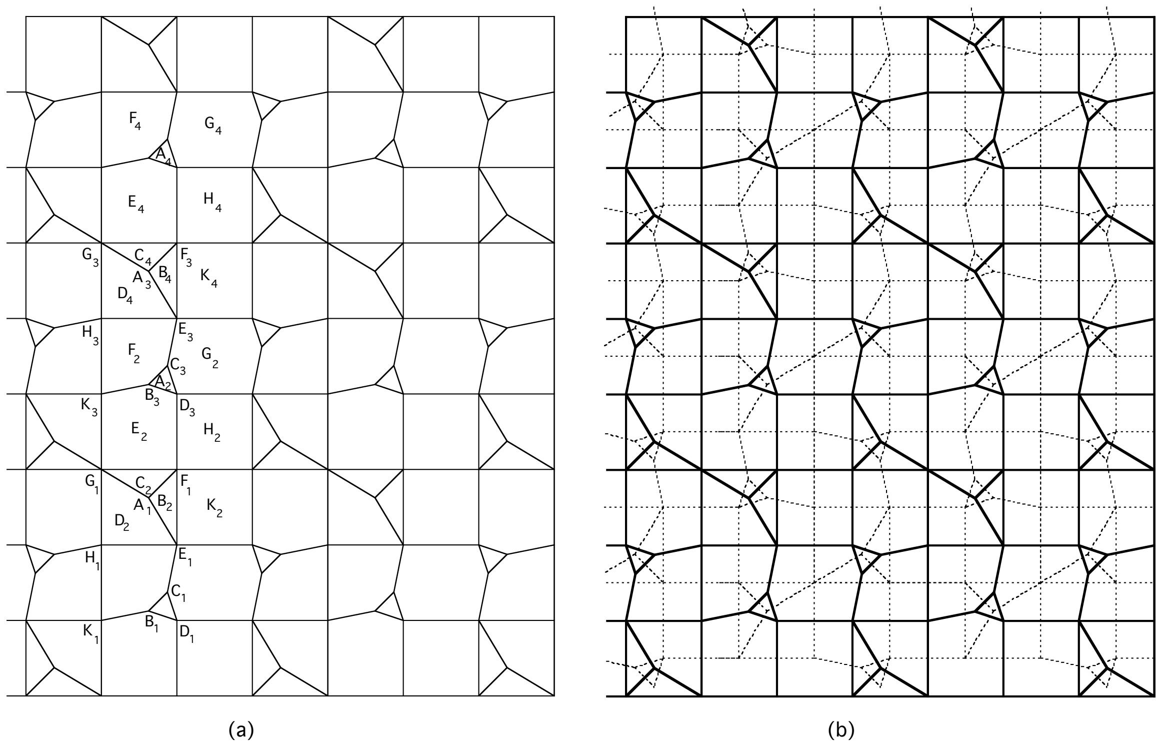


Figure 3. A selfdual tiling T of infinite rank. The labels in (a) indicate one selfduality δ of T which is of infinite rank; the application of δ increases the subscripts by 1. (b) The metric selfduality of T; two copies of T (that is, T and its dual T*) are shown in dual position. Analogous illustrations will be used in several other figures.

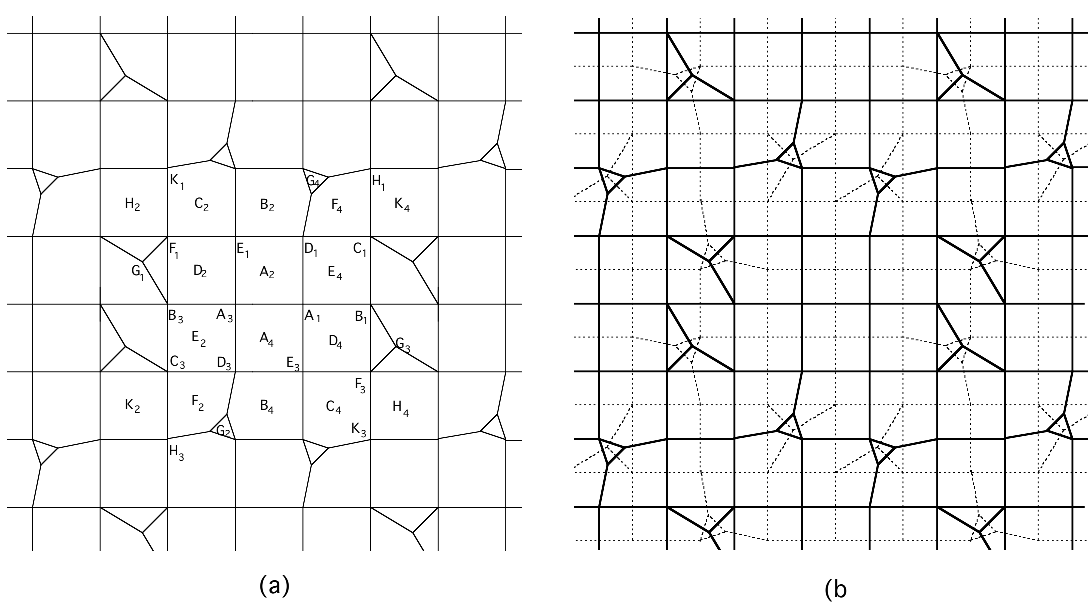


Figure 4. A selfdual tiling T of rank 4. The interpretation of the two parts of this figure is as in Figure 3, except that the subscripts are to be taken mod 4.

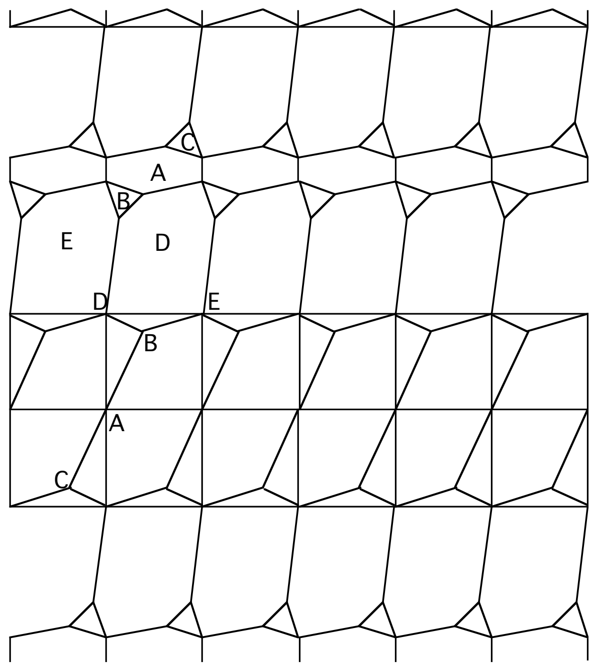


Figure 5. A selfdual tiling of rank 2. A selfduality is indicated by the balanced labelling of a few vertices and tiles. This tiling is not metrically selfdual since the small triangles (such as B and C) are too close to each other to be able to contain an isometric image of the two corresponding trivalent vertices.

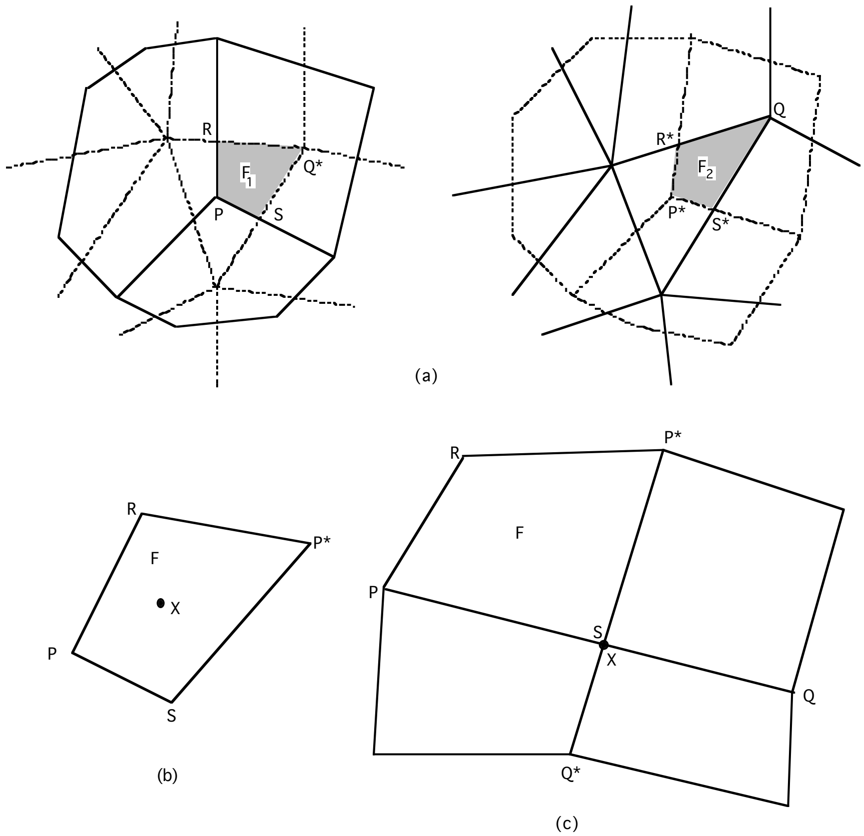
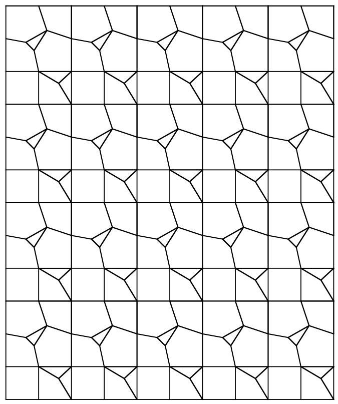
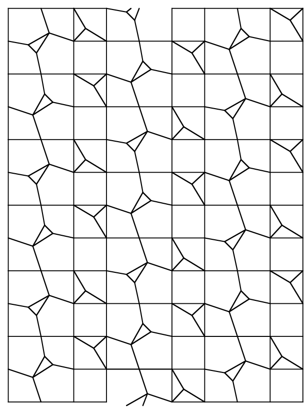


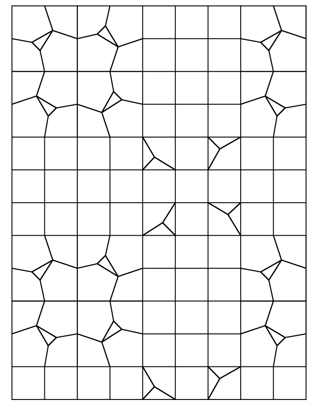
Figure 6. An illustration of the steps in the proof of Theorem 1; details are explained in the text.



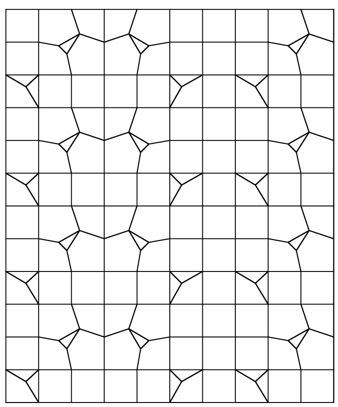
(a) p1



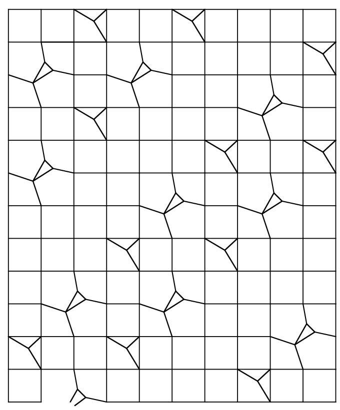
(b) p2



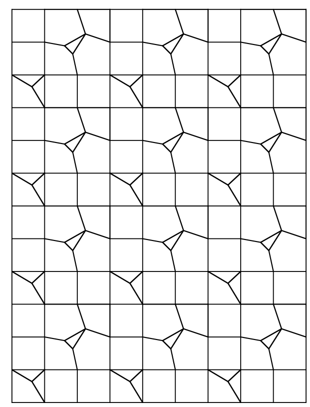
(c) p4



(d) pm

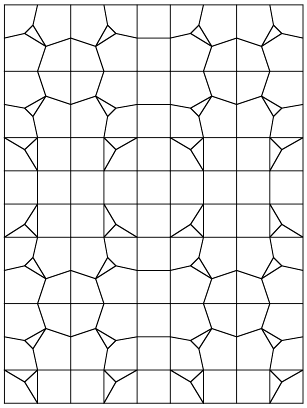


(e) pg

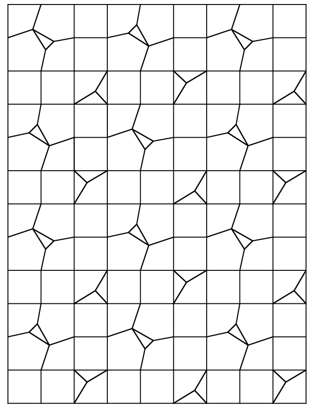


(f) cm

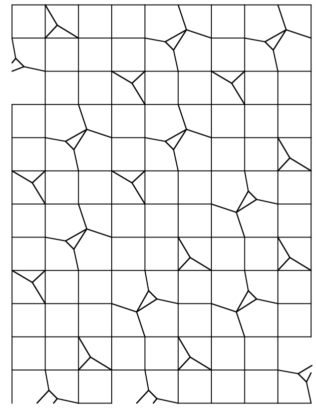
Figure 7, first part. Periodic metrically selfdual tilings of rank 2 with various symmetry groups. Except for (l), all tilings are harmonious.



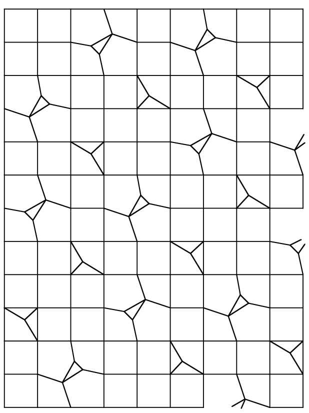
(g) pmm



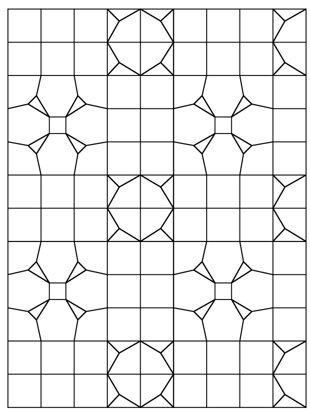
(h) pmg



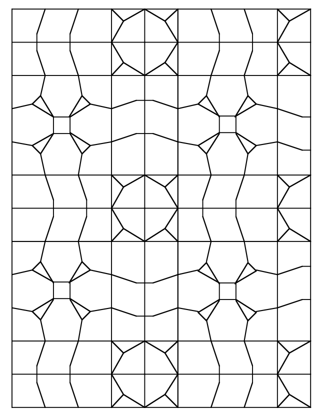
(i) pgg



(j) cmm

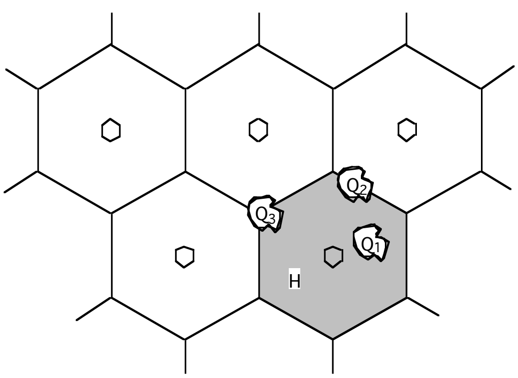


(k) p4m

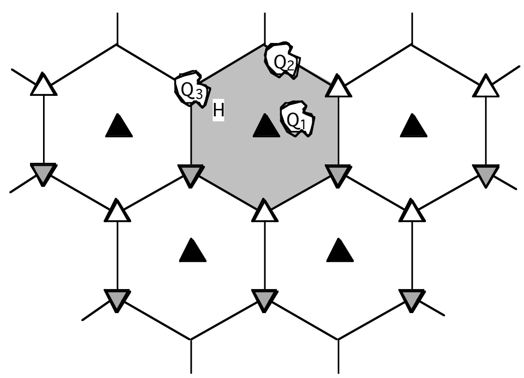


(l) p4g

Figure 7, second part.

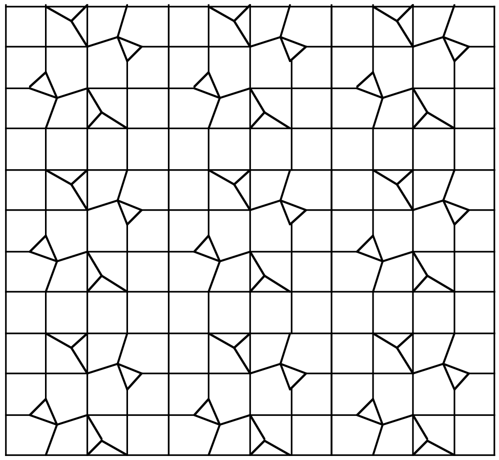


(a)

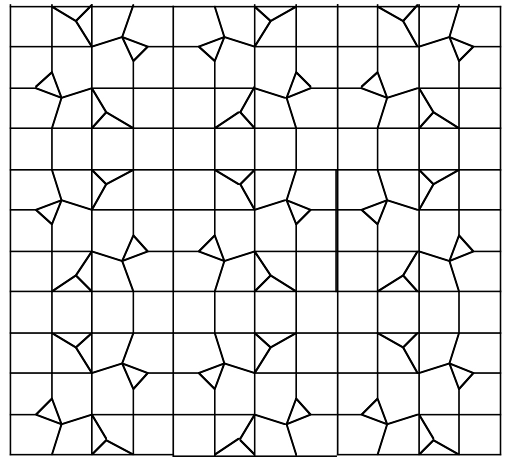


(b)

Figure 8. Illustrations of the steps in the proof of Theorem 2; details are explained in the text.

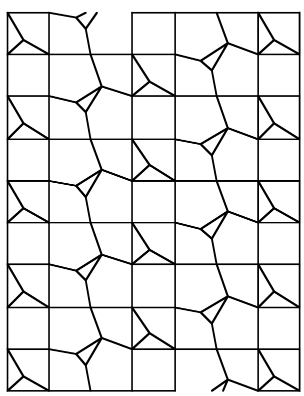


p2

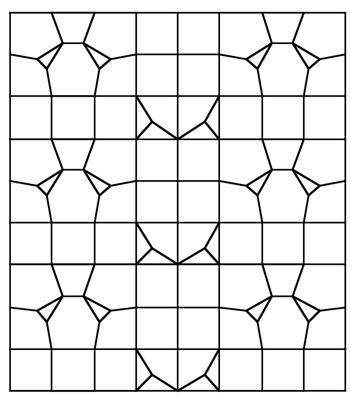


cmm

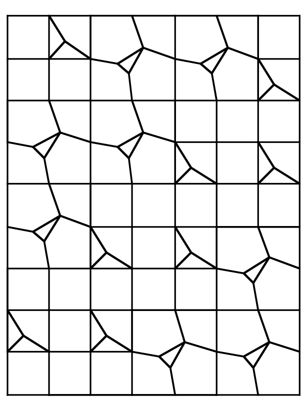
Figure 9. Periodic metrically selfdual and harmonious tilings of rank 4, with various symmetry groups.



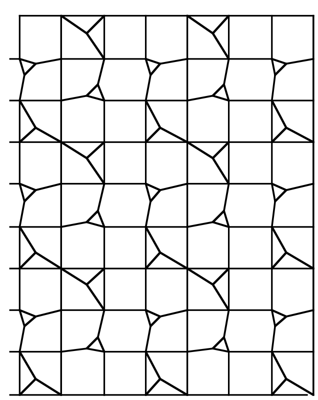
p1



pm



pg



p2

Figure 10. Periodic metrically selfdual and harmonious tilings of rank ∞, with various symmetry groups.

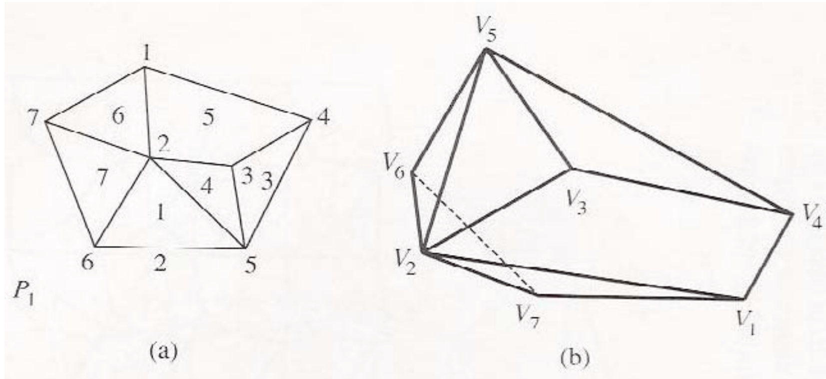


Figure 11. A Schlegel diagram (a), and a view from above (b), of a combinatorially selfdual polyhedron P_1 of rank 2, with 7 vertices and such that $A(P_1) = C_1$ and $D(P_1) = C_2$. In (a), a balanced labelling is indicated; the vertices in (b) have the coordinates given in the text.

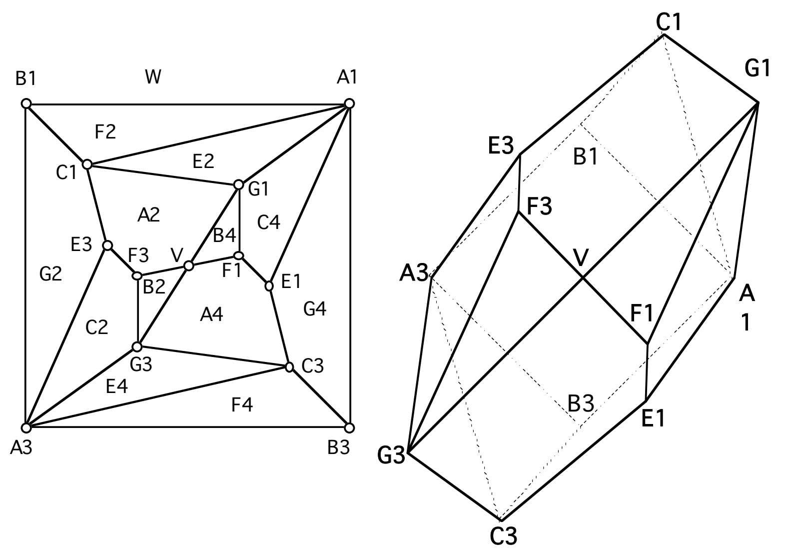
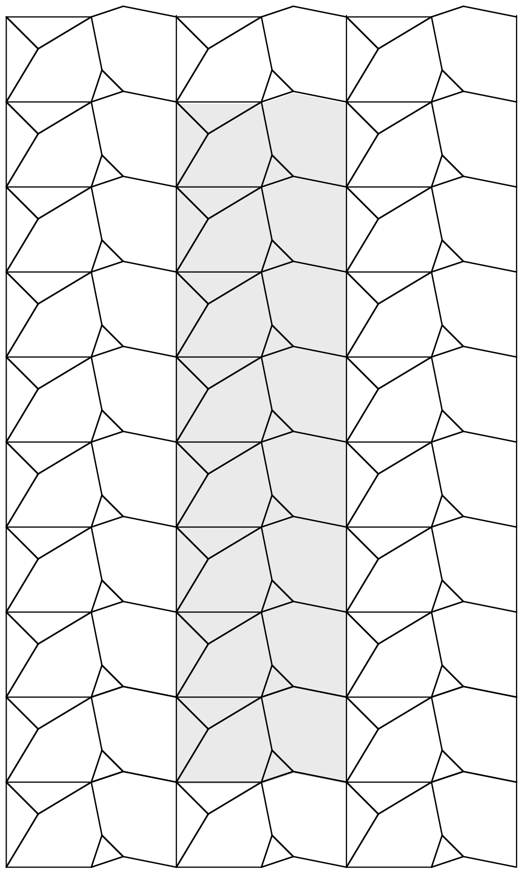
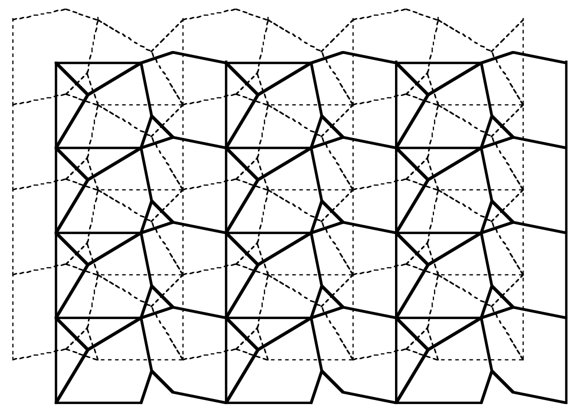


Figure 12. A Schlegel diagram (a), and a projection (b), of a combinatorially selfdual polyhedron P_2 of rank 4, with 13 vertices and such that $A(P_2) = C_2$ and $D(P_2) = C_4$.

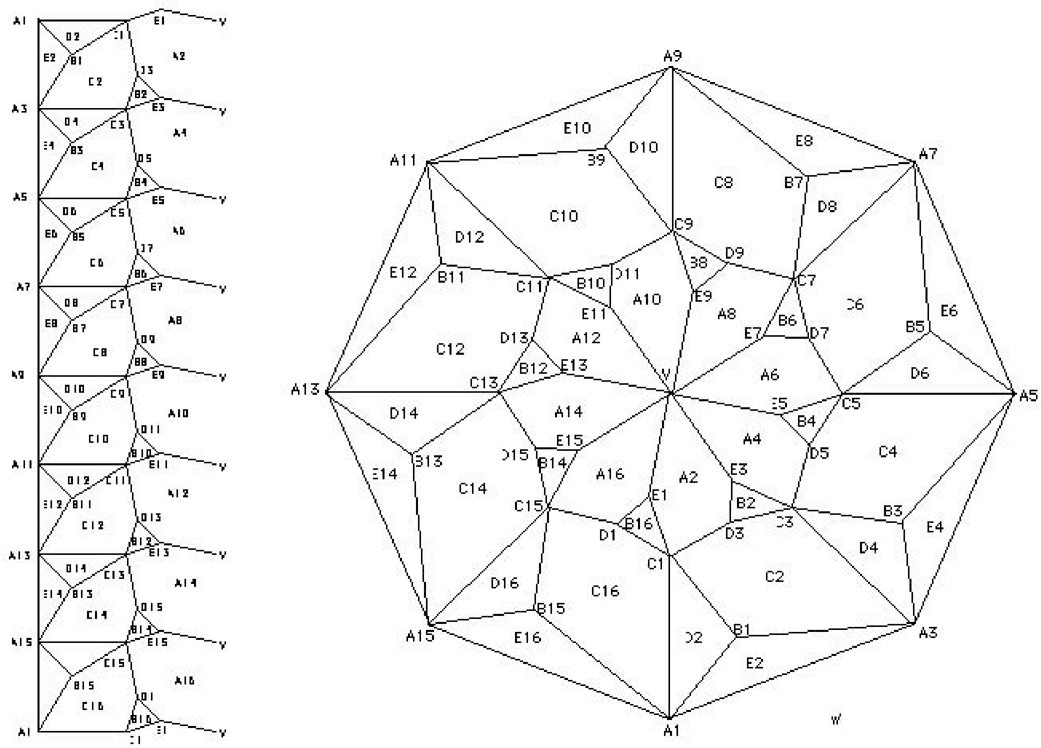


(a)



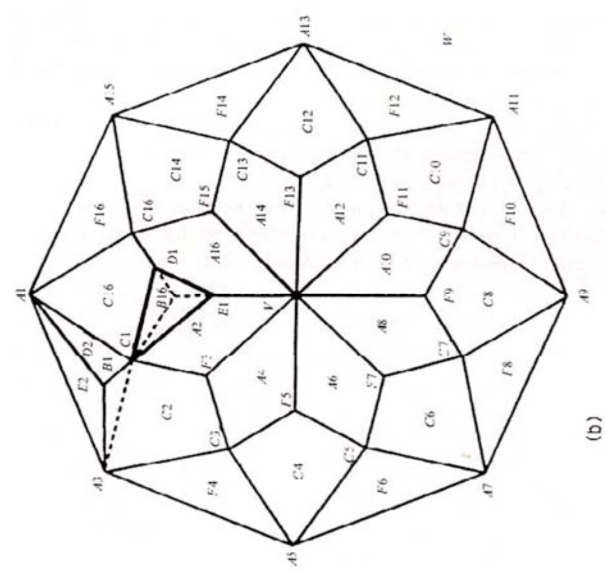
(b)

Figure 13 (first part). An illustration of the proof of Theorem 3. (a) A selfdual tiling T (of infinite rank), and a "strip" of T (indicated by shading) which is used in part (c) to construct a selfdual polyhedron. (b) The tiling T from part (a) is metrically selfdual. (c) The strip from part (a), with labels to indicate the identifications that are needed for the construction. (d) The graph formed by the identifications indicated in part (c); it can also serve as a Schlegel diagram of the polyhedron constructed in the text. The subscripts of the labels taken mod 16 indicate a selfduality of the polyhedron, which has rank 16, has automorphism group C_8 and selfduality group C_{16}.



(c) (d)

Figure 13 (second part).



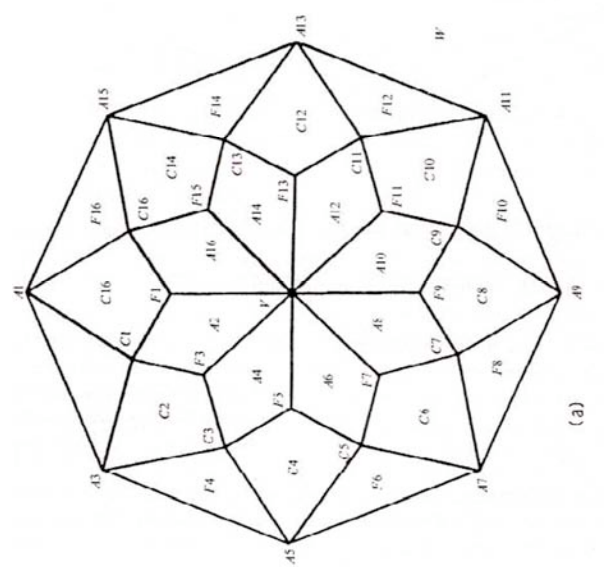


Figure 14. (a) A Schlegel diagram of the polyhedron T_n used in the proof of Theorem 4. (b) An illustration of the process of cutting of the vertices F_j , for the proof of Theorem 4. For simplicity, only one vertex F_j has been cut off.

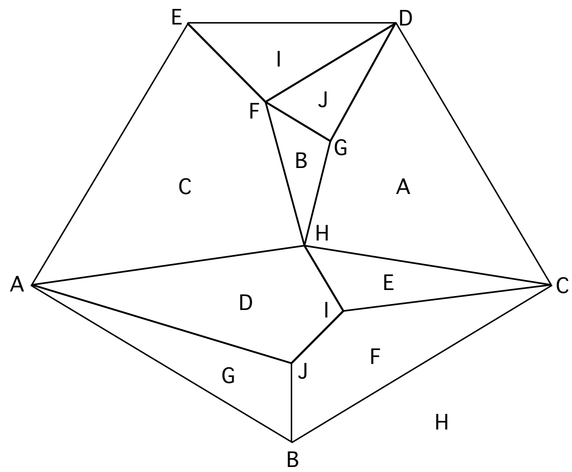


Figure 15. A Schlegel diagram of a polyhedron R_2 .

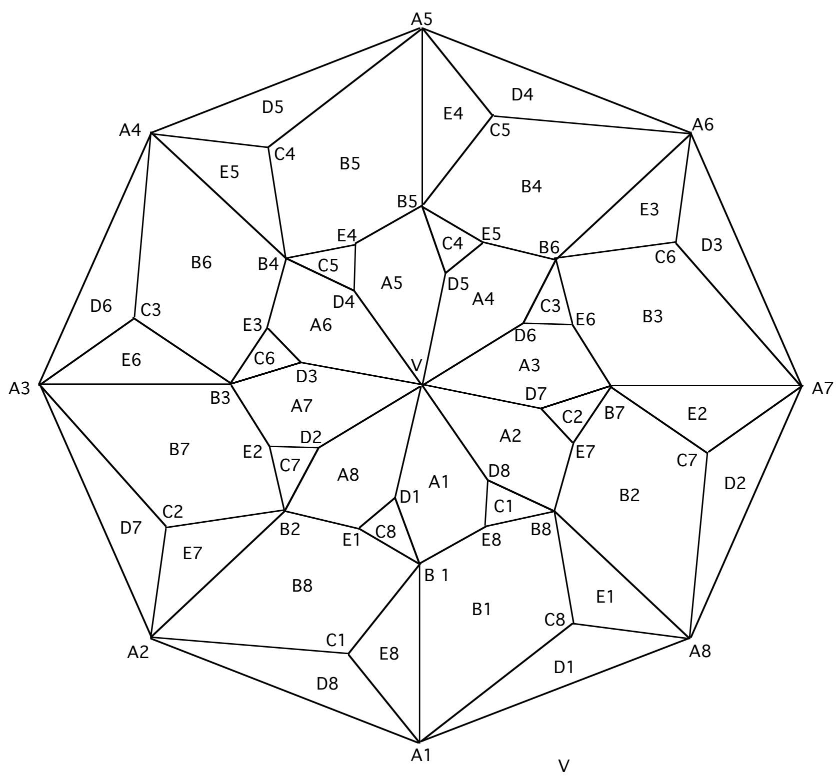


Figure 16. A Schlegel diagram of a polyhedron R_n (for n = 8) with 5n+1 vertices, such that $S(R_n) = [n]^+$ and $S(R_n^H) = [2,n]^+/[n]^+$. In this diagram balanced labelling is used.

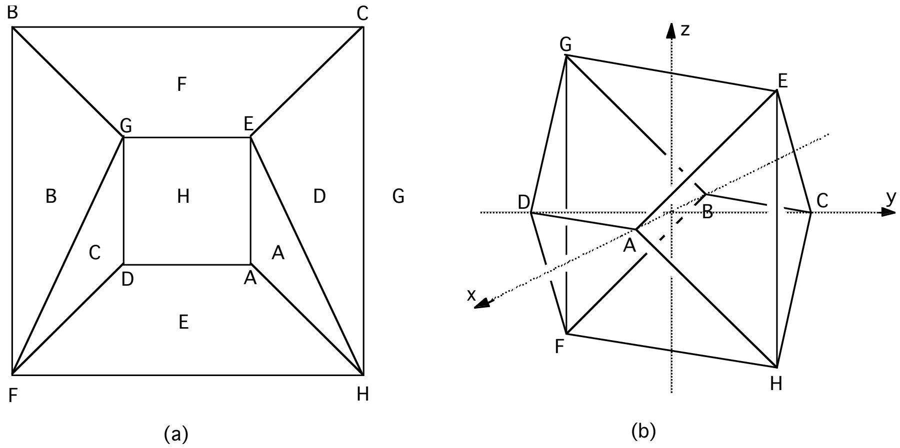


Figure 17. The selfdual centrally symmetric polyhedron C_2 of rank 2. (a) A Schlegel diagram of the polyhedron, with balanced labelling. (b) A projection of a harmoniously selfdual, centrally symmetric realization of C_2, with vertices A = (1,0,0), B = (–1,0,0), C = (0,1,0), D = (0,–1,0), E = (1,1,1), F = (–1,–1,–1), G = (–1,–1,1), H = (1,1,–1).

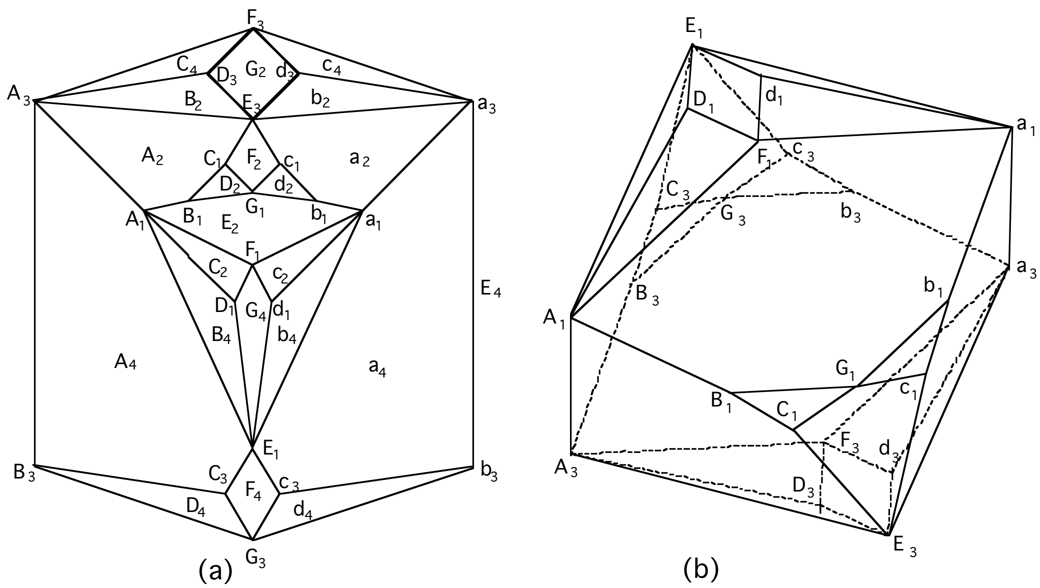


Figure 18. The selfdual centrally symmetric polyhedron C_4 of rank 4. (a) A Schlegel diagram of the polyhedron, with labels indicating a selfduality of rank 4. A capital letter with a subscript is used for each vertex, and the selfduality map increases the subscript by 1 (mod 4). (b) A projection of a centrally symmetric metrically selfdual realization, with vertices A_1 = (60,0,60), A_3 = (60,0,–60), B_1 = (30,30,45), B_3 = (30,–30,–45), C_1 = (18,42,33), C_3 = (18,–42,–33), D_1 = (10,–40,50), D_3 = (10,40,–50), E_1 = (0,–60,30), E_3 = (0,60,-30), F_1 = (0,–20,70), F_3 = (0,20,–70), G_1 = (0,40,40), G_3 = (0,–40,–40). The vertices labelled by lower case letters differ from the the ones marked with the same upper case letter by the sign of the x-coordinate, and all coordinates should be divided by 60. The isometry mapping of C_4 onto its polar C_4*, which is needed for the selfduality of rank 4, is a rotation about the x-axis that carries the z-axis to the y-axis.

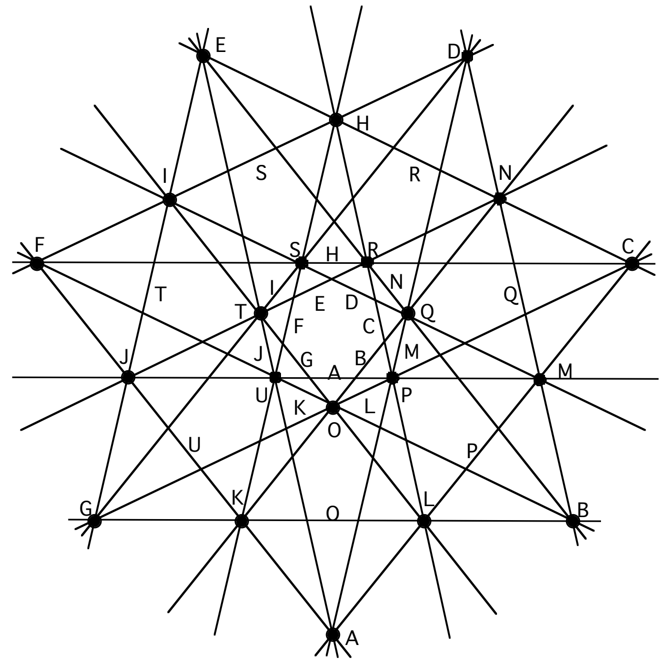
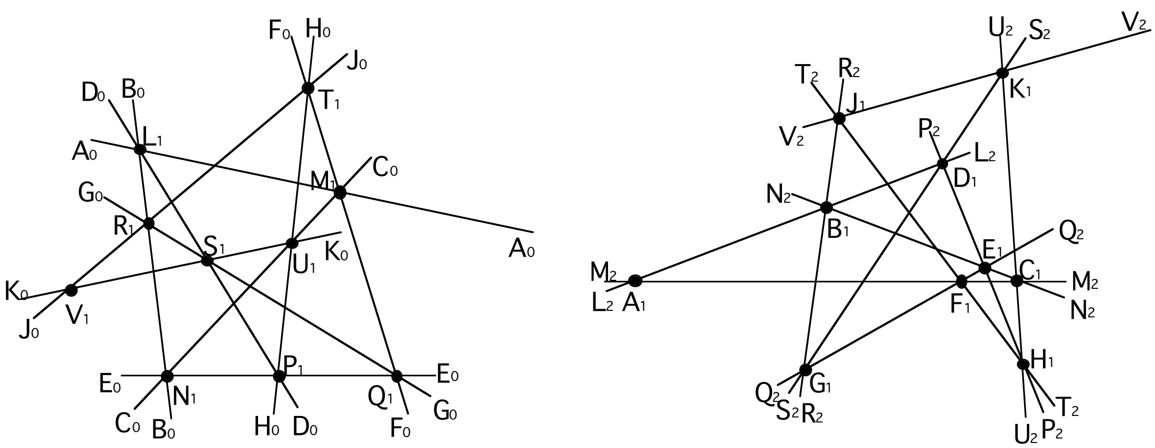


Figure 19. A selfdual configuration (21_4); a balanced labelling is shown.



(a) **F** (b) **G**

Figure 20. Fragments **F** and **G** of configurations used in the construction of the selfdual configuration described in the text.

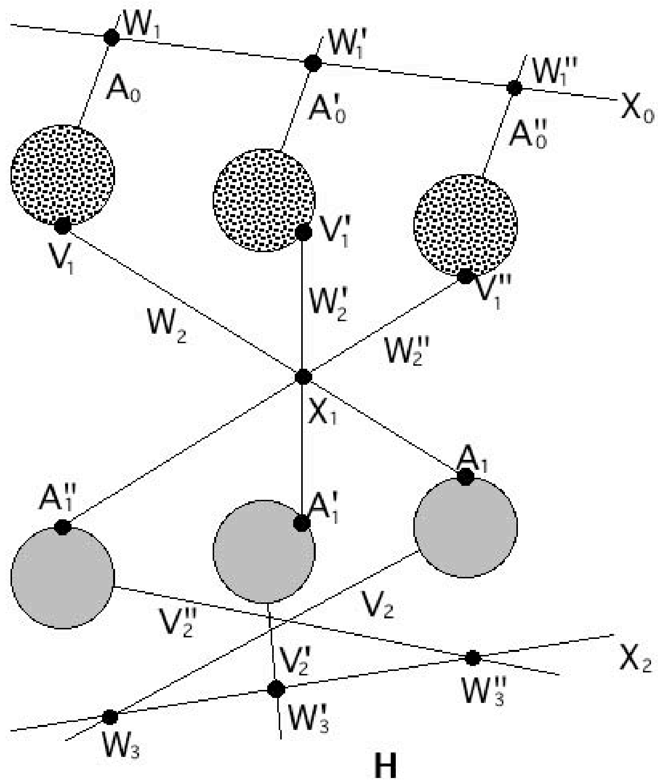
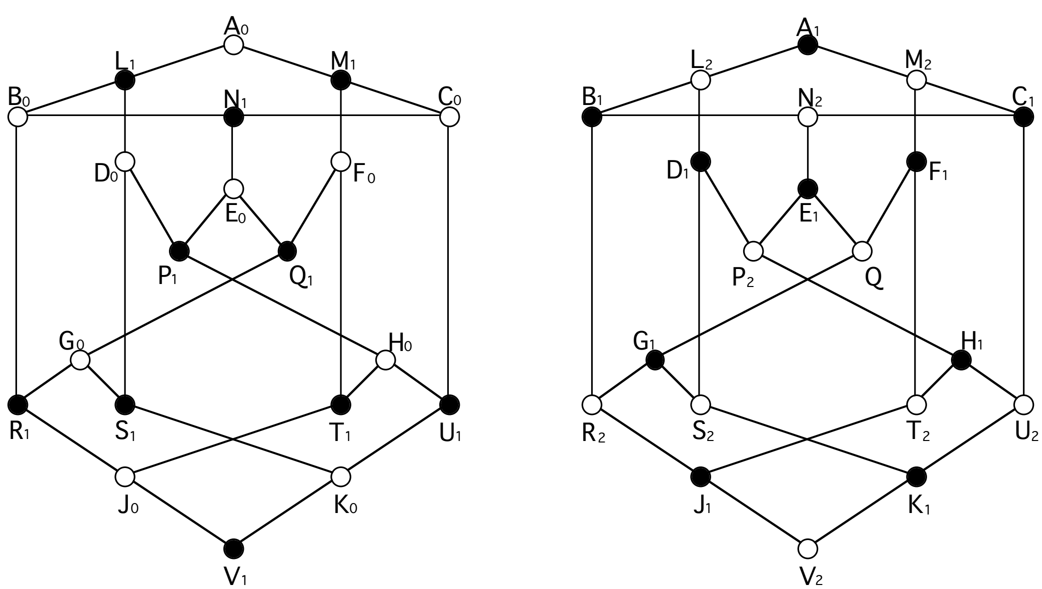


Figure 21. A larger fragment, **H**, of a configuration, used in the proof of Theorem 6.



(a) Levi-diagram of **F**.

(b) Levi-diagram of **G**.

Figure 22. Levi diagrams of the fragments **F** and **G** . The solid dots represent points, the hollow ones represent lines.

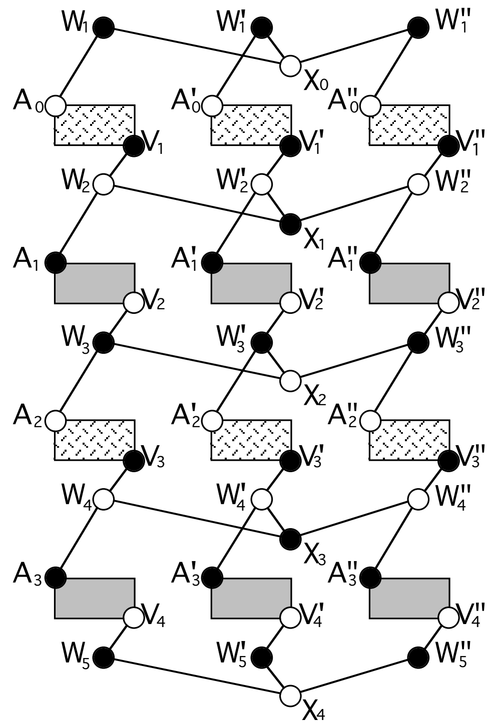


Figure 23. Levi diagram of the fragment **H**.

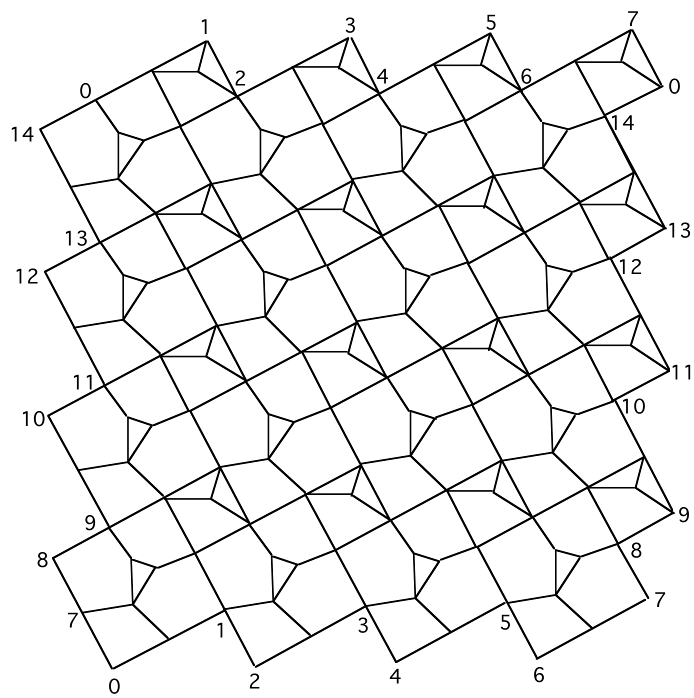


Figure 24. Suitable period parallelograms of a selfdual tiling of the plane can be used to construct selfdual tilings of the torus. Here a tiling with symmetry group p1 and infinite rank is used to define a selfdual tiling of the torus of rank 8.

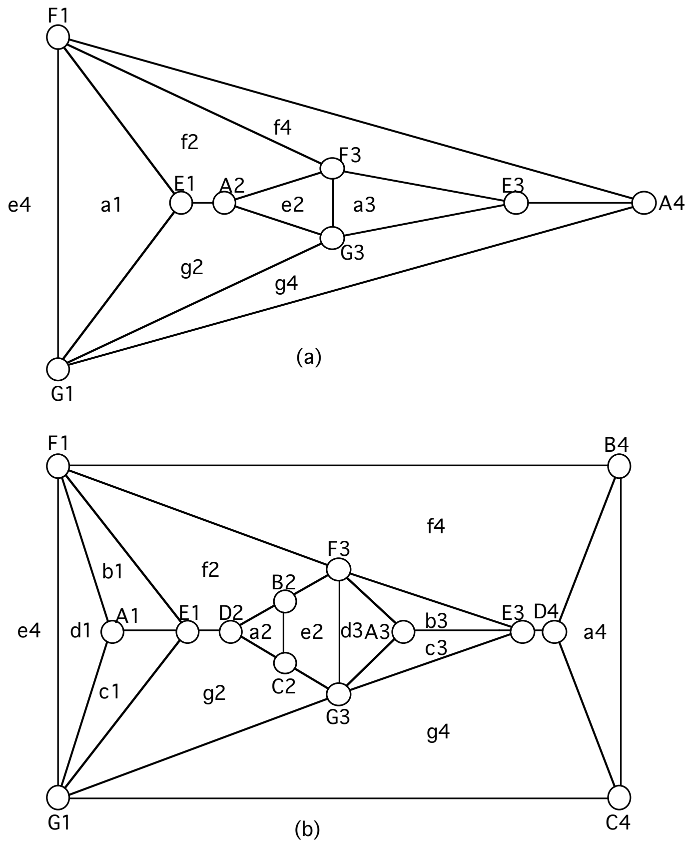


Figure 25. (a) Another Schlegel diagram of the selfdual polyhedron C_2 of Theorem 4, and a selfduality of C_2 rank 4, which increases all subscripts by 1 (mod 4). (b) The selfdual polyhedron of rank 4 described by Jendrol [1989] can be obtained from the selfdual polyhedron in part (a) by subdividing the faces a_1 and a_3, and simultaneously truncating the vertices A_2 and A_4.