# Idiot-proof tiles

BRANKO GRÜNBAUM* AND G. C. SHEPHARD

MacKinnon [4] introduced the interesting concept of an idiot-proof tile. This is a polyomino-shaped tile $P$ which has the property that wherever two copies are placed ("by an idiot") on the square grid, then—so long as the copies do not overlap or enclose a region ("even an idiot would not do that")—it is possible to adjoin further copies of $P$ so as to complete a tiling of the whole plane. MacKinnon proved that the L-triomino is an idiot-proof tile, and indicated how one could prove that the W-pentomino has the same property. The purpose of this note is to extend MacKinnon's results, and to formulate some open problems.

To begin with it is helpful to refine the concept slightly. Let us say that a polyomino $P$ is $k$-idiot proof ($k$-*IP*) if

(i) given any $k$ copies of $P$ on the square grid (which do not overlap or enclose a region) it is always possible to find a tiling of the whole plane by copies of $P$, incorporating the given ones, and

(ii) there exists an arrangement of $k+1$ copies of $P$ on the square grid which cannot be incorporated in any tiling of the plane by copies of $P$.

Thus MacKinnon's definition corresponds to that of a $k$-*IP* tile with $k \geqslant 2$. Conventionally we say that $P$ is 0-*IP* if these exists no tiling of the plane by copies of $P$.

The first, and obvious, question is this: what values of $k$ are possible? Defining a *straight n-omino* as consisting of $n$ collinear squares fused together, and a *zigzag polyomino* as the natural generalization of the L-triomino and W-pentomino to polyominoes of arbitrary sizes, we can state our results as a pair of theorems:

THEOREM 1. Every straight $n$-omino is 3-*IP* if $n \geqslant 3$, and is 4-*IP* if $n = 2$ (the domino).

THEOREM 2. Every zigzag $n$-omino ($n \geqslant 3$) is 2-*IP*.

We conjecture that every polyomino, other than those mentioned in the theorems, is either 1-*IP* or 0-*IP*. Supporting evidence for the truth of this conjecture is provided by the following diagrams which show how two copies of each kind of tetromino or pentomino can be arranged so that the tiling cannot be completed. (This also shows that the $T$-tetromino is 1-*IP*, contrary to the statement in [4]).

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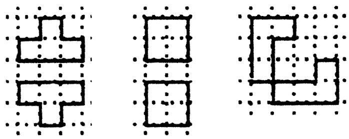
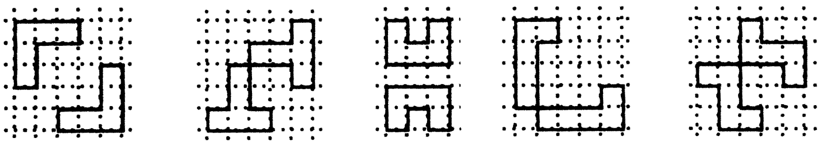
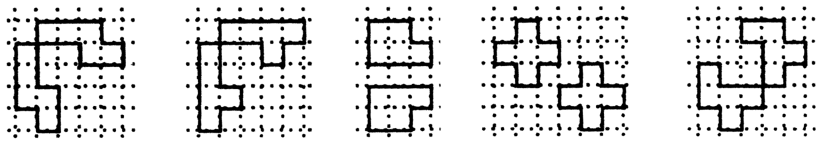
*This really is the limit*

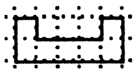
"Erratum: *The treatment of limits in the calculus: are we becoming careless?* The publishers regret that the paper was published without the following corrections and additions . . ." From the *Int. J. Math. Educ. Sci. Technol.* 1990, Vol. 21, no. 6, sent in by S. Brian Edgar.



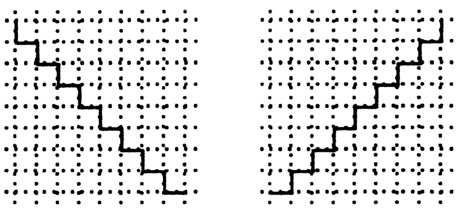




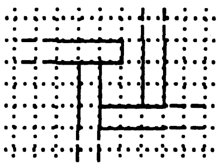
Similar arrangements are possible for the 33 hexominoes different from the straight and the zigzag. In fact, the same is true for each $n$-omino with $n \geqslant 7$ we tested, but we know of no method to prove the conjecture in general. As can be seen from the data in Table 9.4.2 of [3], the smallest value of $n$ for which an $n$-omino is 0-*IP* is 7, since—as the reader may like to verify—the tile below is 0-*IP*, and all hexominoes are known to be $k$-*IP* for $k = 1$, 2 or 3.



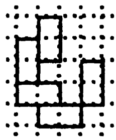
The proofs of the theorems follow similar lines to those suggested by MacKinnon. By a *grid line* we mean any horizontal or vertical line in the grid, and by a *grid zigzag* we mean a stepped line running diagonally from top left to bottom right, or from top right to bottom left, like this.



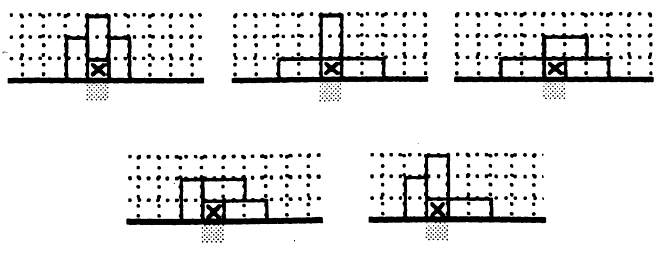
For Theorem 1 we remark that wherever three straight $n$-ominoes $P$ are given, the plane can be divided by a grid line into two half planes, one containing one copy of $P$ and the other containing two copies of $P$. It is then almost trivial to see how the tiling of each half-plane can be carried out using further copies of $P$. If $n \geqslant 3$ then four copies of $P$ arranged as shown below cannot be part of a tiling of the plane, hence proving the first assertion of the theorem.



For $n = 2$ we remark that this set of five dominoes cannot be continued to a tiling of the plane:

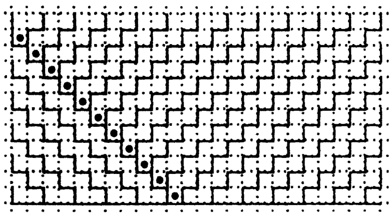


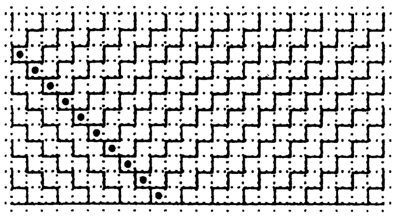
We deduce that the domino is $k$-*IP* with $k \leqslant 4$. If we take any four dominoes that neither overlap nor enclose a region, either it is possible to separate them by a grid line into two pairs, or one can be separated from the other three. If a half-plane contains one or two dominoes then, as above, it is possible to complete a tiling of the half-plane by further dominoes. Hence the only case to be resolved is that in which there are three dominoes in a half-plane $H$. Again, a tiling of $H$ can be trivially completed unless the three dominoes form an arrangement in which they separate a single square on the boundary of $H$ from the rest of $H$. This can happen in any of the five ways shown below (or their mirror images). In each case it is easily seen that all of $H$ except the square marked $x$ can be tiled by further dominoes; since the shaded square cannot be occupied by the single domino contained in the half-plane $H^*$ complementary to $H$, the tiling of the plane can be complemented by adding the domino covering $x$ and the shaded square, and then completing the tiling of $H^*$. Thus the domino is 4-*IP*, and Theorem 1 is proved.



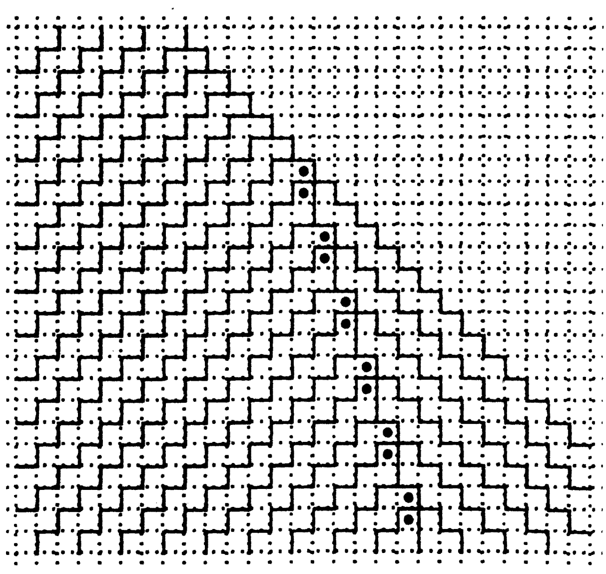
The proof of Theorem 2 uses the zigzag constructions shown in the following diagrams, and the observation that a zigzag strip, infinite in one direction, can be tiled by copies of any zigzag polyomino $Z$. If two copies of $Z$

are in such positions that they can be separated by a grid line, then each half-plane defined by the grid line can be tiled by copies of $Z$—in fact, by infinite zigzags, as indicated below. Moreover, this can be done regardless of the position of $Z$ relative to the grid line. (In this diagram and the following one, the solid dots indicate the various positions that the zigzag may have relative to the boundary of the halfplane.)

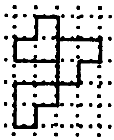


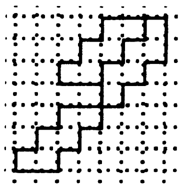


If the two copies of $Z$ cannot be separated by a grid line, then they can be separated by a grid zigzag, and the tilings of the half planes defined by this zigzag can be completed using the method indicated below.



Finally, we remark that for all $n \geqslant 3$ there exist arrangements of three $n$-zigzags which cannot be continued to tilings of the whole plane. The arrangements for $n = 3$ and $n = 6$ show how this may be done. This completes the proof of Theorem 2.





The above illustrates the fact that polyominoes (and the related polyiamonds and polyhexes) have many interesting properties, and there remain several unsolved problems in this area. So far as we are aware, nothing is known about idiot-proof polyiamonds and polyhexes, and it would seem worth investigating whether analogues of our results hold in these cases. For further information on polyominoes one should consult the remarkable book [2], or Section 9.4 of [3]. Analogous questions can be posed for *n-blocks* (each consisting of $n$ squares from a grid, not necessarily connected). It is known that each 4-block is *k-IP* for some $k \geqslant 1$ (see [1], or Section 12.2 of [3]), which implies that each 2-block is *k-IP* for some $k \geqslant 2$.

*References*

1. D. Coppersmith, Each four-celled animal tiles the plane. *J. Comb. Theory* A40 (1985), 444–449.
2. S. W. Golomb, *Polyominoes*, Allen & Unwin, London, 1966.
3. B. Grünbaum and G. C. Shephard, *Tilings and Patterns*, Freeman, New York, 1987.
4. N. MacKinnon, Some thoughts on polyomino tiles. *Math. Gaz.* 74 (1990), 31–33.

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# The best shape for a tin can

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Some time ago, I came across a book intended to popularise mathematics, whose last chapter dealt with the calculus of one variable. Its final section, evidently intended to climax the whole work, solved the problem of designing the proportions of a tin can so as to obtain the greatest volume out of a given