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A MEASURE OF ASYMMETRY FOR CONVEX SURFACES *

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This note gives a proof for a bound of an asymmetry measure of convex sets in E^n that was conjectured by G. v. Sz. NAGY [2] and proved by him for $n=2$ and 3.

Let K be a compact convex body in E^n and let c be the centroid of its surface, i. e., of a mass distributed uniformly over the $(n-1)$-dimensional surface of K. Take two parallel hyperplanes and consider the ratio of the distance from c to the hyperplane that lies farthest away from it and the distance from c to the other hyperplane. The maximum of this ratio as the hyperplanes take all possible directions will be called A(K). This is evidently a measure of the asymmetry of K. It takes the value 1 only for symmetric bodies. In the other direction it is bounded by the number $2n-1$. This bound is never attained except for $n=1$, but there are bodies that have an asymmetry measure arbitrarily close to it. We enunciate the result in the following theorem. (The corresponding problem for the volume was solved by MINKOWSKI (see BONNESEN-FENCHEL [1], pp. 52-53); the bounds in this case are 1 and n).

THEOREM. $1 \leqq A(K) < 2n-1$ $(n \geqq 2)$. $A(K)=1$ *if and only if* K *is a symmetric body.*

We will make the proof of the theorem depend on a lemma. Suppose we have two parallel hyperplanes supporting K on each side. Let these hyperplanes be the coordinate planes $x=0$ and $x=1$ in some coordinate system, and let $\mu(t)$ denote the

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part of the surface mass of the body that lies in the closed slab $0 \leqq x \leqq t$, the function μ being normalized so that $\mu(1) = 1$.

LEMMA. $\mu(t) > \frac{1}{2} t^{n-1}$ *for* $0 < t < 1$ $(n \geqq 2)$.

We first prove the theorem with the help of the lemma. The distance from c to the plane $x = 1$ is

$$\int_0^1 (1-t)\, d\mu(t) = \int_0^1 \mu(t)\, dt > \frac{1}{2n}$$

measured in the coordinate unit. Hence the corresponding ratio is less than $2n - 1$. However, by compactness, there must exist some pair of hyperplanes for which the ratio is the measure of asymmetry. Thus we have proved that $A(K) < 2n - 1$. But it is an elementary matter to verify that for cones with base in the plane $x = 1$ and vertex at $x = 0$ one has $\mu(t) = \left(\frac{1}{2} + \varepsilon\right) t^{n-1}$, $t < 1$, where one can get as small a positive value for ε as one may want. This finishes the proof of the theorem.

In the proof of the lemma we use one result from the general theory of convex bodies, namely that if one convex body contains an other convex body, then the ($(n-1)$-dimensional) surface measure of the former is greater than or equal to that of the latter (see BONNESEN-FENCHEL [1], p. 47). We arrange our convex body K as before between the two hyperplanes $x = 0$ and $x = 1$ and we consider its intersection with the hyperplane $x = t$. Now consider a cone which is generated by the rays from this intersection to some point where $x = 0$ supports K. Denote the intersection of this cone by the slab $0 \leqq x \leqq 1$ with C and the corresponding normalized surface measure by $\nu(t)$. To get the real surface measure comparable to $\mu(t)$ for K, we may have to multiply by some constant $k > 0$. Now we use the result on convex bodies referred to above and get

$$\mu(t) \geqq k\nu(t), \quad k(1 - \nu(t)) \geqq 1 - \mu(t)$$

from which we readily deduce

$$\mu(t) \geqq \nu(t) > \frac{1}{2} t^{n-1}$$

the last inequality according to a remark in the proof of the theorem. Hence the lemma is proved.

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[1] T. Bonnesen und W. Fenchel, *Theorie der konvexen Körper*. Berlin 1934.

[2] Gy. v. Sz. Nagy, *Schwerpunkt von konvexen Kurven und von konvexen Flächen*. Portug. Math. **8**, 1949, p. 17-22.