Shouldn't We Teach **GEOMETRY?**

Branko Grünbaum

The point of view stressed by Dieudonné sees geometry in almost all currently active and important branches of mathematics. Naturally, from such a point of view, we teach geometry whenever we teach mathematics. This view goes far back—was it not customary to call every mathematician "geometer"?—but somehow, I cannot find much solace or joy in this view. It seems to me that to call all the activities described by Dieudonné by the name "geometry" is rather analogous to—and as justified as—calling the activities of a French chef "agriculture". Indeed, in both cases much love, thought and effort is given to specially chosen and suitable objects (meats, vegetables, theorems) while converting them into delicacies for the true connoisseurs. And in both cases, many of the "raw materials" must be wasted if the only destiny deemed worthwhile for them is the transformation into gourmet dishes, or highly abstract mathematics.

Dropping the simile, it seems to me that the situation of geometry in colleges and universities is much less satisfactory than the opinions of my colleagues might have led you to believe. To justify this assertion, I should probably first give a brief explanation of what I mean by "geometry". A good point of departure is provided by the illustration in Figure 1, which I found in a card store a short time ago. I hope that it conveys to you the same idea it conveys to me—that geometry is a special way of replacing objects of the real world by "simpler", "idealized" "figures and shapes", and then investigating the mutual relations of these. This, I believe, is what geometry—"intuitive geometry"—is, and should be. Geometry arose in such manner and—as I hope to elaborate shortly—this is how meaningful and interesting questions continue to arise. In Greek times this process of abstraction led to the points, segments, circles, triangles and all the other objects of Euclidean geometry, and in the course of its development gave the ancients the opportunity to create the deductive system we know as Euclidean geometry.[(1)]

Unfortunately, in later times this great achievement of the ancients was perverted to a religion. And even the Age of Reason did not dispel the superstition of the divine or *a priori* root of Euclidean geometry until about 150 years ago for the most "radical" among mathematicians,—and not at all, to this day, for most mathematicians and laymen alike. Very few people are conscious of the fact that all geometry —and Euclidean geometry in particular—is a product of *our* thinking and represents just one of the ways in which we try to communicate about our surroundings and understand certain aspects of reality.

Anybody who thinks he can "see," or find in nature, a Euclidean point (or a straight line, or a differentiable curve) needs urgent medical attention.

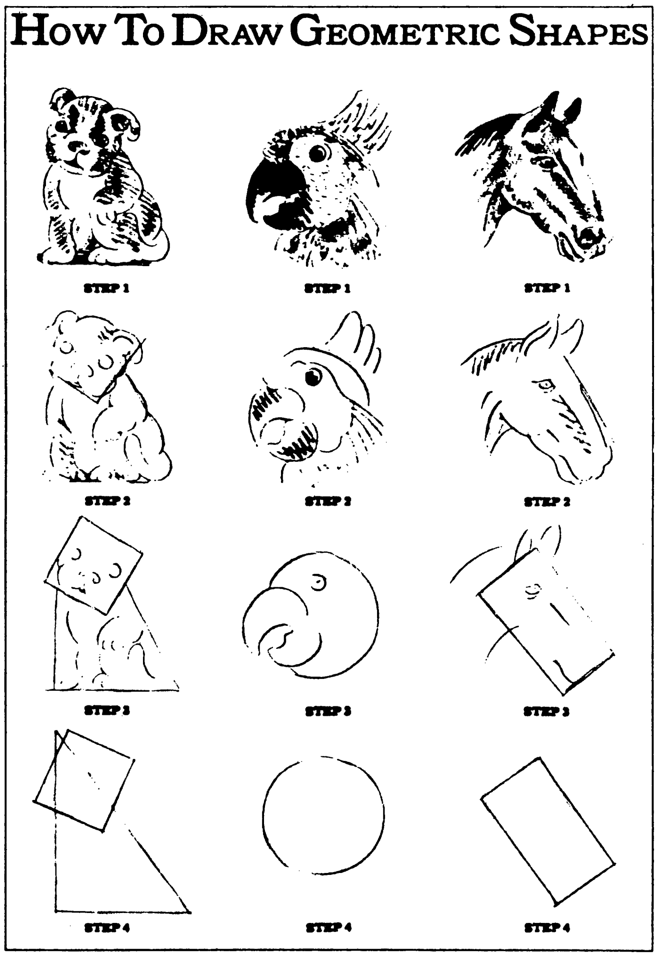

Figure 1

Geometry is one of the few approaches our senses give us to the outside world. Ever since Euclid (or even before) some "experimental" discovery—when idealized, generalized and digested—led to certain geometric results or problems which, in turn, often led to far-reaching developments in geometric or other directions. Whether it is the diagonal of the square leading to irrationals; or the smoothness and continuity of curves to derivatives; or the renewed interest in "foundations" to "non-Euclidean geometries" and to the axiomatic method; or—more recently—properties of convex polyhedra (arising in linear programming) to a vast field stretching from combinatorics to commutative algebra; or any other of the myriad of analogous cases—it is always some relatively simple, easily imaginable, intuitive context which generates the whole field, often so great in extent that the roots become hard to discern.

Most mathematicians are totally unaware of the fact that the elementary, intuitive approach to geometry continues (and will continue) to generate mathematically profound and interesting problems and results. There is no argument against the assertion that all questions in the geometry of points and straight lines in the plane are reducible to (and possibly even solvable by) linear algebra. But many attractive and genuine problems become, under such a transformation, artless conglomerations of symbols and provide no hint as to why one would ever want to spend even a moment thinking about them. A simple example will surely persuade you of the justification of this claim, if you attempt to carry out the translation into linear algebra: If S is a finite set of points in the (real Euclidean) plane, then there is a straight line containing all points of S or there is a straight line containing two and only two points of S. There is no completely trivial geometric proof of this fact (which is known as Sylvester's theorem, although Sylvester did not manage to prove it); but I am certain that you will not be able to come up with an algebraic proof which is simpler than the geometric ones.[(2)]

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The opinion that the interest in visual-geometric aspects of a topic is lost or negligible just because the topic can be translated into algebraic language is as fallacious as the (long prevailing) view that if a problem is solvable in a finite number of steps, further consideration must be boring—the whole challenging field of computational complexity refutes such a stance.

It is a rather unfortunate fact that much of the creative introduction of new geometric ideas is done by non-mathematicians, who encounter geometric problems in the course of their professional activities. Not finding the solution in the mathematical literature, and often not finding even a sympathetic ear among mathematicians, they proceed to develop their solutions as best they can and publish their results in the journals of their disciplines. Crystallography is a prime example of this phenomenon, but other examples range from physics and biology to geography and anthropology, from industrial design to art. All these lead to very challenging mathematical problems—more specifically, to intuitively accessible geometric problems—which could provide a fertile ground for the development of mathematical theories.[(3)]

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But what is happening instead, and why do I see geometry in colleges and universities in a very gloomy light? The simple answer is that we teach very little geometry, and that what we do teach is rather misleading. On the one hand, we often follow "tradition" and pretend to be teaching the "classical" Euclidean geometry, holding it up as an example of a logically perfect deductive science. But we all know that this is intellectually dishonest and mathematically impractical. This led to Dieudonné's famous slogan, "Down with Euclid." Unfortunately, this slogan was generally understood as a call for eliminating all elementary geometry, and Dieudonné's insistence on teaching linear algebra in high school as a substitute for geometry did not help matters, either.

On the other hand, many of the courses we teach and which have "geometry" in their name, are good examples of the "rule of the tool" syndrome: A certain technique is found to be applicable to geometric problems of some kind; soon the interest switches to development of the technique, and then—by actions if not by words—only those geometric problems are deemed important which are tractable by the (by now well-developed) technique. For example, the applicability of calculus to some questions in the theory of curves and surfaces led to differential geometry, with all its modern ramifications; simple observations concerning the numbers of points in which lines, circles, ellipses and other simple curves in the plane intersect each other led to the towering generalizations of contemporary algebraic geometry. Now, I have nothing against C_∞ manifolds, or p-adic fields, or any other developments in these areas—provided the narrow applicability of these disciplines and their very special preconditions are acknowledged and admitted, and the viability and validity of other points of view is granted. But this is where, in practice, the system breaks down. Even if (as is often the case) the teacher is aware of various other aspects of the geometric objects he is exploring, the students in his course on, say, manifolds learn only very few (usually rather trivial) background facts and examples. The results are similar to those that would be obtained in studying the Iliad in the original by somebody who knows only 200 words of Greek, and learns no more. Possibly, he could get the general drift of the action; certainly, he could master all the details of the meter, and other superficial aspects. Would it then be very surprising if—when he became teacher—the erstwhile student were to put undue stress on the meter in the Iliad?

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What is missing from our curricula in colleges (as well as in high schools) is the accent on the many aspects of reality which are susceptible of meaningful geometric interpretation. Equally absent are the elementary geometric considerations which—besides intrinsic interest—can contribute in very significant ways to the students' comprehension of topics in other sciences, in engineering and in art, as much as help guide future progress in mathematics. As Dieudonné has said, geometry is a great tool, a powerful stimulus, and a trustworthy guide. However, geometry can render all these services only to people who know some geometry. Disparaging the importance of the visual, intuitive—even tactile—aspects of geometry and urging their replacement by tool-oriented techniques certainly will not make the future role of geometry any easier. Such an attitude is inherently as absurd as the promotion of soundless music, or verbal rendering of paintings. Just as nobody argues against writing music—and a few fortunate individuals can even enjoy and "hear" music in this form—would the act of writing music not be utterly ridiculous if the composer were oblivious to (or even ignorant of) sounds? Figure 2 shows a "score" for some movements in a ballet (in one of the systems of notation); how much of the excitement of the real thing does it convey to you? But such replacement of visually accessible information by formal symbols is precisely what most mathematicians are inflicting upon themselves and upon their students regarding geometry.

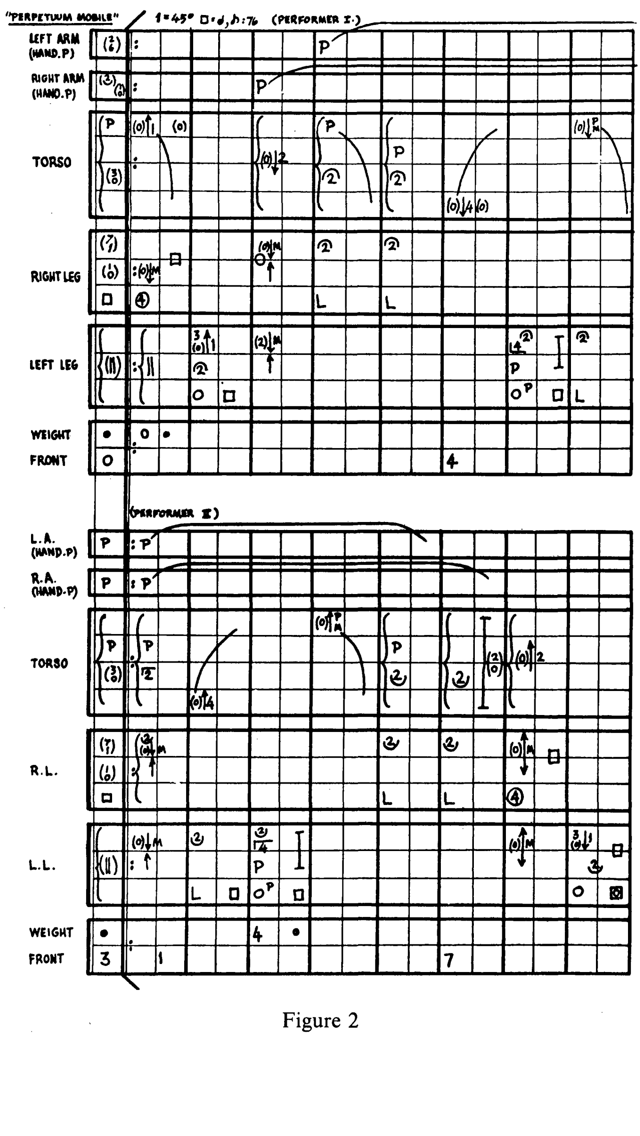

Figure 2

I believe that—at least in the United States—we are at the moment caught in a quandary from which there is no easy exit and for which the very existence of a satisfactory solution is rather uncertain.

On the one side it must be noted by anybody willing to give the matter even minimal attention, that an understanding of the spatial figures and shapes, and their mutual relationships, is becoming increasingly important to scientists as well as to engineers, and certainly to mathematicians. Even the three-dimensional space of everyday experience is much more complicated than we mostly give it credit for; we hardly stand a chance of understanding any of the more complicated "spaces" needed in fields ranging from astronomy to nuclear physics, from biology to art, from space travel to highway design,—if we are as ignorant of the ordinary geometry as most of us are.

I believe that geometry "works" so well for the rest of mathematics and for other fields because the "seeing" aspect of it coevolved with our other mental capabilities ever since we descended from the trees (or possibly even longer). But whether this is so or not—why deny to our students both the pleasure and the utility of exploring the figures and shapes of our space?

On the other side, I am afraid that the proponents of "intuitive geometry" among mathematicians are so few in number and so dispersed and unorganized that they fall well below the "critical mass" needed to sustain any development and changes in attitudes. There seems to be no doubt that intuitive geometry will continue to develop—the question is whether it will find a place among mathematical disciplines, or will it continue to flourish mainly among non-mathematicians.

Even the well-intentioned college professor can hardly be expected to be able (or willing) to develop by himself a suitable course on a topic with which he is largely unfamiliar and for which hardly any texts are available; therefore additional generations of students grow up with the same distorted perspectives. And it has to be admitted that there is very little motivation for potential authors to write (and for publishers to publish) books for which at the moment there seems to be almost no demand.

Clearly, if any solutions are to develop they can come about only through the dedicated work of many individuals. Let's hope that the discussions of the panel have contributed to the understanding of the problem, and will help motivate people to an increased effort towards its solution.

NOTES

1. For a very lively discussion of the nature of Greek geometry, see the articles by S. Unguru ("On the need to rewrite the history of Greek mathematics," Archive for History of Exact Sciences 15(1975/76), pp. 67–114, and "Some reflections on the state of the art," Isis 70(1979), pp. 555–565), B. L. van der Waerden ("Defence of a "shocking" point of view," Archive for History of Exact Sciences 15(1975/76), pp. 199–210), H. Freudenthal (*ibid.* 16(1976/77), pp. 189–200), and A. Weil ("Who betrayed Euclid," *ibid.* 19(1978), pp. 91–93).
2. Relatively simple proofs of Sylvester's theorem are presented in H. S. M. Coxeter's book "Introduction to Geometry" (Wiley, New York 1969). For the history and for references to the abundant literature, as well as for other topics of a similar nature, see the author's "Arrangements and Spreads" (CBMS Regional Conference Series in Mathematics, Number 10, American Mathematical Society 1972).

3. G. C. Shephard and the author have written several survey articles and a book illustrating these facts. The two most recent papers have just been published; they are "A hierarchy of classification methods for patterns" (Zeitschrift für Kristallographie 154(1981), 163–187) and "Tilings with congruent tiles" (Bulletin of the American Mathematical Society N.S. 3(1980), 951–973). The book "Tilings and Patterns" is being published by W. H. Freeman and Co., San Francisco.

Branko Grünbaum is Professor of Mathematics at the University of Washington in Seattle. He was born in Yugoslavia in 1929, and educated in Yugoslavia and in Israel. After receiving his Ph.D. from the Hebrew University in Jerusalem in 1958, he spent two years at the Institute for Advanced Study in Princeton, NJ. Since then he has taught at Hebrew University, Michigan State University, and the University of Washington. His main fields of interest are geometry (in particular, convex sets and polytopes, tilings and patterns, arrangements of lines, and other areas of intuitive geometry) and combinatorics (in particular, graph theory). He has published two books and some 120 research and survey papers on these topics. He received the Lester R. Ford award for 1975 and the Carl B. Allendoerfer award for l978 from the Mathematical Association of America. He has served on the editorial boards of "Aequationes Mathematicae," "Combinatorica," "Discrete Mathematics," "Geometriae Dedicata," "Israel Journal of Mathematics" and "Structural Topology." Since 1974, he has served as a lecturer in the Program of Visiting Lecturers of the Mathematical Association of America.

From the next article:

. . . the specific question I want to address is whether the Bourbaki approach is fundamentally antithetical to geometry.

. . . to speak of the "death of geometry" at the post-secondary or any other level is clearly an exaggeration, it nevertheless reflects a reality.

Pólya, in his wisdom, developed an art of plausible reasoning, rather than a science of problem solving.