Satins and Twills: An Introduction to the Geometry of Fabrics

A mathematical investigation into patterns of weaving reveals subtle problems in combinatorics and geometry.

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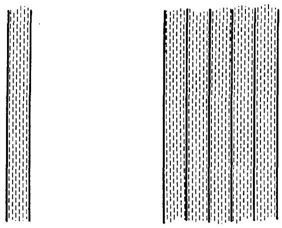
Weaving is one of the oldest activities of mankind and so it is hardly surprising that there exists a vast literature on the subject. But this literature is almost entirely concerned with the practical aspects of weaving; any treatment of the theoretical problem of designing fabrics with prescribed mathematical properties is conspicuously absent. And this is so in spite of the fact that many fabrics which are mathematically interesting were discovered empirically long ago by practitioners of the weaver's craft. One wonders how geometers can fail to be fascinated by the diagrams of fabrics that abound in the literature. Yet, so far as we are aware, the only papers that attempt to treat fabric design from a mathematical point of view are those of Lucas who published about a century ago, an isolated paper of Shorter which appeared in 1920, and a series of three papers by Woods published in 1935. All three authors were concerned principally with satins (or sateens), a type of fabric which we shall discuss in the third section of this paper.

The "geometry of fabrics", as we shall call it, involves ideas from elementary geometry, group theory, number theory and combinatorics. There is a large number of open problems, to some of which we shall draw attention in the following pages.

In order to make the subject manageable, it will be necessary to idealize the concept of a fabric. For example, we shall always assume that our fabrics are unbounded, that is, that they continue indefinitely in every direction. Thus edge-effects and selvedges (which are of great concern to the practical weaver) will be entirely ignored here. A fabric will consist of "strands" woven together and we shall only discuss those fabrics in which the strands are straight and lie in one of two directions, usually at right-angles to each other. Without these restrictions there are many other possibilities about which extremely little seems to be known.

As there is no accepted terminology, it will be necessary to begin by defining the words we shall use. A **strand** (see Figure 1(a)) is a doubly infinite open strip of constant width, that is, the set of points of the plane which lie strictly between two parallel straight lines. For purposes of visualization it is best to think of a strand as a strip of paper, or similar material of zero (or

(a) A strand (an open infinite strip) shaded to show its direction. (b) A layer of strands. Every point of the plane belongs to a single strand or to the boundaries of two strands.



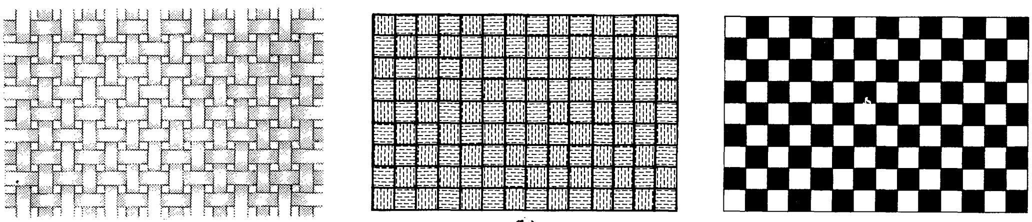
(a) (b)

FIGURE 1.

negligible) thickness. In diagrams we shall sometimes use some method of shading, as in FIGURE 1(a), to indicate the direction of the strand. This will be essential for the correct interpretation of the diagram when only small portions of a strand are visible as in FIGURE 2(b). By a **layer** we mean a collection of congruent and disjoint parallel strands such that each point of the plane either belongs to (the interior of) one of the strands or is on the boundary of two adjacent strands (see FIGURE 1(b)).

The word **fabric** will be used in a mathematical sense to mean, roughly speaking, two layers of congruent strands in the same plane E such that the strands of different layers are nonparallel and they "weave" over and under each other in such a way that the fabric "hangs together." To be precise, "weaving" means that at any point P of E which does not lie on the boundary of a strand, the two strands containing P have a stated **ranking**, that is to say, one strand is taken to have precedence over the other, and this ranking is the same for each point P contained in both strands. This concept may be conveniently expressed by saying that one strand **passes over** the other, in accordance with the obvious practical interpretation. By saying that the fabric **hangs together** we mean that it is impossible to partition the set of all strands into two nonempty subsets so that each strand of the first subset passes over every strand of the second subset.

In FIGURE 2(b) we give a diagramatic representation of the commonest and most familiar of all fabrics, known variously as the **over-and-under, plain, calico** or **tabby weave**. Here the shading not only indicates the direction of each strand, but also shows which strand (a horizontal or a vertical one) passes over the other at each point of the plane. In order to avoid any possible misinterpretation we also give in FIGURE 2(a) a sketch of the same fabric. Here the strands have been "separated" for clarity—this diagram may be regarded as representing the "real" fabric corresponding to the "idealized" or "mathematical" fabric of FIGURE 2(b).



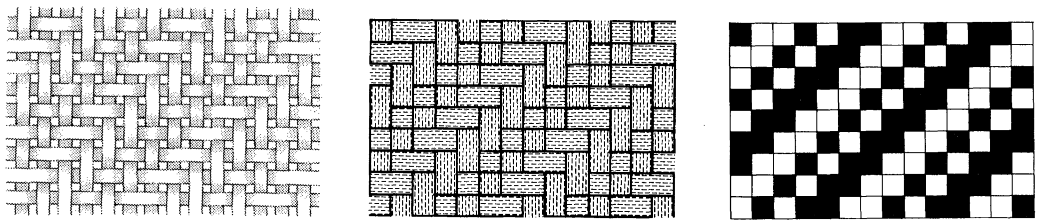
(a) (b) (c)

The most common and best-known type of fabric known as the over-and-under, plain, calico or tabby weave. (a) A sketch of the "real fabric." (b) The idealized fabric consisting of two superimposed layers of strands. (c) A design for the fabric. A white square means that a weft strand passes over a warp strand, and a black square that a warp strand passes over a weft strand.

FIGURE 2.

Traditionally the two layers in a fabric are called the **warp** and the **weft** (or **woof**). In a real fabric the warp runs lengthwise and the weft from side to side. In diagrams it is conventional to draw the warp vertically and the weft horizontally. Here it will be convenient to use the terms warp and weft in this sense.

A simple and convenient method of representing a fabric is by means of a **design** (also called a **diagram** or **draft** by some authors). This is constructed in the following way. We begin with the regular tiling of the plane by unit squares. Each square is the intersection of a row of squares (corresponding to a weft strand) and a column of squares (corresponding to a warp strand); according to the more usual convention we color the square white if the weft strand passes over the warp strand, and we color it black if the warp strand passes over the weft strand. Thus the design may be regarded as indicating the appearance that the fabric would have if the weft strands were colored white and the warp strands were colored black. For example, in FIGURE 2(c) we show a design for the plain weave; the design can be easily obtained from FIGURE 2(b) by replacing vertical shading by black and horizontal shading by white. Another example appears in FIGURE 3. The fabric shown in this figure is called a balanced twill of period six and is an example of a large class of fabrics of practical importance known as twills. A discussion of twills and their properties will be given in the next section. For an application of a computer to the drawing of fabric designs, see Huff [5].



(a) (b) (c)

A balanced twill of period six: (a) is a sketch of the "real fabric," (b) is the idealized fabric, and (c) is a design for this fabric.

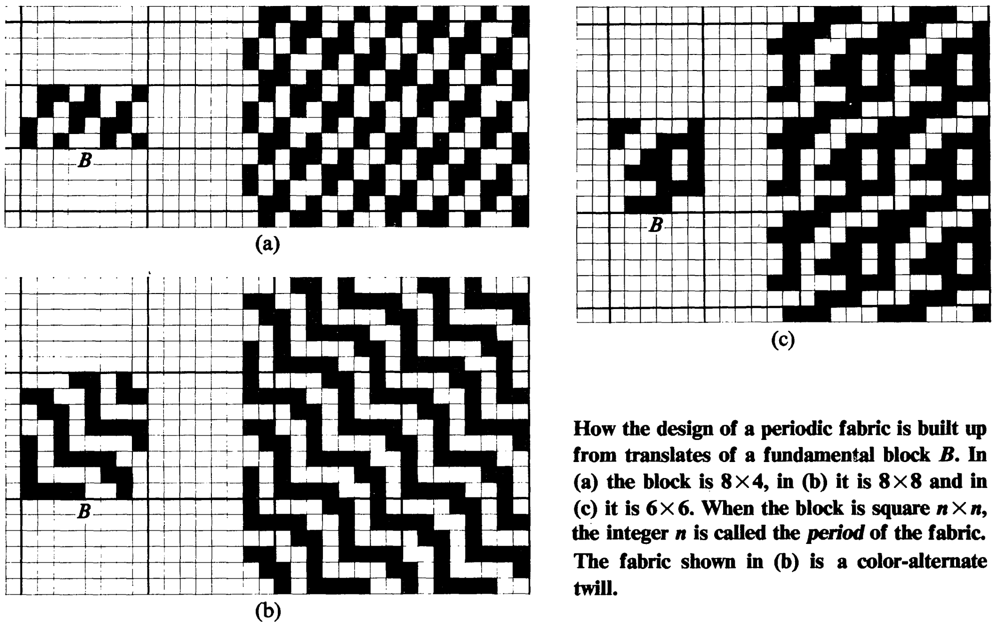
FIGURE 3.

Sometimes it is convenient to use coordinates for the squares in a design. We set up a coordinate system so that the centers of the squares lie on the standard integer lattice and then refer to the square with center (x, y) as the **(x, y)-square**.

By a **symmetry** of a fabric $\mathcal{F}$ in the plane E we mean any isometry that maps each strand of $\mathcal{F}$ into a strand of $\mathcal{F}$ and either preserves all the rankings or reverses them all. Thus it consists of an isometry σ in E (a translation, rotation, reflection, or glide-reflection) possibly followed by a reflection τ in the plane E. The operation τ reverses the ranking of the strands at each point, converting one strand which passes over another into one that passes under it. All the symmetries of a fabric $\mathcal{F}$ form a group under composition called the **symmetry group** of $\mathcal{F}$ and denoted by $S(\mathcal{F})$. The elements of $S(\mathcal{F})$ can be divided into two types: those which do not involve τ and so do not alter the rankings of the strands, and those which involve τ and reverse the rankings. The former type may be said to **preserve the sides** of the fabric and the set of all such forms a normal subgroup $S_0(\mathcal{F})$ of $S(\mathcal{F})$. The latter type may be said to **interchange the sides** of $\mathcal{F}$.

The design D of a fabric $\mathcal{F}$ also has a symmetry group $S(D)$ and each element of $S(D)$ corresponds to a symmetry of $\mathcal{F}$, though not necessarily of the same kind. For example, a translation in $S(D)$ corresponds to a translation in $S(\mathcal{F})$, but a rotation through 90° in $S(D)$ corresponds to a similar rotation of $\mathcal{F}$ combined with the reflection τ. Among others, the designs of satins in FIGURE 12(b) each possess 4-fold rotational symmetries. Each 90° rotation corresponds to a symmetry operation on the fabric which interchanges its sides. This relates to the fact that on one side of the fabric the warp strands are largely visible while on the other the weft strands predominate, and so accounts for the well-known property of satins that their two sides are often dissimilar in appearance. Other symmetries of a fabric $\mathcal{F}$ correspond to isometries which map D onto itself with the colors black and white interchanged. For example, for the design of FIGURE 3(c) there are rotations through 180° which reverse the direction of each row and map the design onto itself if the colors are interchanged. Such operations also correspond to symmetries of $\mathcal{F}$ which interchange its sides.

In almost all the fabrics $\mathcal{F}$ that we shall consider, $S_0(\mathcal{F})$ (and therefore $S(\mathcal{F})$) will contain translations in at least two nonparallel directions. The fabric will then be called **periodic.** The design of a periodic fabric can always be obtained from a **fundamental** $n \times m$ **block** of squares, suitably colored, by translations in horizontal and vertical directions through multiples of n and m units (see FIGURE 4). Although a fundamental block B determines the design of the fabric uniquely, in our diagrams it is usually convenient to show a larger part of the design and



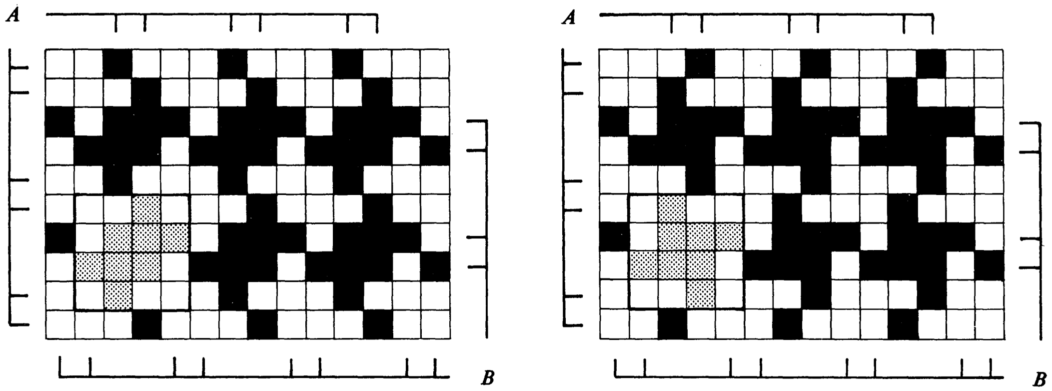
How the design of a periodic fabric is built up from translates of a fundamental block *B*. In (a) the block is 8×4, in (b) it is 8×8 and in (c) it is 6×6. When the block is square *n*×*n*, the integer *n* is called the *period* of the fabric. The fabric shown in (b) is a color-alternate twill.

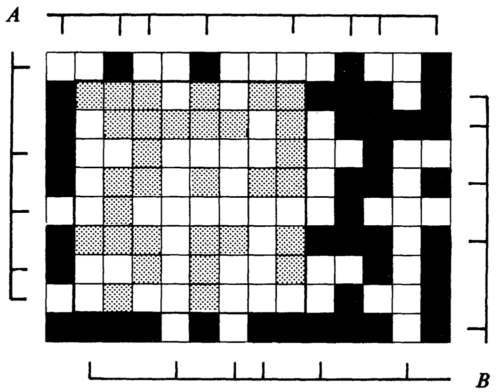
FIGURE 4.

indicate a fundamental block B by using gray and white squares instead of black and white (see, for example, FIGURES 6(a), (d), (h), (i), (j), (k), (l), (m) and (n)). Occasionally, as in the case of certain sponge weaves (see FIGURES 6(b), (c), (e) and (f)), a fundamental block is too big to show on the diagram, and then it will be assumed that the reader will be able to "see" how the design can be continued from the part of it that is given.

The integers m and n (the sides of a fundamental block) are called the **periods** of the fabric. We shall mostly deal with the case where the fundamental block is square, so $m = n$ and the integer n is called the period. Notice that when we say that a fabric is of period n we do not preclude the possibility that it is also of period d where d is any divisor of n. Thus a plain weave is considered to be of period n where n is any even integer. Other terms in use for n are the **order** of the fabric and the **number of ends.** However, we shall not use these terms here.

Not every black and white coloring of a rectangular block of squares is a fundamental block in the design of a fabric, since the requirement that the fabric must "hang together" may be violated in ways that are not immediately apparent. For example, at first glance, the "designs" of FIGURE 5 seem to correspond to fabrics, but this is not so, for in each case the "fabric" will "fall apart." Intuitively the set of strands labelled A can be "lifted off" those labelled B since at each crossing the A strand passes over the B strand.





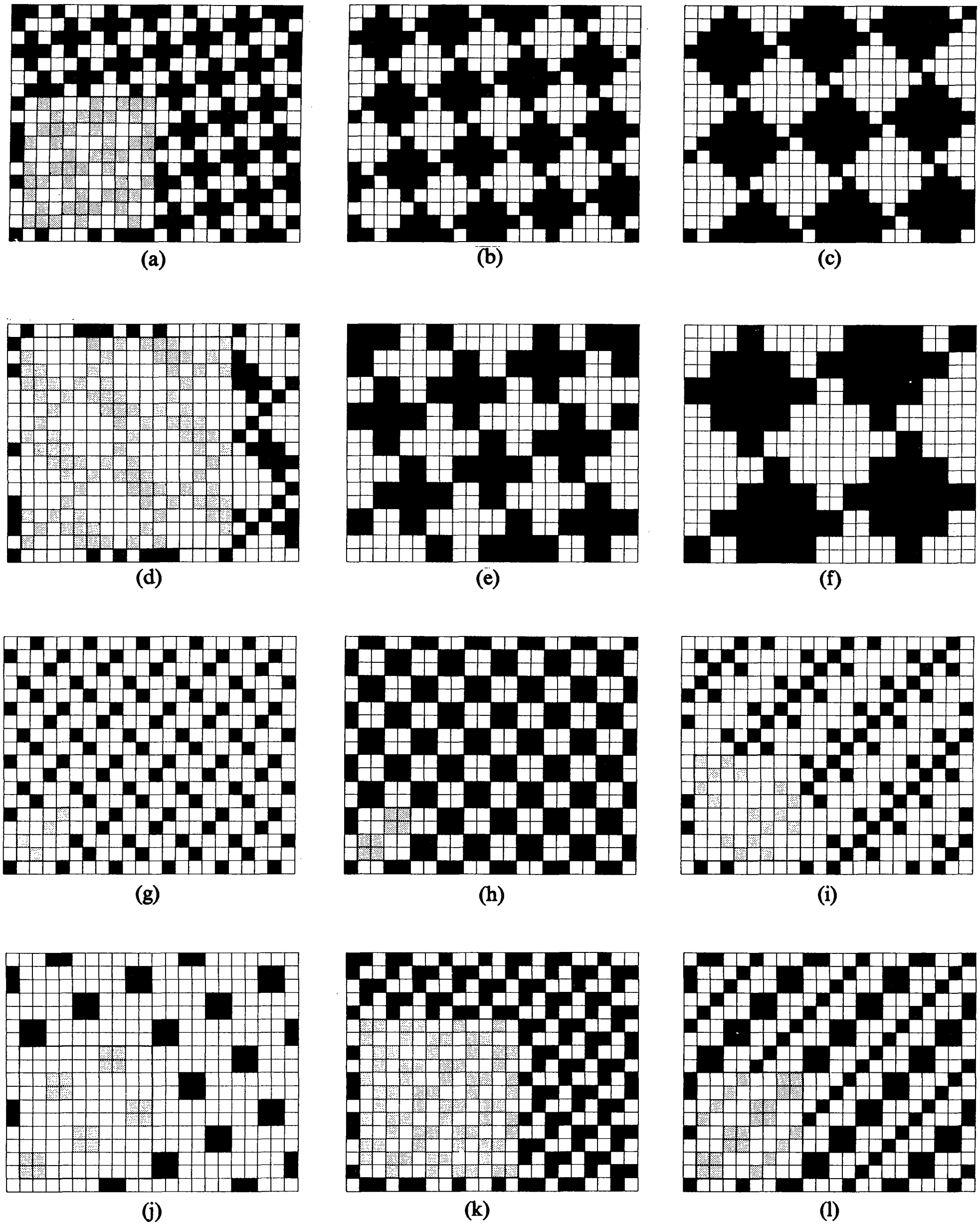
Colorings of the regular square tilings which appear, at first sight, to be designs for fabrics, but they are not. In each case the strands labelled A can be "lifted off" the strands labelled B and so the "fabric" does not "hang together."

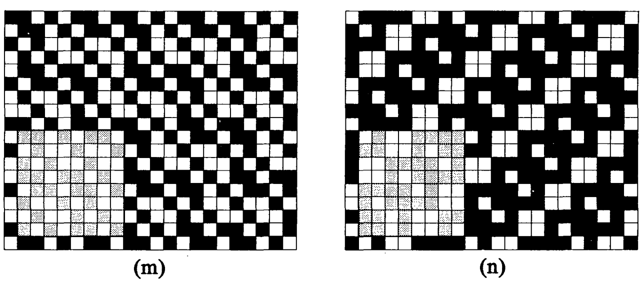
FIGURE 5.

An extremely important class of fabrics, both from a mathematical as well as a practical point of view, will be called "isonemal," a term derived from the Greek words ισος (the same) and νημα (a thread or yarn). A fabric $\mathcal{F}$ is **isonemal** if its symmetry group $S(\mathcal{F})$ is transitive on the strands of $\mathcal{F}$. In other words, for any two strands s_1 and s_2 there exists a symmetry of $\mathcal{F}$ that maps s_1 onto s_2. In terms of the design D of $\mathcal{F}$ this means that any row or column of squares in D can be mapped into any other row or column by either a symmetry of D, or by such a symmetry combined with interchange of the colors black and white. In FIGURE 6 we show examples of isonemal fabrics, and the profusion of possibilities will be apparent. Moreover, as will be seen from the caption, many of these are *actual fabrics* used by the textile industry. The reader to whom these ideas are unfamiliar should convince himself that the fabrics of FIGURE 6 are isonemal, while those of FIGURES 7 and 8 are not.

FIGURE 7 shows that even in fabrics which are not isonemal, it is possible for every strand to weave under and over the other strands in the same "pattern" or "sequence." A fabric with this property is called **mononemal**. The difference between the concepts of isonemality and mononemality can be explained by the observation that mononemality is local—it merely implies that every strand "looks alike"—whereas isonemality is global—the relationship of each strand to the totality of other strands must be the same. Again the reader is urged to verify for himself that the fabrics of FIGURE 7 are mononemal, while those of FIGURE 8 are not. Clearly every isonemal fabric is mononemal.

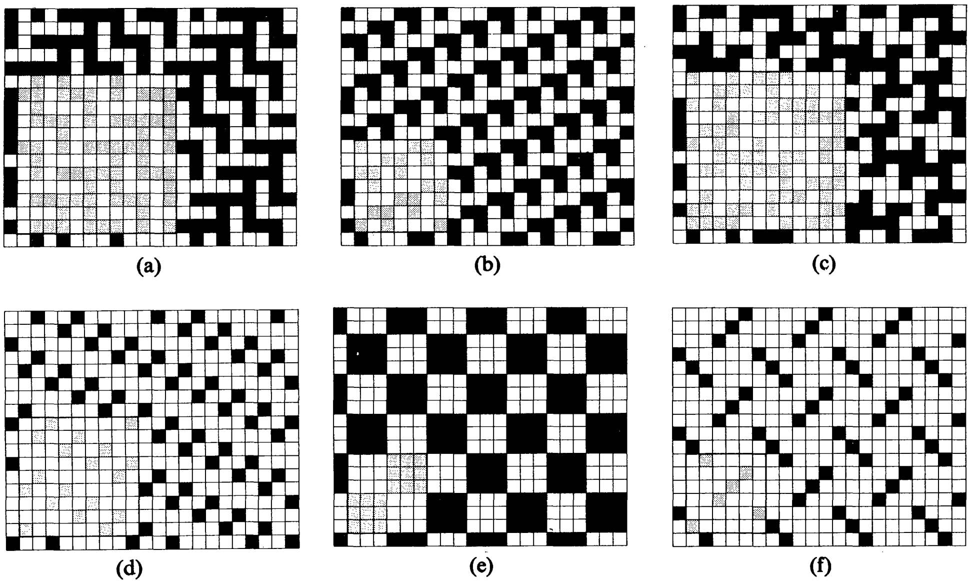
The above ideas lead to a classification of fabrics into three major types: isonemal (I), mononemal but not isonemal (M), and not mononemal (N). For some purposes a finer





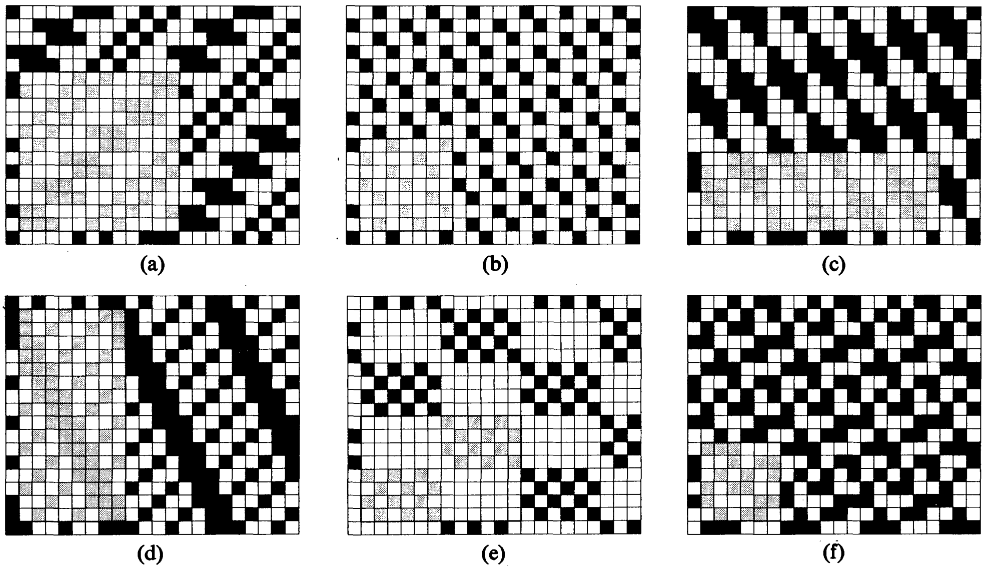
Examples of isonemal fabrics. (a), (b), (c) and (d) are sponge weaves, (e) and (f) are doubled sponge weaves, (g) is satinette, (h) is duck cloth or sailcloth, (j) is a doubled satin, and (l) is a figured twill. (Designs (a), (b), (d), (g), (h) and (l) are from Nisbet [c], FIGURES 232, 233, 240, 167, 21, and 177 respectively.)

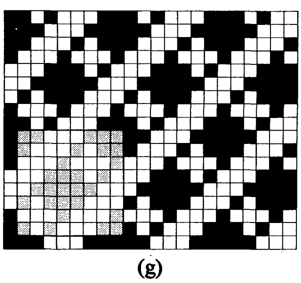
FIGURE 6.



Examples of fabrics which are mononemal but not isonemal. (e) is matt weave and (f) is rice weave. These latter two designs are from Nisbet [c] FIGURES 22 and 168. The first four designs are of fabrics which are both warp-isonemal and weft-isonemal and so are of type M1. The last two are of type M3.

FIGURE 7.





Examples of fabrics which are not mononemal. (a) and (g) are figured twills, (c) is a broken twill, (d) is a combined twill and (f) is a rearranged twill, being obtained from the twill shown in FIGURE 3 by rearranging the warp threads. (Designs (a), (c), (d), (f) and (g) are from Nisbet [c], FIGURES 186, 158, 133, 118 and 181 respectively.) Fabrics (a) and (b) are of type N1, (c) of type N2, (d) of type N3, (e) of type N4, (f) of type N5 and (g) of type N6.

FIGURE 8.

classification is useful and interesting. Referring to FIGURE 9, we see that it is possible for the symmetry group $S(\mathcal{F})$ of the fabric $\mathcal{F}$ to be transitive on the weft strands (each is mapped onto the next one above it by a "step" of 6 squares to the left or right), whereas $S(\mathcal{F})$ is *not* transitive on the warp strands (in fact these form three transitivity classes indicated by the letters X, Y and Z). We express this by saying that $\mathcal{F}$ is **weft-isonemal** but not **warp-isonemal.** On the other hand, the warp strands weave under and over the weft strands in the same pattern or sequence (one over, one under, one over, one under, and so on) and hence, by an obvious extension of the terminology, we may say that it is **warp-mononemal.** Just as an isonemal fabric is mononemal, so a weft-isonemal fabric such as that shown is also weft-mononemal. Hence we see that a fabric can be both warp-mononemal and weft-mononemal without being a mononemal fabric. On the other hand, every mononemal fabric must be both warp-mononemal and weft-mononemal. Similar remarks apply to isonemality; an isonemal fabric is necessarily both warp-isonemal and weft-isonemal, but the converse statement is not generally true.

This terminology enables us to classify fabrics into ten types (I, M1–M3 and N1–N6) as indicated in TABLE 1. The fabrics of FIGURE 6 are all of type I and in FIGURES 7 and 8 we show fabrics belonging to eight of the remaining nine classes. The type that is missing is M2 and we believe, but cannot prove, that no periodic fabrics of this kind exist. More precisely we conjecture that *every periodic mononemal fabric which is warp-isonemal is also weft-isonemal.* The reader may like to try to prove this conjecture; even if he does not succeed he will learn a great deal about the possible structures of different types of fabric.

The ten types of fabrics

Isonemal fabrics

Type I: necessarily warp I and weft I.

Mononemal, but not isonemal, fabrics

Type M1: warp I and weft I.
Type M2: warp I and weft M, *or* warp M and weft I.
Type M3: warp M and weft M.

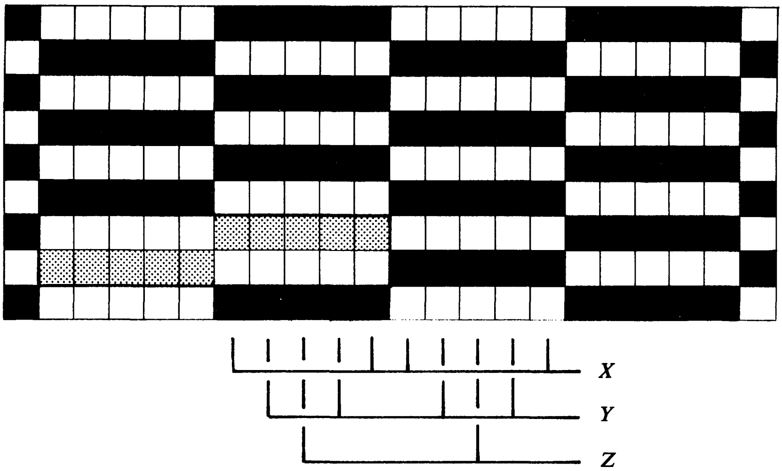
Fabrics which are not mononemal

Type N1: warp I and weft I.
Type N2: warp I and weft M, *or* warp M and weft I.
Type N3: warp I and weft N, *or* warp N and weft I.
Type N4: warp M and weft M.
Type N5: warp M and weft N, *or* weft N and warp M.
Type N6: warp N and weft N.

In the descriptions, I means isonemal, M means mononemal but not isonemal, and N means not mononemal. For example, the fabric whose design is shown in FIGURE 9 is not mononemal, but it is weft-isonemal and warp-mononemal. Hence it is of type N2.

TABLE 1.

Yet another kind of classification arises from considerations of balance; a periodic fabric is called **balanced** if a fundamental block contains equal numbers of black and white squares. Thus if a fundamental block is n by m, then at least one of n or m must be even, and then the block will contain $\frac{1}{2}nm$ white squares and $\frac{1}{2}nm$ black squares. Examples of balanced fabrics are given in FIGURES 2, 3, 5(a),(b), 6(a),(b),(c),(e),(f),(h),(k) and (m). In a balanced fabric equal "amounts" of warp and weft show on each side, a property which is desirable in certain practical applications. For a balanced isonemal fabric it is possible for $S_0(\mathcal{F})$ to be transitive on the strands of $\mathcal{F}$, though not all such fabrics have this property. For example $S_0(\mathcal{F})$ is transitive on the strands for the fabric of FIGURE 6(a) but not in the case of the fabrics of FIGURE 11.



A fabric which is weft-isonemal but not warp-isonemal. It is, however, warp-mononemal and so we see that it is of type N2. (Design from Nisbet [c], FIGURE 16.)

FIGURE 9.

We have already remarked on the great practical and theoretical importance of isonemal fabrics, and the rest of this paper will be devoted to these. The commonest kinds of isonemal fabrics are the twills and satins which will be discussed in the next two sections. After this we shall describe a general method for finding designs of certain kinds of isonemal fabrics of which the twills and satins are special cases. The method will yield many of the fabrics shown in FIGURE 6 (such as the sponge weaves and sailcloth) but not all. At present a completely general method of determining all possible isonemal fabrics of a given period is lacking, and the problem of enumerating such fabrics seems to be completely intractable.

In the above discussion, and also in the rest of this paper, we shall try to adopt terminology which conforms to that in use in the textile industry. Unfortunately this has not always been possible, for not only do authors disagree on the exact meanings of words, but in some cases they formulate their definitions so loosely that we were unable to understand, in a rigorous mathematical sense, exactly what is intended. At the end of the paper we give a list of references concerning the practical aspects of weaving. These are some that we have consulted, but apart from Nisbet [c], which gives a large number of very interesting examples and attempts to be comprehensive, they do not seem to us to be of particular interest or merit; the reader will easily find other works of equal usefulness in any large library or bookshop.

Twills

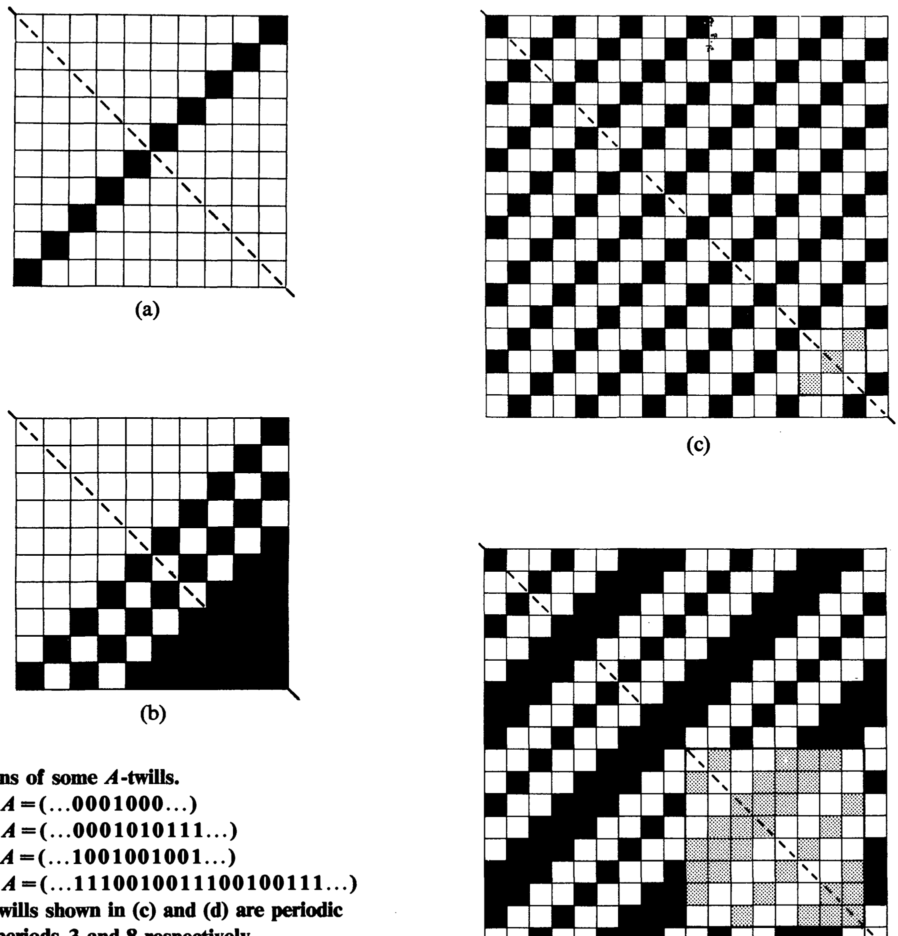
The plain weave of FIGURE 2 may be regarded as the simplest example of the class of fabrics known as twills. These can be easily described by the following scheme. Let $A=(a_i)_{-\infty}^{\infty}$ be any two-way infinite sequence of zeros and ones. A fabric $\mathcal{F}$ is an **A-twill** provided that in its design the (x,y)-square is colored black if $a_{y-x}=1$ and white if $a_{y-x}=0$, or, alternatively, that these relations hold after the design has been turned through 90°. Thus the plain weave is a twill with $A=(\ldots 0\ 1\ 0\ 1\ 0\ 1 \ldots)$ and in FIGURE 10 we show designs of A-twills with $A=(\ldots 0\ 0\ 0\ 1\ 0\ 0\ 0 \ldots)$, $(\ldots 0\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 1 \ldots)$, $(\ldots 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1 \ldots)$ and $(\ldots 1\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ 1 \ldots)$. Of these the last two are periodic and the last one is also balanced.

The colorings of any two rows in the design of a twill differ only in that one is shifted sideways relative to the other. If the rows are adjacent, then the upper row is obtained from the lower by a shift to the right through one unit or, as we shall say, a **1-step** to the right. A similar remark applies to the columns. It is this step-like structure that gives a twill its characteristic appearance—it is covered with diagonal stripes. In fact some authors extend the meaning of the word "twill" to include *any* fabric with a pronounced diagonal stripe, and Shorter [11] even suggests a numerical measure of "twilliness" which indicates the obviousness of such stripes.

Our first result is very simple.

THEOREM 1. *If A is a sequence of zeros and ones which contains at least two pairs of distinct neighbors, then the A-twill $\mathcal{F}$ is an isonemal fabric. Moreover $\mathcal{F}$ is periodic with period n if and only if A is periodic with period n, that is, if and only if $a_i = a_j$ whenever $i \equiv j$* (mod n).

The proof is rather trivial. The existence of two pairs of distinct neighbors is necessary for it is easily verified that $\mathcal{F}$ does not "hang together" if the sequence A is either constant or one of $(\ldots 1\,1\,1\,0\,0\,0 \ldots)$ or $(\ldots 0\,0\,0\,1\,1\,1 \ldots)$. These are the only sequences which have fewer than two pairs of distinct neighbors. Translations of the plane, corresponding to the shifts or steps mentioned above, show that $S(\mathcal{F})$ is transitive on the warp strands (warp-isonemal) and also on the weft strands (weft-isonemal). Clearly $S(\mathcal{F})$ also includes rotations by 180° in three dimensions about lines parallel to $x+y=0$ (one of which is shown dotted in FIGURE 10) and these interchange warp and weft strands, showing them to be equivalent. (Notice that this operation interchanges the sides of the fabric.) The proof of the second part of the theorem is apparent



Designs of some A-twills.
(a) $A = (\ldots 0001000 \ldots)$
(b) $A = (\ldots 0001010111 \ldots)$
(c) $A = (\ldots 1001001001 \ldots)$
(d) $A = (\ldots 1110010011100100111 \ldots)$
The twills shown in (c) and (d) are periodic with periods 3 and 8 respectively.

FIGURE 10.

from the diagrams, where the fact that $a_i = a_j$ when $i \equiv j \pmod{n}$ shows that both $x \to x+n$ and $y \to y+n$ are symmetries of the fabric which therefore has a fundamental block of size $n \times n$. Hence $\mathcal{F}$ is a periodic fabric with period n.

In practical applications the sequences A are invariably taken to be periodic, and there is a standard notation for these twills used by practical weavers. They are denoted by

$$\frac{c_1 \quad c_2 \quad \cdot \quad \cdot \quad c_p}{b_1 \quad b_2 \quad \cdot \quad \cdot \quad b_p}$$

when a period of A consists, in order, of b_1 zeros, c_1 ones, b_2 zeros, c_2 ones ,..., b_p zeros, c_p ones (see [c, Chapter 3]). Thus the fabrics of FIGURES 10(c) and 10(d) are $\frac{1}{2}$ and $\frac{3\ 1}{2\ 2}$ twills, respectively. In particular we note that $\Sigma b_i + \Sigma c_i = n$, and that $\Sigma b_i = \Sigma c_i$ is a necessary and sufficient condition for the twill to be balanced.

It is an interesting combinatorial problem to determine the number $t(n)$ of distinct twills of any given period n. Here two twills are considered identical if one is the image of the other under an isometry. Equivalently, they are identical if their designs can be made to coincide by a rigid motion of the plane possibly followed by an interchange of the colors black and white. In order to give a formula for $t(n)$ we need to use Euler's phi-function $\phi(d)$ which is defined as the number of positive integers less than, and prime to, d (see [6, p. 120]). (Thus, for example, $\phi(12)=4$ since $m=1$, 5, 7 and 11 are the only integers satisfying $1 \leqslant m \leqslant 12$ and $\gcd(m,12)=1$.) It is well known that

$$\phi(d) = d\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_r}\right)$$

where $p_1, p_2, \ldots, p_r$ are the *distinct* prime factors of d. It is also convenient to introduce a function $\rho(n) = \frac{1}{2}(3+(-1)^n)$ which takes the value 1 if n is odd and 2 if n is even. With this notation we can now state the result.

THEOREM 2. *The number $t(n)$ of distinct twills of period n is given by*

$$t(n) = 2^{\frac{1}{2}(n+\rho(n))-2} + \frac{1}{4n}\sum \phi(d)\rho(d)2^{n/d} - 1,$$

where the summation in the second term is over all the positive integer divisors d of n.

n	symbol
2	$\frac{1}{1}$
3	$\frac{2}{1}$
4	$\frac{3}{1}$ $\frac{2}{2}$ $\frac{1\ 1}{1\ 1}$
5	$\frac{4}{1}$ $\frac{3}{2}$ $\frac{2\ 1}{1\ 1}$
6	$\frac{5}{1}$ $\frac{4}{2}$ $\frac{3\ 1}{1\ 1}$ $\frac{2\ 2}{1\ 1}$ $\frac{3}{3}$ $\frac{2\ 1}{2\ 1}$ $\frac{1\ 1\ 1}{1\ 1\ 1}$
7	$\frac{6}{1}$ $\frac{5}{2}$ $\frac{4\ 1}{1\ 1}$ $\frac{3\ 2}{1\ 1}$ $\frac{4}{3}$ $\frac{3\ 1}{2\ 1}$ $\frac{2\ 2}{2\ 1}$ $\frac{2\ 1\ 1}{1\ 1\ 1}$
8	$\frac{7}{1}$ $\frac{6}{2}$ $\frac{5\ 1}{1\ 1}$ $\frac{4\ 2}{1\ 1}$ $\frac{3\ 3}{1\ 1}$ $\frac{5}{3}$ $\frac{4\ 1}{2\ 1}$ $\frac{3\ 2}{2\ 1}$ $\frac{3\ 1\ 1}{1\ 1\ 1}$ $\frac{2\ 2\ 1}{1\ 1\ 1}$ $\frac{4}{4}$ $\frac{3\ 1}{3\ 1}$ $\frac{2\ 2}{3\ 1}$ $\frac{2\ 2}{2\ 2}$ $\frac{2\ 1\ 1}{2\ 1\ 1}$ $\frac{1\ 2\ 1}{2\ 1\ 1}$ $\frac{1\ 1\ 1\ 1}{1\ 1\ 1\ 1}$

Symbols for all distinct twills of period n up to $n=8$.

TABLE 2.

The proof of this theorem involves standard methods, and we only give a brief outline here. The reader familiar with Pólya's Theorem and its applications, see [**6**, Chapter 5], will appreciate that $t(n)$ can be interpreted as the number of bracelets of n beads, each black or white, subject to the additional proviso that two bracelets are not to be counted as distinct if one can be obtained from the other by interchange of the two colors. The cycle index of the corresponding group of the bracelet is

$$\frac{1}{2n}\left(ns_1s_2^{(n-1)/2}+\sum\phi(d)s_d^{n/d}\right)$$

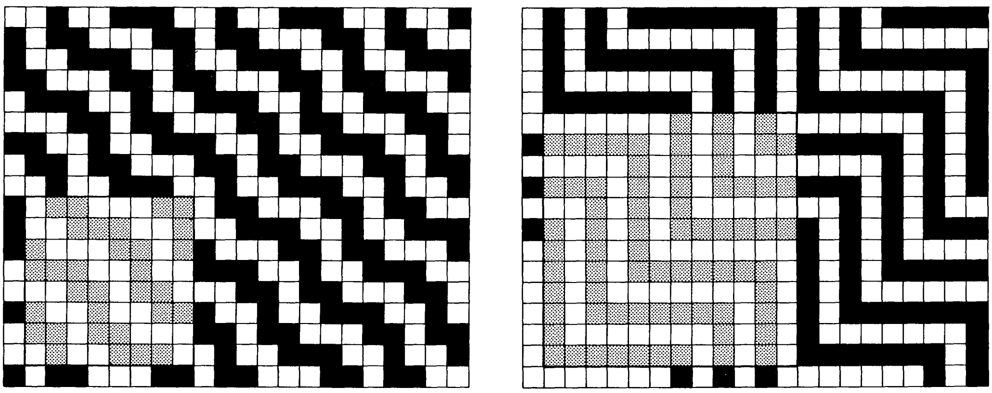
or

$$\frac{1}{2n}\left(\tfrac{1}{2}ns_1^2s_2^{(n-2)/2}+\tfrac{1}{2}ns_2^{n/2}+\sum\phi(d)s_d^{n/d}\right)$$

according as n is odd or even. From this the result follows by application of a generalization of Pólya's Theorem (see [**6**, p. 157]). The function $\rho(n)$ is introduced to enable the values for both even and odd n to be written in one compact formula.

Theorem 2 yields, for example, $t(1)=0$, $t(2)=1$, $t(3)=1$, $t(4)=3$, $t(5)=3$, $t(6)=7$, $t(7)=8$ and $t(8)=17$. These figures can be easily verified by actual construction (see TABLE 2).

In FIGURE 4(b) and FIGURE 11 we show three isonemal fabrics which are not twills but are what we shall call **color-alternate twills**. In these, each row is obtained from the one below it by a 1-step to the right and an interchange of colors black and white. In other words, starting from a



(a) (b)

Two color-alternate twills. Each row is obtained from the one below it by a 1-step to the right and an interchange of the colors black and white.

FIGURE 11.

two-way infinite sequence A, the *odd* rows of the fabric are colored in the same way as for an A-twill (as described at the beginning of this section) while the *even* rows are obtained by reversing the colors in the corresponding rows of the A-twill. However, unlike the "ordinary" twills, in a color-alternate twill the sequence A has to be chosen very carefully if the resulting fabric is to be isonemal. We shall explain how this can be done in the fourth section of this paper.

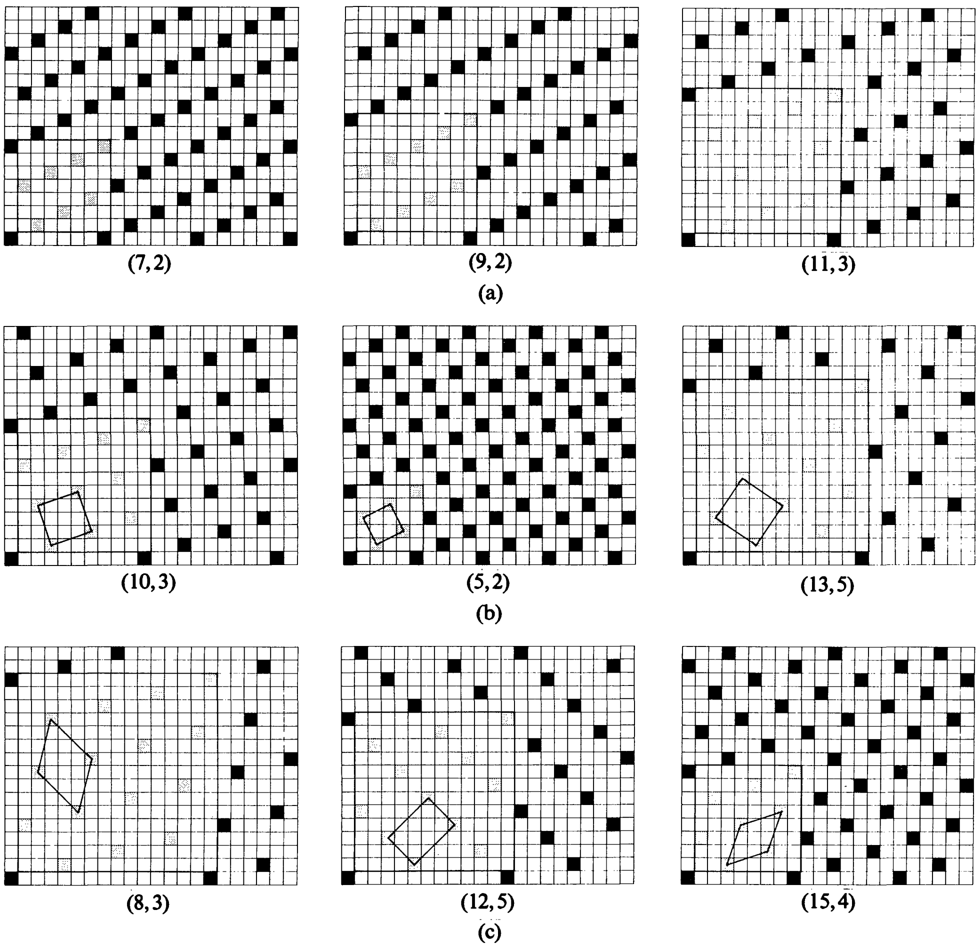
It is strange that color-alternate twills seem to have been rarely, if ever, used in practice, and we can find no record of them in the literature. They have a characteristic and attractive appearance which may be described as a modified herring-bone effect.

Satins

The next type of fabric to be discussed is known as a **satin** or **sateen**. (Authorities differ on the distinction between the meanings of these two words.) An (n,s)**-satin** is a periodic fabric of period n, in which the fundamental $n \times n$ block contains just one black square in each row, and the position of that square is displaced from one row to the next above it by a step of s units to the right (an s-step) (see FIGURE 12). Alternatively, an (n,s)-satin can be defined as one for which the (x,y)-square in the design is colored black if and only if $sy \equiv x \pmod{n}$. Of course, exactly similar considerations will apply if the roles of the colors black and white are interchanged.

We observe that unless s is prime to n the resultant satin does not "hang together," and that there is no loss of generality in assuming that $1 < s < \frac{1}{2}n$. The left inequality arises from the fact that $s=1$ corresponds to the twills $\frac{1}{n-1}$ discussed in the previous section, and the right inequality because any (n,s)-satin is a mirror-image of an $(n,n-s)$-satin.

An easy counting argument (see Shorter [11]) shows that if an (n,s)-satin is rotated counterclockwise through 90° (interchanging warp and weft) then we obtain either an (n,t)-satin or an $(n,n-t)$-satin, where t is the (unique) solution of the congruence, $st \equiv \pm 1 \pmod{n}$ that satisfies $1 < t < \frac{1}{2}n$. By putting $s = t$ we obtain the following fundamental result.



Examples of (n,s)-satins. The pair (n,s) is indicated near each diagram: (a) mononemal (not isonemal) satins, (b) square isonemal satins, (c) symmetric isonemal satins. Of the three latter that are shown, two are rhombic and one is rectangular.

FIGURE 12.

THEOREM 3. *An (n,s)-satin is isonemal if and only if* $s^2 \equiv \pm 1 \pmod{n}$.

The two cases lead to isonemal satins with essentially different properties. If $s^2 \equiv -1 \pmod{n}$, then the satin is said to be **square**, and the symmetry group of the design contains 4-fold rotations but no reflections or glide-reflections (see FIGURE 12(b)). The name "square" comes from the fact that the centers of the black squares form a lattice of which one of the fundamental parallelograms is a square. (In FIGURE 12(b) we have indicated one such square for each of the three fabrics.) These square isonemal satins are characterised by the fact that -1 is a quadratic residue modulo n.

If, on the other hand, $s^2 \equiv +1 \pmod{n}$ then the satin is called **symmetric**. Its symmetry group contains reflections and 2-fold rotations but no 4-fold rotations (see FIGURE 12(c)). The name arises since each design is symmetric in a line parallel to $x=y$. The symmetric satins are characterised by the fact that $+1$ is a quadratic residue modulo n. Symmetric satins can be divided into two classes, **rectangular** and **rhombic** (or **diamond**) satins, according to the possible shapes of the fundamental parallelograms of the lattices of centers of the black squares. (In FIGURE 12(c) it will be seen that the first and third fabrics are rhombic, while the second is rectangular.) Woods [**12**, p. T307] briefly discusses rectangular and rhombic satins which are not necessarily isonemal. It is easy to distinguish between these two classes:

THEOREM 4. *A symmetric isonemal satin is rectangular if n is even and* $s^2 \equiv +1 \pmod{2n}$. *Otherwise it is rhombic.*

We give an outline of the proof of this theorem, leaving the details to be filled in by the reader. As before, we refer to each square in the satin by its coordinates, so the (x,y)-square is colored black if and only if $sy \equiv x \pmod{n}$. Let us consider the family of parallel lines $x+y=$constant that contain black squares. One of these is $x+y=0$, and another is $x+y=s+1$ (because the $(s,1)$-square is black). The line immediately to the right of $x+y=0$ is $x+y=d$, where $d=\gcd(s+1,n)$. The satin will be rectangular if and only if the $(\frac{1}{2}d,\frac{1}{2}d)$-square is black (see FIGURE 12(c)) which implies that d must be even, so n is even and s is odd. But the $(\frac{1}{2}d,\frac{1}{2}d)$-square is black if and only if $\frac{1}{2}ds \equiv \frac{1}{2}d \pmod{n}$, that is, $\frac{1}{2}d(s-1) \equiv 0 \pmod{n}$. Substituting for d we get the equivalent condition $n|\gcd(\frac{1}{2}(s^2-1), \frac{1}{2}n(s-1))$, or $n|\frac{1}{2}(s^2-1)$ (because s is odd and therefore $\frac{1}{2}n(s-1)$ is necessarily a multiple of n). This can be rewritten as the stated condition $s^2 \equiv 1 \pmod{2n}$, so completing the proof of the theorem.

In order to enumerate the isonemal satins for a given n we have to determine the number of solutions of the congruence and inequality $s^2 \equiv \pm 1 \pmod{n}$, $1 < s < \frac{1}{2}n$, and this is easily achieved using known results on quadratic residues (see, for example, Bachmann [**2**, pp. 172, 187, 198]).

THEOREM 5. *For a given n the number of distinct isonemal (n,s)-satins is $u(n)+v(n)$, where $u(n)$ is the number of square satins and $v(n)$ is the number of symmetric satins. If $n=2^{\alpha}p_1^{\beta_1}p_2^{\beta_2}\cdots p_j^{\beta_j}$ is the factorization of n into distinct primes $2,p_1,p_2,\ldots,p_j$, then*

$$u(n) = \begin{cases} 0 & \text{if } \alpha \geqslant 2 \text{ or if } p_i \equiv 3 \pmod{4} \\ & \text{for some } i \text{ with } 1 \leqslant i \leqslant j, \\ 2^{j-1} & \text{if } \alpha \leqslant 1 \text{ and if } p_i \equiv 1 \pmod{4} \\ & \text{for } i=1,2,\ldots,j, \end{cases}$$

and

$$v(n) = \begin{cases} 2^{j-1}-1 & \text{if } \alpha=0 \text{ or } \alpha=1, \\ 2^j-1 & \text{if } \alpha=2, \\ 2^{j+1}-1 & \text{if } \alpha \geqslant 3. \end{cases}$$

Thus, for example, the smallest n for which there exist two distinct isonemal square satins is 65, the satins being (65, 8) and (65, 18). There is also a (rhombic) symmetric (65, 14)-satin, and 65 is the smallest n for which both square and symmetric satins exist. The smallest value of n for which there exists more than one symmetric satin is 24, and the satins are (24, 5), (24, 7) and (24, 11). Two of these are rhombic and one is rectangular, see TABLE 3.

The number of distinct mononemal (but not isonemal) satins of period n can also be found (see Lucas [8]).

n	s	n	s	n	s
5	2a	42	5, 11, 13c	73	2, 3, 4, 5, 6, 7, 8, 10, 11, 13, 14,
7	2	43	2, 3, 4, 5, 6, 8, 9, 10, 12, 15		15, 16, 17, 19, 25, 27a, 31
8	3b	44	3, 5, 7, 13, 21c	74	3, 5, 7, 9, 11, 13, 19, 23, 31a
9	2	45	2, 4, 7, 8, 14, 19b	75	2, 4, 7, 8, 11, 13, 14, 17, 26b, 29
10	3a	46	3, 5, 7, 11, 17	76	3, 5, 7, 9, 13, 21, 23, 27, 37c
11	2, 3	47	2, 3, 4, 5, 6, 7, 9, 10, 11, 13, 15	77	2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 25, 34b
12	5c	48	5, 7b, 11, 17c, 23b	78	5, 7, 17, 19, 25c, 29
13	2, 3, 5a	49	2, 3, 4, 5, 6, 9, 13, 17, 18, 20	79	2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 14, 15, 18, 19, 23,
14	3	50	3, 7a, 9, 13, 19		27, 28, 29, 32
15	2, 4b	51	2, 4, 5, 7, 8, 11, 16b, 20	80	3, 7, 9b, 11, 13, 17, 19, 31c, 39b
16	3, 7b	52	3, 5, 7, 9, 11, 25c	81	2, 4, 5, 7, 8, 11, 13, 14, 17, 26, 31, 32, 35
17	2, 3, 4a, 5	53	2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 14,	82	3, 5, 7, 9a, 11, 13, 17, 21, 23, 31
18	5		17, 23a	83	2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 16, 17, 18, 19,
19	2, 3, 4, 7	54	5, 7, 13, 17		20, 22, 24, 27, 30
20	3, 9c	55	2, 3, 4, 6, 7, 12, 13, 16, 19, 21b	84	5, 11, 13c, 19, 25, 29c, 41c
21	2, 4, 8b	56	3, 5, 9, 13b, 15c, 17, 27b	85	2, 3, 4, 6, 7, 8, 9, 11, 13a, 16b, 18, 22, 23, 24,
22	3, 5	57	2, 4, 5, 7, 10, 11, 13, 16, 20b		26, 29, 38a
23	2, 3, 4, 5, 7	58	3, 5, 7, 9, 11, 15, 17a	86	3, 5, 7, 9, 11, 13, 15, 21, 25, 27
24	5b, 7c, 11b	59	2, 3, 4, 5, 6, 7, 8, 9, 11, 14, 18	87	2, 4, 5, 7, 8, 10, 13, 14, 16, 17, 19, 23, 28b, 37
25	2, 3, 4, 7a, 9		19, 24, 25	88	3, 5, 7, 9, 13, 15, 17, 19, 21b, 23c, 43b
26	3, 5a, 7	60	7, 11c, 13, 19c, 29c	89	2, 3, 4, 5, 6, 7, 8, 9, 12, 13, 14, 16, 17, 20, 23,
27	2, 4, 5, 8	61	2, 3, 4, 5, 6, 7, 8, 9, 11a, 13, 16,		24, 25, 27, 28, 29, 34a, 36
28	3, 5, 13c		17, 21, 22, 24	90	7, 11, 17, 19c, 23, 29
29	2, 3, 4, 5, 8, 9, 12a	62	3, 5, 7, 11, 13, 15, 23	91	2, 3, 4, 5, 6, 8, 9, 11, 12, 16, 19, 20, 22, 25,
30	7, 11c	63	2, 4, 5, 8b, 10, 11, 13, 17, 20		27b, 31, 32, 36
31	2, 3, 4, 5, 7, 11, 12	64	3, 5, 7, 11, 15, 19, 23, 31b	92	3, 5, 7, 9, 11, 15, 17, 19, 21, 33, 45c
32	3, 5, 7, 15b	65	2, 3, 4, 6, 7, 8a, 9, 12, 14b, 17,	93	2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 19, 22, 25, 32b,
33	2, 4, 5, 7, 10b		18a, 19, 21		34
34	3, 5, 9, 13a	66	5, 7, 17, 23c, 25	94	3, 5, 7, 9, 11, 13, 15, 23, 33, 35, 39
35	2, 3, 4, 6b, 8, 11	67	2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13,	95	2, 3, 4, 6, 7, 8, 9, 11, 13, 14, 17, 18, 23, 29,
36	5, 11, 17c		14, 16, 18, 23, 29		31, 39b, 41, 42
37	2, 3, 4, 5, 6a, 7, 8,	68	3, 5, 7, 9, 11, 13, 19, 33c	96	5, 7, 11, 13, 17b, 23, 31c, 47b
	10, 13	69	2, 4, 5, 7, 8, 11, 13, 19, 20, 22b,	97	2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 17, 18, 19, 20,
38	3, 5, 7, 9		28		21, 22a, 23, 25, 26, 28, 30, 33, 35
39	2, 4, 5, 7, 14b, 16	70	3, 9, 11, 13, 17, 29c	98	3, 5, 9, 13, 17, 19, 25, 27, 37, 41
40	3, 7, 9c, 11b, 19b	71	2, 3, 4, 5, 6, 7, 8, 11, 15, 16, 17,	99	2, 4, 5, 7, 8, 10b, 13, 16, 17, 19, 23, 28, 29, 32,
41	2, 3, 4, 5, 6, 9a, 11, 12,		20, 21, 22, 23, 26, 28		40
	13, 16	72	5, 7, 11, 17c, 19b, 23, 35b	100	3, 7, 9, 13, 17, 19, 27, 29, 39, 49c

A list of all the (n, s)-satins with $n < 100$. If the value of s is followed by a, b or c, then the satin is isonemal. The letter a means that the satin is square, b that it is rhombic and c that it is rectangular.

TABLE 3.

THEOREM 6. *The number $w(n)$ of mononemal (but not isonemal) satins of period n is given by* $w(n)=\frac{1}{4}[\phi(n)-2u(n)-2v(n)-2]$, *where ϕ is Euler's phi-function.*

The proof of Theorem 6 depends upon the observation that to each mononemal (but not isonemal) satin of period n correspond exactly four distinct integers less than, and prime to, n, namely s, $n-s$, t and $n-t$ in the notation used above. On the other hand, isonemal satins of either kind correspond to two such integers, and the correction -2 arises from the exclusion of the plain weave.

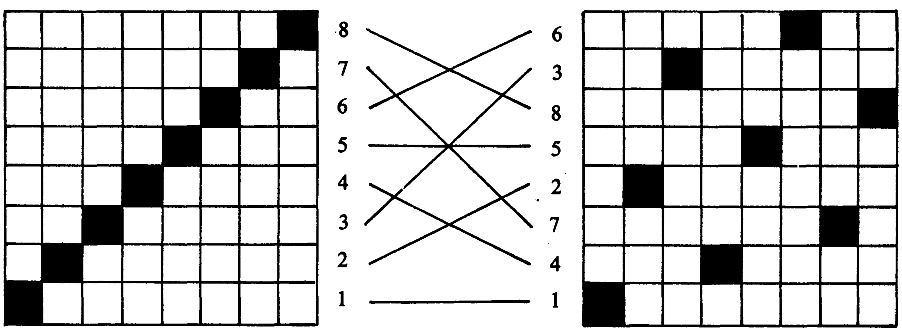
In TABLE 3 we list all possible satins, both mononemal and isonemal, for values of $n \leqslant 100$. This extends the table in Lucas [**9**], besides giving additional information. (Note that the corresponding table in Lucas [**8**] contains many errors.) We are indebted to M. G. Shephard for help with the computations needed in the preparation of TABLE 3. Examination of this table reveals an arithmetical curiosity concerning satins. Let $n'>1$ be any divisor of n and s' be the remainder on dividing s by n'. Then if the (n,s)-satin is square, or symmetric, then the (n',s')-satin is also square, or symmetric, respectively (compare Woods [**12**, p. T306]). Thus, for example, as the $(48,17)$-satin is symmetric, so is the $(12,5)$-satin, and as the $(85,38)$-satin is square, so is the $(17,4)$-satin. The proof of this result is an elementary exercise.

We remark, in conclusion, that a related family of isonemal fabrics can be constructed from the isonemal square satins by the process of "doubling." For this we replace every square in the design by a 2×2 block of squares all colored in the same way. Thus the fabric shown in FIGURE 6(j) is obtained by doubling the $(5,2)$-satin of FIGURE 12(b).

Twillins

Nisbet [**c**] remarks on the fact that a fundamental block of an (n,s)-satin can be obtained from that of an $\frac{1}{n-1}$ twill by suitably rearranging the rows (weft strands) or columns (warp strands) (see FIGURE 13). He then generalizes this procedure by starting from any twill, and so obtains a class of fabrics called **rearranged twills**. For example, FIGURE 8(f) shows a fabric obtained by rearranging the warp strands of the twill $\frac{1\ 2}{2\ 1}$ of FIGURE 3. Most rearranged twills are, as in this example, not isonemal (or even mononemal) and this suggests the following interesting combinatorial problem: *How can one determine the twills $\mathcal{T}$ and the rearrangements of the weft (or warp) strands of $\mathcal{T}$, which lead to isonemal fabrics?* Any isonemal fabric which can be constructed in this way will be called a **twillin** being a generalization of both a twill and a satin. The purpose of this section is to explain how *all* twillins of a given period n can be found.

Let us begin by considering which permutations of the weft strands of a twill are admissible. Let B_t be a fundamental $n\times n$ block for the twill $\mathcal{T}$, and let B_f be a fundamental block obtained from B_t by permuting its rows. Without loss of generality we may suppose that the first row of B_f is the first row of B_t, and the second row of B_f is the $(s+1)$st row of B_t $(1<s<n-1)$. Then this



Rearranging the rows (weft strands) of a twill so as to form a satin.

FIGURE 13.

second row is obtained from the first by an s-step to the right (see FIGURE 13 for the case $s=3$, $n=8$). If the resulting fabric is to be weft-isonemal, then it is clearly necessary that every other row of B_f is obtained from the previous row by an s-step to the right with the same value of s. Thus the permutation of the rows may be written

$$\begin{pmatrix} 1 & 1+s & 1+2s & \cdots & 1+(n-1)s \\ 1 & 2 & 3 & \cdots & n \end{pmatrix}$$

where the integers in the top row are reduced modulo n. Note that s must be prime to n since otherwise the numbers in the top row of the above array would not be distinct and so we would not have a proper permutation. We shall use the term (n,s)-twillin for an isonemal fabric of period n constructed by applying s-steps to the rows of B_t in this way.

Now let us consider how the twill $\mathfrak{F}$ can be chosen so that the resulting fabric is isonemal. In order to illustrate the method we shall consider in detail the special case $n=8$, $s=3$. We begin with an 8×8 block of squares and number the squares in the first (lowest) row with the integers $1,2,\ldots,8$. These numbers are repeated in the other rows using 3-steps to the right between adjacent rows. The resulting 8×8 array will be called an $(8,3)$-**number square** (see FIGURE 14). We observe that every row and every column of this square contains all the integers $1,2,\ldots,8$ just once—this is a consequence of the fact that s was chosen prime to n. Our objective is to convert this number square into a fundamental block for a twillin by replacing each integer by a color (black or white).

For a design produced in this way to be mononemal it is necessary and sufficient that the sequence of colors in each column (warp strand) should either be the same as the sequence of colors in each row (weft strand) or should be so after the colors black and white have been interchanged. This can be achieved in several ways. Let us begin, for example, by seeing if it is possible for the first column of the number square (read upwards) to correspond to the first row. We write these thus,

first row:	1	2	3	4	5	6	7	8
first column:	1	6	3	8	5	2	7	4

and note that the coloring will be the same if 2 and 6, and also 4 and 8, represent the same color. To put it another way, if the above scheme is thought of as representing a permutation and this permutation is written in cycle notation (1)(2 6)(3)(4 8)(5)(7), then all integers in the same cycle must represent the same color. Let us label the six cycles, $A,B,\ldots,F$ and make the corresponding substitutions in the number square of FIGURE 14. We obtain the block labelled I in FIGURE 15. This will be called an $(8,3)$-**letter square**. By construction it has a property corresponding to mononemality, namely that if the plane is covered by translates of this block so as to form a tiling $\mathfrak{T}$ of square tiles labelled with the letters $A,B,\ldots,F$, then each row and column of tiles in $\mathfrak{T}$ will contain the same sequence of letters in the same order. In fact, in this case, a much stronger property corresponding to isonemality also holds; the symmetry group of $\mathfrak{T}$ (that is, the group of isometries which map each tile of $\mathfrak{T}$ onto one bearing the same letter) is transitive on the rows and columns of $\mathfrak{T}$. This transitivity property is a consequence of the fact that $3^2\equiv 1 \pmod 8$: It will always occur whenever we are constructing (n,s)-fabrics with $s^2\equiv \pm 1 \pmod n$. The proof of this assertion follows in exactly the same way as Theorem 3.

In the letter square we now substitute the colors black or white for each of the letters $A,B,\ldots,G$. However we do this, subject only to the overriding condition that the resultant fabric must "hang together," we will obtain a fundamental $n\times n$ block for an $(8,3)$-twillin. Hence if we can determine all possible letter squares, it will be possible to obtain the designs of all $(8,3)$-twillins by systematic substitution of colors for letters.

At first sight it appears that there are a great number of possibilities, but many of these can be eliminated immediately. For one thing, we can ignore all twills and satins since these have already been described and enumerated in the previous two sections, and for another, very many designs are repeated. We do not, of course, consider as distinct fundamental blocks that can be

4	5	6	7	8	1	2	3
7	8	1	2	3	4	5	6
2	3	4	5	6	7	8	1
5	6	7	8	1	2	3	4
8	1	2	3	4	5	6	7
3	4	5	6	7	8	1	2
6	7	8	1	2	3	4	5
1	2	3	4	5	6	7	8

The (8,3)-number square.

FIGURE 14.

D	*E*	*B*	*F*	*D*	*A*	*B*	*C*
F	*D*	*A*	*B*	*C*	*D*	*E*	*B*
B	*C*	*D*	*E*	*B*	*F*	*D*	*A*
E	*B*	*F*	*D*	*A*	*B*	*C*	*D*
D	*A*	*B*	*C*	*D*	*E*	*B*	*F*
C	*D*	*E*	*B*	*F*	*D*	*A*	*B*
B	*F*	*D*	*A*	*B*	*C*	*D*	*E*
A	*B*	*C*	*D*	*E*	*B*	*F*	*D*

I

D	*A*	*D*	*E*	*B*	*A*	*B*	*C*
E	*B*	*A*	*B*	*C*	*D*	*A*	*D*
B	*C*	*D*	*A*	*D*	*E*	*B*	*A*
A	*D*	*E*	*B*	*A*	*B*	*C*	*D*
B	*A*	*B*	*C*	*D*	*A*	*D*	*E*
C	*D*	*A*	*D*	*E*	*B*	*A*	*B*
D	*E*	*B*	*A*	*B*	*C*	*D*	*A*
A	*B*	*C*	*D*	*A*	*D*	*E*	*B*

II

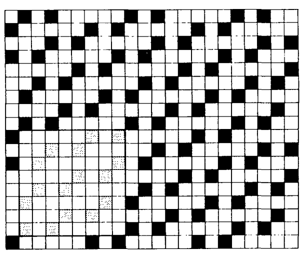
The two (8,3)-letter squares that can be derived from the number square shown in FIGURE 14.

FIGURE 15.

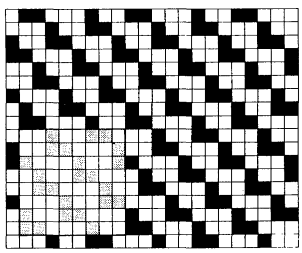
made to coincide by cyclic rearrangement or reversal of the rows or columns, or by interchanging the colors black and white.

From the letter square I of FIGURE 15 substituting colors for letters yields only four twillins, namely those shown in FIGURES 16(a), (b), (c) and (d). Below each diagram we have indicated an allocation of colors by the simple device of separating the letters which represent each of the two colors by a hyphen. By way of example, we note that for the letter square I, *A-BCDEF* is a satin, *B-ACDEF* is a twill, and *AB-CDEF* is identical with *AD-BCEF* and also with *ADEF-BC* since each of these is an isometric image of the others.

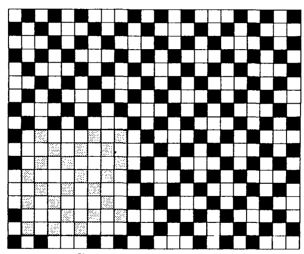
To construct other letter squares, instead of identifying the colors in the first row with those in the first column, we do so after applying a cyclic permutation to the latter. Equivalently we may use any of the other seven columns in the number square. Further possibilities arise if we read the columns *downwards* instead of *upwards*, so eight more cases need to be considered. Some of these lead to letter squares that can only represent twills (for example if we identify 1 2 3 4 5 6 7 8 with 4 7 2 5 8 3 6 1) and we can reject these. Systematic investigation of the possibilities leads to just two letter squares, namely those shown in FIGURE 15. (The second of



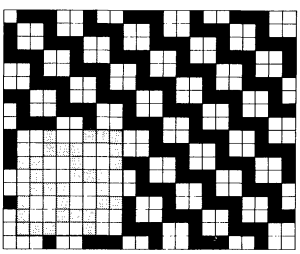
(a) I *AC-BDEF*



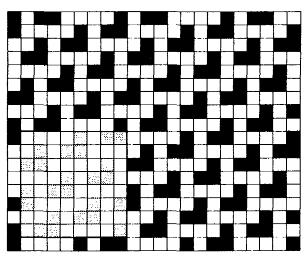
(b) I *AB-CDEF*



(c) I *ACE-BDF*



(d) I *ABC-DEF*



(e) II *BC-ADE*

Five designs for (8,3)-twillins derived from the letter squares of FIGURE 15. In each case we have indicated one possible method of substituting colors for letters.

FIGURE 16.

these comes from identifying the first row 1 2 3 4 5 6 7 8 with the second column 5 8 3 6 1 4 7 2 of the number square read downwards.) These lead in turn to the five fundamental blocks of FIGURE 16; for each we have indicated one possible allocation of colors.

There is still another possibility which we have not yet considered. The fabric can be isonemal if there is an isometry which maps the rows of the design onto the columns with colors reversed. This cannot happen in the case of an (8,3)-twillin, but does occur for (10,3)-twillins (see FIGURE 17). Identifying the first row of the number square with the second column (read upwards), we obtain

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 9 & 6 & 3 & 10 & 7 & 4 & 1 & 8 & 5, \end{matrix}$$

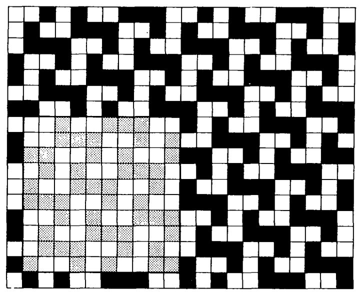
leading to the permutation that can be written as three cycles (1 2 9 8)(3 6 7 4)(5 10). Instead of allocating the same color to all the numbers in a cycle, we do so alternately; this is possible since all the cycles are of even length. Thus we write A for 1 and 9, and A' (the opposite color) for 2 and 8, and so on. This leads to the letter square of FIGURE 17(b), and in FIGURE 17(c) we show an example of the fundamental block of a design obtained from this.

4	5	6	7	8	9	10	1	2	3
7	8	9	10	1	2	3	4	5	6
10	1	2	3	4	5	6	7	8	9
3	4	5	6	7	8	9	10	1	2
6	7	8	9	10	1	2	3	4	5
9	10	1	2	3	4	5	6	7	8
2	3	4	5	6	7	8	9	10	1
5	6	7	8	9	10	1	2	3	4
8	9	10	1	2	3	4	5	6	7
1	2	3	4	5	6	7	8	9	10

(a)

B'	C	B'	B	A'	A	C'	A	A'	B
B	A'	A	C'	A	A'	B	B'	C	B'
C'	A	A'	B	B'	C	B'	B	A'	A
B	B'	C	B'	B	A'	A	C'	A	A'
B'	B	A'	A	C'	A	A'	B	B'	C
A	C'	A	A'	B	B'	C	B'	B	A'
A'	B	B'	C	B'	B	A'	A	C'	A
C	B'	B	A'	A	C'	A	A'	B	B'
A'	A	C'	A	A'	B	B'	C	B'	B
A	A'	B	B'	C	B'	B	A'	A	C'

(b)



(c)

The construction of a (10,3)-twillin for which the symmetry operations which map rows into columns interchange the colors black and white. The fabric in (c) is given by the coloring A-BC. It is also given by the coloring B-AC, while the sponge weave of FIGURE 6(c) is given by the coloring C-AB.

FIGURE 17.

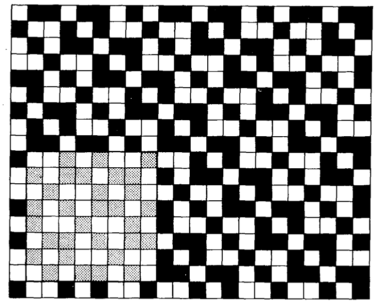
An analogous method to that described above can be used to construct **color-alternate twillins.** (The color-alternate twills mentioned at the end of Section 2 are examples of these with the value $s=1$.) A color-alternate (n,s)-twillin is defined as an isonemal fabric of period n in which each row is obtained from the one below it by an s-step to the right and an interchange of the colors black and white. For example, let us consider the case $n=8$ and $s=3$. Starting from the number square of FIGURE 14 we add primes to the integers in alternate rows (to signify that the colors are interchanged) (see FIGURE 18(a)) and then construct a permutation as before by comparing one of the columns with a row. In the example shown in FIGURE 18 we have identified the first row with the first column read upwards to obtain the permutation (1)(2 6′)(3)(4 8′)(5)(7). Again allocate letters $A,B,\ldots,G$ to these six cycles, and so obtain the letter-square of FIGURE 18(b). As usual, a prime indicates that the colors must be reversed; thus B is substituted for 2 and 6′, which means that B' is substituted for 2′ and 6. From the letter square any allocation of colors to the various letters (subject only to trivial restrictions) will yield an isonemal fabric, that is, a color-alternate twillin. An example of such a fabric is given in FIGURE 18(c) along with the corresponding allocation of colors. As before, there are many possibilities to be explored, though the number of distinct fabrics obtained in this way is not large.

4′	5′	6′	7′	8′	1′	2′	3′
7	8	1	2	3	4	5	6
2′	3′	4′	5′	6′	7′	8′	1′
5	6	7	8	1	2	3	4
8′	1′	2′	3′	4′	5′	6′	7′
3	4	5	6	7	8	1	2
6′	7′	8′	1′	2′	3′	4′	5′
1	2	3	4	5	6	7	8

(a)

D′	*E′*	*B*	*F′*	*D*	*A′*	*B′*	*C′*
F	*D′*	*A*	*B*	*C*	*D*	*E*	*B′*
B′	*C′*	*D′*	*E′*	*B*	*F′*	*D*	*A′*
E	*B′*	*F*	*D′*	*A*	*B*	*C*	*D*
D	*A′*	*B′*	*C′*	*D′*	*E′*	*B*	*F′*
C	*D*	*E*	*B′*	*F*	*D′*	*A*	*B*
B	*F′*	*D*	*A′*	*B′*	*C′*	*D′*	*E′*
A	*B*	*C*	*D*	*E*	*B′*	*F*	*D′*

(b)



(c)

The construction of a color-alternate (8,3)-twillin. The design of (c) is given by the coloring *ABDEF-C*.

FIGURE 18.

The constructions described above for twillins and color-alternate twillins are very simple and lead to many attractive designs for fabrics. For example, the designs of FIGURES 16(c), (d), 17 and 18 seem to be especially pleasing and we find it hard to believe that they have not been used by some practical weaver—yet we can find no mention of them in the literature.

Although the above method enables us to construct all twillins and color-alternate twillins, it does not lead to a solution of the problem of enumerating these fabrics. In fact, since the same twillin can arise in many different ways and there seems to be no way of deciding just how many such ways, the enumeration problem seems completely intractable. And, of course, one must remember that, as we remarked earlier, the twillins and color-alternate twillins compose only a small part of the large class of isonemal fabrics.

New Viewpoints and Questions

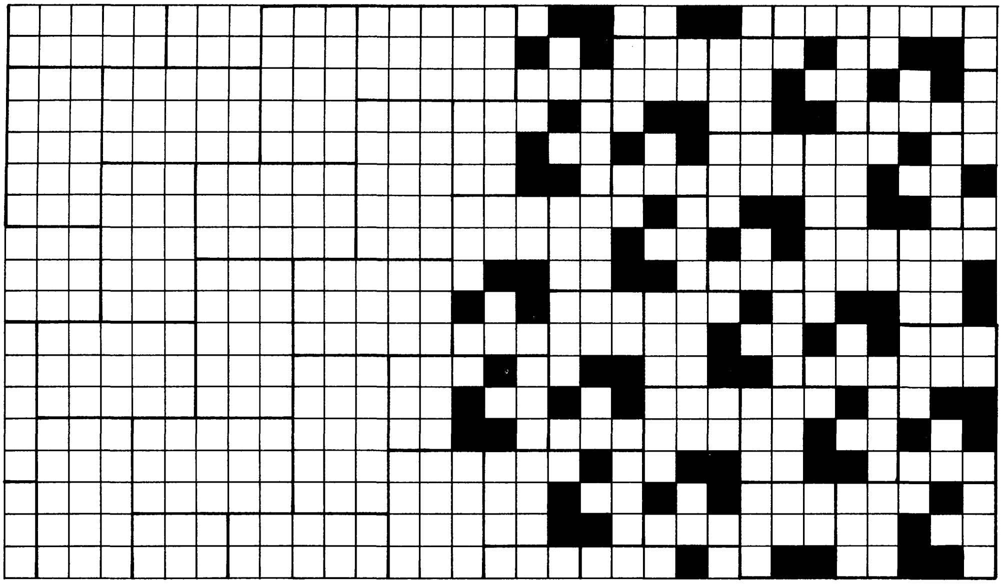
In this section our outlook changes. Instead of constructing isonemal fabrics with a given period, we suppose that we are given *any* block Q of squares colored black and white, and we ask whether this block can form part of the design of an isonemal fabric. The following result, which implies that the answer is in the affirmative, hints at the enormous number of isonemal fabrics of a given period that exist.

THEOREM 7. *Let p and q be relatively prime integers. Then any $p \times q$ rectangular block B of black and white squares is part of the design of a twillin of period $2pq$.*

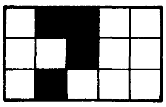
If a block Q of squares is not of the required shape, we can apply the theorem by determining the smallest $p \times q$ block B (with p and q relatively prime) that contains it. In particular, if Q is square of side k, then we can take $p = k$, $q = k+1$, and the period of the twillin is then $2k(k+1)$.

The constructive proof of Theorem 7 is very simple. It is based on the existence of isohedral tilings $\mathfrak{T}$ by rectangular tiles (see FIGURE 19(b) for tiles of size 3×5). Given any 3×5 block B such as that in FIGURE 19(a), we replace each tile of $\mathfrak{T}$ by a copy of B as shown. It is easily verified that the resulting design is that of a fabric (in that it "hangs together") unless B consists entirely of black or white squares, and that the fabric is a twillin of period $2pq$. The exceptional (monochromatic) blocks are easily dealt with by considering suitable satins.

It is probable that, in general, the period $2pq$ cannot be greatly reduced, since each strand must contain copies of the p rows of B (each of length q) and of the q columns of B (each of length p). However, for small values of p and q, better results may be obtained by *ad hoc* methods. For example, every 2×2 block Q is contained in the design of either a twill $\frac{1}{2}$ or the duck weave (FIGURE 6(h)). These are of periods 3 and 4 respectively, which improves the estimate $2(2^2+2)=12$ given by the theorem. We do not know the minimum period for 3×3



(b) (c)



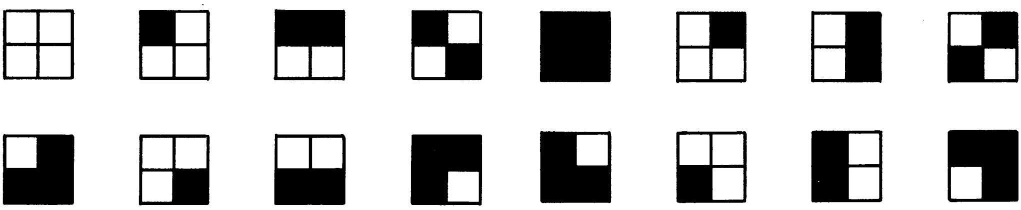
(a)

The construction of a fabric design which incorporates any given block of squares colored black and white. Here a 3×5 block (a) is given. In (c) we show how copies of this block may be substituted for the tiles in a tiling (b) to obtain the design of a (24,11)-twillin.

FIGURE 19.

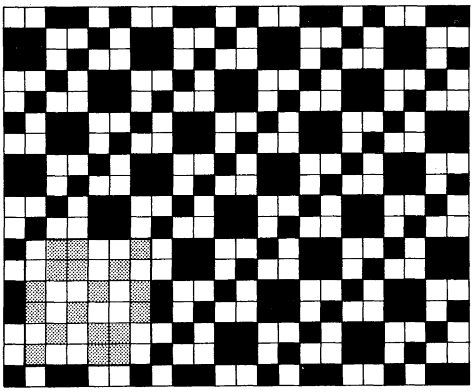
blocks, though we should not be surprised if it can be reduced to a value much less than that $(2(3^2+3)=24)$ given by the theorem.

A related but apparently very difficult problem is to determine for each k the minimal period of an isonemal fabric $\mathcal{F}$ which is k-**universal** in that it contains every $k\times k$ block of squares colored black and white in all possible ways. That universal fabrics exist is an easy consequence of THEOREM 7, for we need only "stack" all possible $k\times k$ blocks together to form a large block Q, and then proceed as before. But the period of the fabric obtained in this way is clearly wildly larger than necessary. There are two possible interpretations of this problem. Of the 16 possible colorings of a 2×2 block (see FIGURE 20) only four are essentially distinct in the sense that all the others can be obtained from these four by a rigid motion or an interchange of colors. Four such essentially different blocks are shown in the top row of the diagram. We can ask either for *all* sixteen blocks to occur in the design of a fabric $\mathcal{F}$ (in which case we shall call $\mathcal{F}$ **strongly**

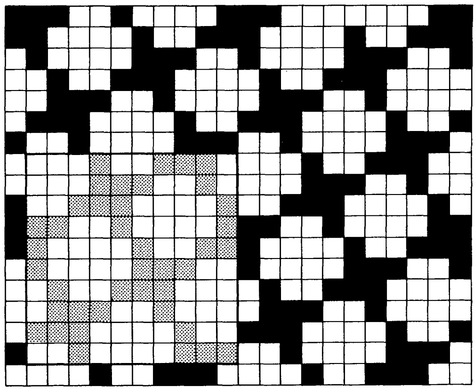


The sixteen different 2×2 blocks of black and white squares. Only four blocks are essentially different (for example, the first four in the top row); all the other blocks can be obtained from these by a rigid motion or, possibly, by interchange of colors.

FIGURE 20.



(a)



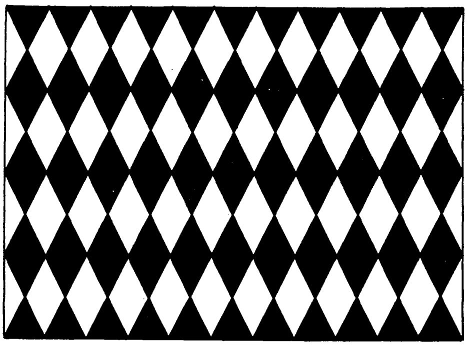
(b)

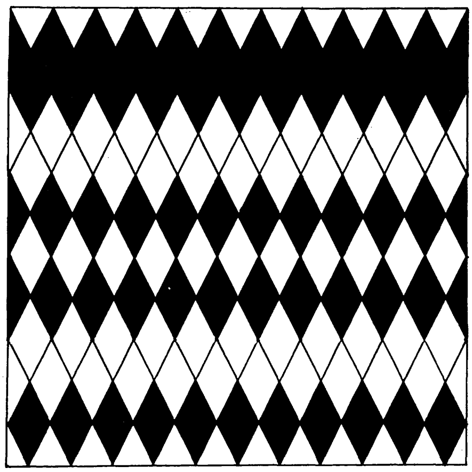
Designs of two 2-universal fabrics. That shown in (b) is a strongly universal twillin. It is believed that these universal fabrics have the smallest possible periods (6 for 2-universal and 10 for strongly 2-universal).

FIGURE 21.

2-universal) or for only four essentially distinct ones. (An example of a fabric which is 2-universal in this latter sense is shown in FIGURE 6(n).) The theory of pantactic squares (see Astle [1] and Bouwkamp et al. [3]) is clearly relevant to the discovery of strongly universal fabrics, and we remark on the curious fact that the minimum period of a strongly 2-universal fabric *only just* fails to be four! The "design" of FIGURE 5(b) is strongly 2-universal, but unfortunately does not represent a fabric that "hangs together." The strongly 2-universal fabric of least period that we have been able to find has period 10 (see FIGURE 21(b)), while for a 2-universal fabric (not strongly 2-universal) the corresponding period is 6 (see FIGURE 21(a)). For 3-universal fabrics we have no results, or conjectures of any kind, and this remains a completely open field for investigation.

Throughout the whole of this paper we have restricted attention to fabrics in which the warp and weft strands are perpendicular to each other. This is not necessary, and there exist isonemal fabrics with oblique strands (see FIGURE 22). In fact every fabric whose design admits, as a symmetry, reflection in a line parallel to $x+y=0$ or $x-y=0$ remains isonemal if its strands are made oblique. Thus all the twills, the symmetric isonemal satins, and many of the twillins, can be made into "oblique fabrics."





Two examples of isonemal fabrics with oblique strands.

FIGURE 22.

We conclude with some general remarks. It is clear that the material in this paper is only the beginning of a large subject; generalizations in many directions are possible and most of these are completely unexplored. Why is class M2 of mononemal fabrics empty? How many distinct (n,s)-twillins exist for small values of n and s? What are the possible symmetry groups of each of the ten types of fabric? Are there any interesting 2-isonemal fabrics (those in which the strands form two transitivity classes under the operations of the symmetry group) apart from the mononemal satins and those that can be obtained by "doubling" any isonemal fabric?

There is no need to restrict attention to the plane. For example a fabric in the shape of a torus can be constructed from two sets of "annular" strands, or even from just two strands if these are allowed to "spiral" round the torus. Recently Jean J. Pedersen has constructed isonemal fabrics on polyhedral surfaces [**10**], but there still remain many open problems concerning fabrics on manifolds and other surfaces in three dimensional space.

Yet another possibility is to investigate fabrics in which the strands lie in more than two directions. (Practical examples of these occur in basketry.) Some results on such fabrics are already known and will be described in a forthcoming paper by the authors. We can already say that in this case a large number of new isonemal fabrics exists.

We are grateful to the referees for suggesting several improvements to this paper, and to Paul J. Campbell for drawing our attention to the work of Lucas [7, **8**, **9**] and Woods [**12**] concerning fabrics. Both these authors mention some earlier literature, mainly concerned with satins, but this seems to be almost inaccessible.

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