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Isohedral tilings of the plane by polygons[1]

BRANKO GRÜNBAUM AND G. C. SHEPHARD:

Dedicated to Hugo Hadwiger on his seventieth birthday

1. Introduction and background

Since antiquity, artists and architects as well as mathematicians have been interested in finding the shapes of polygonal tiles that can be used to tile the plane *monohedrally*, that is, using only tiles that are congruent (directly, or reflectively) to each other. Many papers have considered this problem or parts of it, mostly by exhibiting examples of various monohedral tilings (see, besides the papers mentioned below, [1, 3, 4, 5, 6, 7, 8, 13, 18, 19, 20, 21, 27, 40]), but a complete list of tiles that admit such tilings is still unknown. Claims occasionally made for the completeness have all been based either on error or else on (usually tacit) restrictions imposed on the tiles or tilings. For example, the early work of the MacMahons [31, 32, 33, 34] was restricted to what we shall call isohedral edge-to-edge tilings in which only directly congruent tiles are allowed. Gardner [10] published an expository survey of the problem and what was thought, at that time, to be its complete solution. However, the incorrectness of this assumption was pointed out by several readers (see [11] and the up-to-date survey [39]).

There are several natural variants of the problem – a fact that contributes to its interest, to its difficulty, and to the confusion in the literature. To explain these variants it is necessary to introduce some terminology. We restrict attention to plane tilings τ in which each tile is a closed topological disk. The *vertices* of τ are the points which belong to three or more tiles, and the *edges* of τ are the arcs into which the vertices partition the boundaries of the tiles. To prevent confusion, for a polygonal tile we use the words *corners* and *sides* (instead of the more usual words "vertices" and "edges"). Thus an n-gon has n corners, and n sides each of which is a straight-line segment joining two corners.

Now consider the following conditions (in order of increasing strength) that

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can be placed on the intersections of tiles in a polygonal tiling τ:

I.1 The intersection of any two tiles is a connected set.

I.2 The intersection of any two tiles is contained in a side of each.

I.3 The intersection of any two tiles is either empty, or a corner, or a side of each.

Imposing condition I.1 eliminates tilings such as that of Figure 1(a) (the tile used here was discovered by Voderberg [41, 42]). A tiling which satisfies I.2 is called *proper* and the adoption of this condition eliminates tilings such as those of Figure 1(b). One of the advantages of restricting attention to proper tilings is that, so far as isohedral tilings are concerned, it reduces the possibilities to a finite number of types (see below). If all the tiles in τ are convex, then τ is necessarily proper. A tiling which satisfies I.3 is called *edge-to-edge*, and this condition excludes tilings such as that of Figure 1(c).

Other variants of the problem depend upon the extent in which requirements of symmetry are imposed. Two reasonable conditions are:

S.1 Tilings must be *periodic*, that is, the symmetry group $S(\tau)$ of τ must contain translations in at least two non-parallel directions.

S.2 Tilings must be *isohedral*, that is, the symmetry group $S(\tau)$ must act transitively on the tiles of τ.

Although non-periodic monohedral tilings by polygons are easy to find, the following problem is still unsolved: *Does there exist a polygonal tile T which is aperiodic, that is, admits a tiling of the plane but admits no periodic tiling?* Interest in this problem has been stimulated by Roger Penrose's recent discovery [12, 17] of a *pair* of tiles T_1, T_2 that form an *aperiodic set* (that is, there exists a tiling of the plane using only polygons congruent to T_1 and T_2, but no such tiling is periodic).

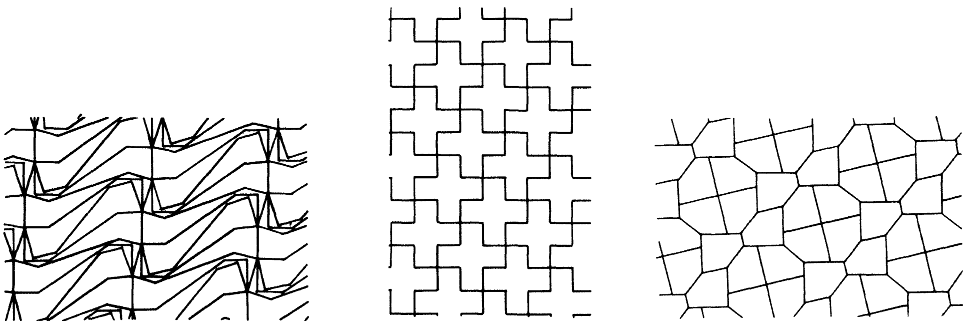

(a) (b) (c)

Figure 1

To enumerate monohedral tilings that satisfy S.1 it is necessary to find all polygons *T* with the property that, using tiles congruent to *T*, it is possible to form a "patch" of tiles of which translates can tile the plane. Although the determination of all such tiles *T* has not been carried out, several systematic (if laborious!) approaches are conceivable.

The present paper is mainly concerned with proper isohedral tilings by arbitrary polygons, that is, with polygonal tilings satisfying conditions I.2 and S.2. The history of the problem of determining such tilings is of interest and we shall review it briefly.

The eighteenth of Hilbert's famous problems [26] asks whether (in our terminology) there is a tile that admits a monohedral tiling of the 3-dimensional space, but admits no isohedral tiling. From the context (see also [38]) it appears that Hilbert assumed that the corresponding planar problem has a negative solution. A similar (although rather vaguely expressed) assumption was made earlier by Fedorov [5, § 64]. The same opinion was shared by K. Reinhardt, who was Hilbert's assistant during part of the years of World War I, and whose dissertation [36] investigated planar tilings by polygons. (We shall mention some of the results of this dissertation below.) Later, Reinhardt [37] found examples of 3-dimensional tiles that admit only non-isohedral tilings, and in the same paper he asserted that no such tiles exist in the plane. He even announced that a paper proving this assertion was in preparation – but Heesch [23] found a counterexample, a tile that admits a periodic tiling of the plane but no isohedral tiling. Heesch's tile (see Figure 2) is non-convex, and leads only to improper tilings, but it was the first example to demonstrate that conditions S.1 and S.2 are genuinely distinct. Variants of Heesch's tile were given in [22, 24, 35]. Another counterexample, which is not related to these and uses a convex tile, will be described below. Milnor's recent account [35] of developments related to Hilbert's eighteenth problem devotes a section to monohedral tilings but goes no further than Heesch's contribution [23] from 1935.

One of the aims of Reinhardt's thesis [36] was the determination of all convex tiles that admit monohedral tilings of the plane; he expected to establish Hilbert's conjecture that each such tile admits isohedral tilings. Reinhardt observed that all

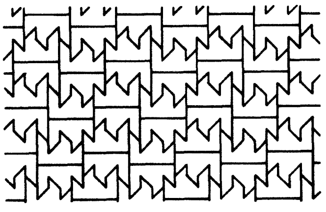
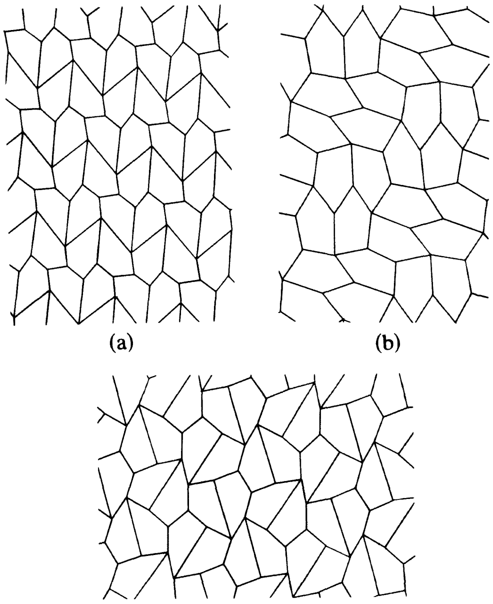


Figure 2

triangles and all quadrangles admit monohedral tilings, that all hexagons with that property may be grouped into three families, and he found five such families of pentagons. He acknowledged the possible omission of some types of pentagons but asserted that their discovery "could be done by the above method; but carrying out such a discussion is highly cumbersome, very laborious, and offers little satisfaction. Moreover, there is a certain probability that no other types of pentagon [besides the five families] will be discovered." [36, p. 85] Actually, although he chose to consider only tilings that are edge-to-edge (I.3), Reinhardt's list is incomplete; he even missed some tiles that admit monohedral edge-to-edge tilings in which the sequences of valences around all the tiles are the same (such as, for example, those of Figures 3(a) and 3(c)).

The first complete list of convex polygons that admit isohedral tilings appears in the book by Heesch and Kienzle [25], reporting on work done in part by Heesch during the nineteen-thirties. They also give examples of non-convex polygons with at most six sides that admit isohedral tilings, without insisting that the tilings are proper.

A different approach was followed by Kershner [28]; he restricted attention to convex polygons and attempted to find all such tiles that admit monohedral tilings. Although – as we have remarked above – he did not succeed in this task, or even in the enumeration of all tiles that admit periodic tilings, he did produce


(a) (b)

(c)

Figure 3

three families of convex pentagons that admit periodic, but not isohedral, tilings (see Figure 3). He thus improved on Heesch's example, and showed that S.1 and S.2 are distinct even for edge-to-edge tilings by convex tiles. Moreover, the example in Figure 3(a) (unlike Heesch's) uses only directly congruent tiles. Kershner's three families, illustrated in Figure 3, should have been included in Reinhardt's list.

Another example of a pentagon that admits a monohedral but no isohedral tiling is shown in Figure 1(c). This was discovered as recently as 1975 by Richard James (see [11, 39]).

In the works of Reinhardt, Heesch-Kienzle, Kershner and others, for each tile that admits an isohedral tiling one example of such a tiling is exhibited. However, it is clearly of greater interest to find *all* the possible types of isohedral tilings that are admitted by a given polygonal tile. The present paper is devoted to this task. All polygonal types of tiling satisfying I.2 and S.2 will be enumerated and the results will be described in detail in the next section.

2. The classification of isohedral tilings by convex polygons

Our classification depends heavily on the notion of *isohedral types* of tilings introduced in our paper [15]. We shall assume that the reader is familiar with its methods and terminology. The classification in [15] deals with tiles of all shapes and thus – for polygonal tiles – with both proper and improper tilings. We now introduce a refinement of this classification which seems appropriate for proper polygonal isohedral tilings.

We shall say that two proper polygonal tilings τ_1 and τ_2 are of the same *polygonal isohedral type* if the following two conditions hold.

P.1 τ_1 and τ_2 are of the same isohedral type in the sense of [15]. This condition can be stated as follows: there exists a group isomorphism $\sigma : S(\tau_1) \to S(\tau_2)$, and a combinatorial isomorphism $\varphi : \tau_1 \to \tau_2$ (which maps the tiles, edges and vertices of τ_1 onto the tiles, edges and vertices of τ_2 and preserves inclusion), such that

$$\begin{tikzcd}
\tau_1 \arrow[r, "\varphi"] \arrow[d, "s"'] & \tau_2 \arrow[d, "\sigma(s)"] \\
\tau_1 \arrow[r, "\varphi"] & \tau_2
\end{tikzcd}$$

is a commutative diagram for all symmetries $s \in S(\tau_1)$.

P.2 A vertex v in τ_1 is a corner of a tile T in τ_1 if and only if $\varphi(v)$ is a corner of the tile $\varphi(T)$ of τ_2.

Expressed more simply, condition P.2 means that a tile T of τ_1 has the same number of sides as the tile $\varphi(T)$ of τ_2; moreover, if any side e of T is made up of two or more edges $e_1, e_2, \ldots$ of τ_1, then the corresponding side $\varphi(e)$ of $\varphi(T)$ is made up of the same number of edges $\varphi(e_1)$, $\varphi(e_2), \ldots$ of τ_2.

The main purpose of this paper is to establish the following result:

THEOREM 1. *In the case of convex tiles, there exist* 14 *polygonal isohedral types of tiling with triangular tiles*; 56 *polygonal isohedral types of tiling with quadrangular tiles*; 24 *polygonal isohedral types of tiling with pentagonal tiles*; 13 *polygonal isohedral types of tiling with hexagonal tiles. There are no other proper polygonal isohedral tilings by convex tiles*; *in particular no types by n-gons with* $n \geq 7$.

The last part of the theorem is the easiest to prove, even with "isohedral" replaced by "monohedral." It is a consequence of the fact that monohedral tilings satisfy *Euler's theorem for tilings* and hence the tiles have at most six edges and so at most six sides. Proofs of Euler's theorem for tilings may be found, for example, in [3, 16, 29, 36]. However, arguments that purport to prove it in many other papers are spurious; recent examples are [30, 43].

The details of the other assertions of Theorem 1 are displayed in Tables I to IV, and examples of these tilings by convex tiles appear in the diagrams following the tables. Hence in order to complete the proof of the theorem it is only necessary to explain how the tables were constructed.

The first stage has, in effect, already been carried out in [15]. From the ninth column of Table I of [15] we see that out of the total of 81 isohedral types, 47 can be realized by convex tiles, that is, by convex polygonal tiles. These yield the 47 types of edge-to-edge tilings listed in the tables that follow, and identified by an asterisk placed near their reference numbers in Column (1) of the tables. All the non-edge-to-edge types can be derived from these 47 by examining each in turn and deciding whether it can also be realized by polygons with fewer sides. In effect, we let some of the interior angles of a polygon take the value π, so that the corresponding vertex is no longer a corner of the tile. The process of examining each case is straightforward, but laborious; see the remarks about Column (10) below.

In Column (2) of the tables we indicate the *net*, or *topological type* of the tiling; this is one of the eleven *Laves tilings* (see, for example, [24, 29]; for more information about these and the following technical details see [15]). Column (3) contains the *incidence symbol* $[A; B]$, where A is a *tile symbol* and B is an *adjacency symbol* in the sense of [15]. In Column (4) we give the international symbol for the *symmetry group* $S(\tau)$ of the tiling, and in Column (5) the *induced*

Table I
Isohedral tilings by triangles.

List number	Net	Incidence symbol	Symmetry group	Tile group	Vertex transitivity	Angle relations	Edge transitivity	Aspects	References
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(11)
$^*P_3$-1	$[3.12^2]$	$[ab^+b^-; ab^-]$	$p6m$	$d1$	$\alpha\alpha\beta$	$A = B = \pi/6$	$\alpha\beta\beta$	6	IH 40
P_3-2	$[4^4]$	$[a^+b^+c^+d^+; a^+b^+c^+d^+]$	$p2$	e	$(\alpha)\alpha\alpha\alpha$	$A = \pi$	$\alpha)\beta\gamma(\delta$	$2D$	IH 46
P_3-3		$[a^+b^+c^+d^+; c^-b^+a^-d^+]$	pgg	e	$(\alpha)\alpha\alpha\alpha$	$A = \pi$	$\alpha)\beta\alpha(\gamma$	$2D, 2R$	IH 51
P_3-4		$[a^+b^+c^+d^+; b^-a^-c^+d^+]$	pgg	e	$\alpha\alpha\alpha(\alpha)$	$A = C, D = \pi$	$\alpha\alpha(\beta\gamma)$	2	IH 53
P_3-5			pgg	e	$(\alpha)\alpha\alpha\alpha$	$A = \pi$	$\alpha)\alpha\beta(\gamma$	$2D, 2R$	IH 53
P_3-6			pgg	e	$\alpha(\alpha)\alpha\alpha$	$B = \pi$	$(\alpha\alpha)\beta\gamma$	$2D, 2R$	IH 53
P_3-7		$[a^+b^+b^-a^-; a^+b^+]$	pmg	$d1(l)$	$(\alpha)\alpha\alpha\alpha$	$A = \pi, B = D$	$\alpha)\beta\beta(\alpha$	2	IH 69
$^*P_3$-8	$[4.6.12]$	$[a^+b^+c^+; a^-b^-c^-]$	$p6m$	e	$\alpha\beta\gamma$	$A = \pi/3, B = \pi/2$	$\alpha\beta\gamma$	$6D, 6R$	IH 77
$^*P_3$-9	$[4.8^2]$	$[a^+b^+c^+; a^+b^-c^-]$	cmm	e	$\alpha\alpha\beta$	$C = \pi/2$	$\alpha\beta\gamma$	$2D, 2R$	IH 78
$^*P_3$-10		$[ab^+b^-; ab^-]$	$p4m$	$d1$	$\alpha\alpha\beta$	$A = B = \pi/4$	$\alpha\beta\beta$	4	IH 82
$^*P_3$-11	$[6^3]$	$[a^+b^+c^+; a^+b^+c^+]$	$p2$	e	$\alpha\alpha\alpha$	–	$\alpha\beta\gamma$	$2D$	IH 84
$^*P_3$-12		$[a^+b^+c^+; a^-b^+c^+]$	pmg	e	$\alpha\alpha\alpha$	–	$\alpha\beta\gamma$	$2D, 2R$	IH 85
$^*P_3$-13		$[ab^+b^-; ab^+]$	cmm	$d1$	$\alpha\alpha\alpha$	$A = B$	$\alpha\beta\beta$	2	IH 91
$^*P_3$-14		$[aaa; a]$	$p6m$	$d3$	$\alpha\alpha\alpha$	$A = B = C = \pi/3$	$\alpha\alpha\alpha$	2	IH 93

Table II
Isohedral tilings by quadrangles.

List number (1)	Net (2)	Incidence symbol (3)	Symmetry group (4)	Tile group (5)	Vertex transitivity (6)	Angle relations (7)	Edge transitivity (8)	Aspects (9)	Related tilings (10)	References (11)
P_4-1	$[3^6]$	$[a^+b^+c^+d^+e^+f^+; b^-a^-f^+e^-d^-c^+]$	pg	e	$(\alpha)\alpha\alpha\beta(\beta)\beta$	$A=E=B+C=D+F=\pi$	$\alpha)\alpha\beta(\gamma\gamma)(\beta$	$1D$, $1R$	—	IH 2
P_4-2		$[a^+b^+c^+d^+e^+f^+; c^-e^+a^-f^-b^+d^-]$	pg	e	$(\alpha)\beta\alpha(\beta)\alpha\beta$	$B=E$, $C=F$, $A=D=\pi$	$\alpha)\beta(\alpha\gamma)\beta(\gamma$	$1D$, $1R$	—	IH 3
P_4-3		$[a^+b^+c^+d^+e^+f^+; a^+e^+c^+d^+b^+f^+]$	$p2$	e	$\alpha\alpha\beta\beta(\beta)(\alpha)$	$A+B=C+D=E=F=\pi$	$\alpha\beta\gamma(\delta\beta\varepsilon)$	$2D$	—	IH 4
P_4-4			$p2$	e	$(\alpha)\alpha\beta\beta(\beta)\alpha$	$A=E=C+D=B+F=\pi$	$\alpha)\beta\gamma(\delta\beta)(\varepsilon$	$2D$	—	IH 4
P_4-5			$p2$	e	$(\alpha)\alpha\beta(\beta)\beta\alpha$	$A=D=\pi$, $B=E$, $C=F$	$\alpha)\beta(\gamma\delta)\beta(\varepsilon$	$1D$	—	IH 4
P_4-6			$p2$	e	$\alpha(\alpha)\beta\beta(\beta)\alpha$	$B=E=C+D=A+F=\pi$	$(\alpha\beta)\gamma(\delta\beta)\varepsilon$	$2D$	—	IH 4
P_4-7		$[a^+b^+c^+d^+e^+f^+; a^+e^+d^-c^-b^+f^+]$	pgg	e	$(\alpha)\alpha\beta(\beta)\beta\alpha$	$A=D=\pi$, $B=E$, $C=F$	$\alpha)\beta(\gamma\gamma)\beta(\delta$	$1D$, $1R$	—	IH 5
P_4-8			pgg	e	$(\alpha)\alpha\beta\beta(\beta)\alpha$	$A=E=C+D=B+F=\pi$	$\alpha)\beta\gamma(\gamma\beta)(\delta$	$2D$, $2R$	—	IH 5
P_4-9			pgg	e	$\alpha\alpha(\beta)\beta\beta(\alpha)$	$C=F=A+B=D+E=\pi$	$\alpha(\beta\gamma)\gamma(\beta\delta)$	$2D$, $2R$	—	IH 5
P_4-10			pgg	e	$\alpha(\alpha)\beta(\beta)\beta\alpha$	$B=D=C+E=A+F=\pi$	$(\alpha\beta)(\gamma\gamma)\beta\delta$	$2D$, $2R$	—	IH 5
P_4-11			pgg	e	$\alpha(\alpha)(\beta)\beta\beta\alpha$	$B=C=D+E=A+F=\pi$	$(\alpha\beta\gamma)\gamma\beta\delta$	$2D$, $2R$	—	IH 5
P_4-12		$[a^+b^+c^+d^+e^+f^+; a^+e^-c^+f^-b^-d^-]$	pgg	e	$(\alpha)\alpha\beta(\beta)\alpha\beta$	$A=D=B+C=E+F=\pi$	$\alpha)\beta(\gamma\delta)\beta(\delta$	2	—	IH 6
P_4-13			pgg	e	$\alpha(\alpha)\beta\beta\alpha(\beta)$	$B=F=A+E=C+D=\pi$	$(\alpha\beta)\gamma\delta(\beta\delta)$	$2D$, $2R$	—	IH 6
P_4-14			pgg	e	$\alpha\alpha\beta\beta(\alpha)(\beta)$	$A+B=C+D=E=F=\pi$	$\alpha\beta\gamma(\delta\beta\delta)$	$2D$, $2R$	—	IH 6
P_4-15			pgg	e	$(\alpha)\alpha(\beta)\beta\alpha\beta$	$A=C=B+E=D+F=\pi$	$\alpha)(\beta\gamma)\delta\beta(\delta$	$2D$, $2R$	—	IH 6
P_4-16			pgg	e	$(\alpha)\alpha\beta\beta\alpha(\beta)$	$A=F=B+E=C+D=\pi$	$\alpha)\beta\gamma\delta(\beta\delta$	$2D$, $2R$	—	IH 6
P_4-17		$[a^+b^+c^+a^+b^+c^+; a^+b^+c^+]$	$p2$	$c2$	$\alpha(\alpha)\alpha\alpha(\alpha)\alpha$	$B=E=\pi$, $A=D$, $C=F$	$(\alpha\beta)\gamma(\alpha\beta)\gamma$	$1D$		IH 8
P_4-18		$[a^+b^+c^+a^+b^+c^+; a^+c^-b^-]$	pgg	$c2$	$\alpha\alpha(\alpha)\alpha\alpha(\alpha)$	$c=F=\pi$, $A=D$, $B=E$	$\alpha(\beta\beta)\alpha(\beta\beta)$	$1D$, $1R$	—	IH 9
P_4-19			pgg	$c2$	$(\alpha)\alpha\alpha(\alpha)\alpha\alpha$	$A=D=\pi$, $B=E$, $C=F$	$\alpha)\beta(\beta\alpha)\beta(\beta$	$1D$, $1R$	—	IH 9
P_4-20		$[ab^+c^+dc^-b^-; db^+c^+a]$	pmg	$d1(s)$	$\alpha\alpha\alpha(\alpha)(\alpha)\alpha$	$A=B$, $C=F$, $D=E=\pi$	$\alpha\beta(\gamma\alpha\gamma)\beta$	2	—	IH 13
P_4-21			pmg	$d1(s)$	$\alpha\alpha(\alpha)\alpha\alpha(\alpha)$	$A=B=D=E=\frac{\pi}{2}$, $C=F=\pi$	$\alpha(\beta\gamma)\alpha(\gamma\beta)$	1	—	IH 13
P_4-22		$[ab^+b^-ab^+b^-; ab^+]$	cmm	$d2$	$\alpha\alpha(\alpha)\alpha\alpha(\alpha)$	$A=B=D=E=\frac{\pi}{2}$, $C=F=\pi$	$\alpha(\beta\beta)\alpha(\beta\beta)$	1	—	IH 17

P$_4$-23	[$3^4.6$]	$[a^+b^+c^+d^+e^+;\ e^+c^+b^+d^+a^+]$	$p6$	e	$\alpha\beta\gamma\beta(\beta)$	$A=\frac{\pi}{3},\ C=\frac{2\pi}{3},\ E=\pi$	$\alpha\beta\beta(\gamma\alpha)$	$6D$	—	IH 21
P$_4$-24			$p6$	e	$\alpha(\beta)\gamma\beta\beta$	$A=\frac{\pi}{3},\ C=\frac{2\pi}{3},\ B=\pi$	$(\alpha\beta)\beta\gamma\alpha$	$6D$	—	IH 21
P$_4$-25			$p6$	e	$\alpha\beta\gamma(\beta)\beta$	$A=\frac{\pi}{3},\ C=\frac{2\pi}{3},\ D=\pi$	$\alpha\beta(\beta\gamma)\alpha$	$6D$	—	IH 21
P$_4$-26	[$3^3.4^2$]	$[a^+b^+c^+d^+e^+;\ a^-e^+d^-c^-b^+]$	cm	e	$\alpha\alpha\beta\beta(\beta)$	$A+B=C+D=E=\pi$	$\alpha\beta\gamma(\gamma\beta)$	$1D$, $1R$	—	IH 22
P$_4$-27			cm	e	$\alpha\alpha\beta(\beta)\beta$	$A+B=C+E=D=\pi$	$\alpha\beta(\gamma\gamma)\beta$	$1D$, $1R$	—	IH 22
P$_4$-28		$[a^+b^+c^+d^+e^+;\ a^+e^+c^+d^+b^+]$	$p2$	e	$\alpha\alpha\beta(\beta)\beta$	$A+B=C+E=D=\pi$	$\alpha\beta(\gamma\delta)\beta$	$1D$	—	IH 23
P$_4$-29			$p2$	e	$\alpha\alpha(\beta)\beta\beta$	$A+B=D+E=C=\pi$	$\alpha(\beta\gamma)\delta\beta$	$2D$	—	IH 23
P$_4$-30		$[a^+b^+c^+d^+e^+;\ a^-e^+c^+d^+b^+]$	pmg	e	$\alpha\alpha\beta(\beta)\beta$	$A+B=C+E=D=\pi$	$\alpha\beta(\gamma\delta)\beta$	$1D$, $1R$	—	IH 24
P$_4$-31			pmg	e	$\alpha\alpha(\beta)\beta\beta$	$A+B=D+E=C=\pi$	$\alpha(\beta\gamma)\delta\beta$	$2D$, $2R$	—	IH 24
P$_4$-32		$[a^+b^+c^+d^+e^+;\ a^+e^+d^-c^-b^+]$	pgg	e	$\alpha\alpha\beta(\beta)\beta$	$A+B=C+E=D=\pi$	$\alpha\beta(\gamma\gamma)\beta$	$1D$, $1R$	—	IH 25
P$_4$-33			pgg	e	$\alpha\alpha(\beta)\beta\beta$	$A+B=D+E=C=\pi$	$\alpha(\beta\gamma)\gamma\beta$	$2D$, $2R$	—	IH 25
P$_4$-34		$[ab^+c^+c^-b^-;\ ab^-c^+]$	cmm	$d1$	$\alpha\alpha\beta(\beta)\beta$	$A=B=C=E=\frac{\pi}{2},$ $D=\pi$	$\alpha\beta(\gamma\gamma)\beta$	1	—	IH 26
P$_4$-35	[$3^2.4.3.4$]	$[a^+b^+c^+d^+e^+;\ a^+d^-e^-b^-c^-]$	pgg	e	$(\alpha)\alpha\beta\alpha\beta$	$A=\pi,\ B=E,\ C=D$	$\alpha)\beta\gamma\beta(\gamma$	2	—	IH 27
P$_4$-36			pgg	e	$\alpha\alpha\beta(\alpha)\beta$	$A+B=D=\pi$	$\alpha\beta(\gamma\beta)\gamma$	$2D$, $2R$	—	IH 27
P$_4$-37		$[a^+b^+c^+d^+e^+;\ a^+c^+b^+e^+d^+]$	$p4$	e	$\alpha\alpha\beta(\alpha)\gamma$	$A+B=D=\pi,\ C=E=\frac{\pi}{2}$	$\alpha\beta(\beta\gamma)\gamma$	$4D$	—	IH 28
P$_4$-38			p4	e	$(\alpha)\alpha\beta\alpha\gamma$	$A=B+D=\pi,\ C=E=\frac{\pi}{2}$	$\alpha)\beta\beta\gamma(\gamma$	$4D$	—	IH 28
P$_4$-39		$[ab^+c^+c^-b^-;\ ac^+b^+]$	$p4g$	$d1$	$\alpha\alpha\beta(\alpha)\beta$	$A=B=C=E=\frac{\pi}{2},\ D=\pi$	$\alpha\beta(\beta\beta)\beta$	2	—	IH 29
***P$_4$-40**	[3.4.6.4]	$[a^+b^+c^+d^+;\ a^-b^-d^+c^+]$	$p31m$	e	$\alpha\beta\alpha\gamma$	$B=\frac{\pi}{6},\ D=\frac{\pi}{3}$	$\alpha\beta\gamma\gamma$	$3D$, $3R$	—	IH 30
***P$_4$-41**		$[a^+a^-b^+b^-;\ a^-b^-]$	$p6m$	$d1$	$\alpha\beta\alpha\gamma$	$A=C=\frac{\pi}{2},\ B=\frac{\pi}{3},\ D=\frac{2\pi}{3}$	$\alpha\alpha\beta\beta$	6	—	IH 32
***P$_4$-42**	[3.6.3.6]	$[a^+a^-a^+a^-;\ a^-]$	$p6m$	$d2$	$\alpha\beta\alpha\beta$	$A=C=\frac{2\pi}{3},\ B=D=\frac{\pi}{3}$	$\alpha\alpha\alpha\alpha$	3	—	IH 37
***P$_4$-43**	[4^4]	$[a^+b^+c^+d^+;\ a^+b^+c^+d^+]$	$p2$	e	$\alpha\alpha\alpha\alpha$	—	$\alpha\beta\gamma\delta$	$2D$	$A(P_3\text{-}2)$	IH 46
***P$_4$-44**		$[a^+b^+c^+d^+;\ a^-b^+c^-d^+]$	pmg	e	$\alpha\beta\beta\alpha$	$A+D=B+C=\pi$	$\alpha\beta\gamma\delta$	$2D$, $2R$	—	IH 49

Table II (*Continued*)

List number (1)	Net (2)	Incidence symbol (3)	Symmetry group (4)	Tile group (5)	Vertex transitivity (6)	Angle relations (7)	Edge transitivity (8)	Aspects (9)	Related tilings (10)	References (11)
*P$_4$-45	[4^4]	$[a^+b^+c^+d^+;\ c^+b^-a^+d^+]$	*pmg*	*e*	αββα	$A=C,\ B=D$	αβαγ	1*D*, 1*R*	—	IH 50
*P$_4$-46		$[a^+b^+c^+d^+;\ c^-b^+a^-d^+]$	*pgg*	*e*	αααα	—	αβαγ	2*D*, 2*R*	$A(P_3\text{-}3)$	IH 51
*P$_4$-47		$[a^+b^+c^+d^+;\ b^-a^-c^+d^+]$	*pgg*	*e*	αααα	—	ααβγ	2*D*, 2*R*	$A(P_3\text{-}5)$, $B(P_3\text{-}6)$, $D(P_3\text{-}4)$	IH 53
*P$_4$-48		$[a^+b^+c^+d^+;\ a^-b^-c^-d^+]$	*cmm*	*e*	αβγα	$B=C=\frac{\pi}{2},\ A+D=\pi$	αβγδ	2*D*, 2*R*	—	IH 54
*P$_4$-49		$[a^+b^+c^+d^+;\ b^+a^+c^-d^-]$	*p4g*	*e*	αβαγ	$A+C=\pi,\ B=D=\frac{\pi}{2}$	ααβγ	4*D*, 4*R*	—	IH 56
*P$_4$-50		$[a^+b^+a^+b^+;\ a^+b^+]$	*p*2	*c*2	αααα	$A=C,\ B=D$	αβαβ	1*D*	—	IH 57
*P$_4$-51		$[a^+b^+a^+b^+;\ a^-b^+]$	*pmg*	*c*2	αααα	$A=C,\ B=D$	αβαβ	1*D*, 1*R*	—	IH 58
*P$_4$-52		$[ab^+cb^-;\ ab^+c]$	*cmm*	*d*1(*s*)	αααα	$A=B,\ C=D$	αβγβ	2	—	IH 67
*P$_4$-53		$[a^+b^+b^-a^-;\ a^+b^+]$	*pmg*	*d*1(*l*)	αααα	$B=D$	αββα	2	$A(P_3\text{-}7)$	IH 69
*P$_4$-54		$[abab;\ ab]$	*pmm*	*d*2(*s*)	αααα	$A=B=C=D=\frac{\pi}{2}$	αβαβ	1	—	IH 72
*P$_4$-55		$[a^+a^-a^+a^-;\ a^+]$	*cmm*	*d*2(*l*)	αααα	$A=C,\ B=D$	αααα	1	—	IH 74
*P$_4$-56		$[aaaa;\ a]$	*p4m*	*d*4	αααα	$A=B=C=D=\frac{\pi}{2}$	αααα	1	—	IH 76

Table III.
Isohedral tilings by pentagons.
(In the last column of this table and of Table IV, the references beginning HK are to the book [25] by Heesch and Kienzle and those beginning S are to the paper [39] by Schattschneider.)

List number (1)	Net (2)	Incidence symbol (3)	Symmetry group (4)	Tile group (5)	Vertex transitivity (6)	Angle relations. (7)	Edge transitivity (8)	Aspects (9)	Related tilings (10)	References (11)
P_5-1	$[3^6]$	$[a^+b^+c^+d^+e^+f^+;$ $b^-a^-f^+e^-d^-c^+]$	pg	e	$\alpha(\alpha)\alpha\beta\beta\beta$	$B=A+C=\pi$	$(\alpha\alpha)\beta\gamma\gamma\beta$	$1D$, $1R$	$D(P_4\text{-}1)$, $E[P_4\text{-}18]$	IH 2, S-15
P_5-2			pg	e	$(\alpha)\alpha\alpha\beta\beta\beta$	$A=B+C=\pi$	$\alpha)\alpha\beta\gamma\gamma(\beta$	$1D$, $1R$	$E(P_4\text{-}1)$, $D[P_4\text{-}19]$	IH 2, S-17
P_5-3		$[a^+b^+c^+d^+e^+f^+;$ $c^-e^+a^-f^-b^+d^-]$	pg	e	$\alpha(\beta)\alpha\beta\alpha\beta$	$B=A+F=\pi$	$(\alpha\beta)\alpha\gamma\beta\gamma$	$1D$, $1R$	$C[P_4\text{-}20]$, $E[P_4\text{-}19]$	IH 3, S-19
P_5-4		$[a^+b^+c^+d^+e^+f^+;$ $a^+e^+c^+d^+b^+f^+]$	$p2$	e	$\alpha\alpha(\beta)\beta\beta\alpha$	$C=D+E=\pi$	$\alpha(\beta\gamma)\delta\beta\varepsilon$	$2D$	$B(P_4\text{-}3)$, $A(P_4\text{-}4)$, $F(P_4\text{-}6)$	IH 4, HK-P5-1, S-24
P_5-5			$p2$	e	$\alpha\alpha\beta(\beta)\beta\alpha$	$D=C+E=\pi$	$\alpha\beta(\gamma\delta)\beta\varepsilon$	$2D$	$A(P_4\text{-}5)$, $B(P_4\text{-}4)$	IH 4, S-10
P_5-6		$[a^+b^+c^+d^+e^+f^+;$ $a^+e^+d^-c^-b^+f^+]$	pgg	e	$\alpha\alpha\beta\beta\beta(\alpha)$	$F=A+B=\pi$	$\alpha\beta\gamma\gamma(\beta\delta)$	$2D$, $2R$	$C(P_4\text{-}9)$, $D(P_4\text{-}10)$, $E(P_4\text{-}11)$	IH 5, S-12
P_5-7			pgg	e	$\alpha\alpha(\beta)\beta\beta\alpha$	$C=D+E=\pi$	$\alpha(\beta\gamma)\gamma\beta\delta$	$2D$, $2R$	$A(P_4\text{-}8)$, $B(P_4\text{-}11)$, $F(P_4\text{-}9)$	IH 5, S-13
P_5-8			pgg	e	$\alpha\alpha\beta(\beta)\beta\alpha$	$D=C+E=\pi$	$\alpha\beta(\gamma\gamma)\beta\delta$	$2D$, $2R$	$A(P_4\text{-}7)$, $B(P_4\text{-}10)$	IH 5, S-11
P_5-9			pgg	e	$(\alpha)\alpha\beta\beta\beta\alpha$	$A=B+F=\pi$	$\alpha)\beta\gamma\gamma\beta(\delta$	$2D$, $2R$	$C(P_4\text{-}8)$, $D(P_4\text{-}7)$	IH 5, S-16
P_5-10		$[a^+b^+c^+d^+e^+f^+;$ $a^+e^-c^+f^-b^-d^-]$	pgg	e	$\alpha\alpha(\beta)\beta\alpha\beta$	$C=D+F=\pi$	$\alpha(\beta\gamma)\delta\beta\delta$	$2D$, $2R$	$A(P_4\text{-}15)$, $B[P_4\text{-}20]$, $E(P_4\text{-}13)$	IH 6, HK-P5-3, S-20
P_5-11			pgg	e	$\alpha\alpha\beta(\beta)\alpha\beta$	$D=C+F=\pi$	$\alpha\beta(\gamma\delta)\beta\delta$	$2D$, $2R$	$A(P_4\text{-}12)$, $B(P_4\text{-}15)$, $E(P_4\text{-}16)$	IH 6, S-21
P_5-12			pgg	e	$\alpha\alpha\beta\beta(\alpha)\beta$	$E=A+B=\pi$	$\alpha\beta\gamma(\delta\beta)\delta$	$2D$, $2R$	$C(P_4\text{-}13)$, $D(P_4\text{-}16)$. $F(P_4\text{-}14)$	IH 6, S-14
P_5-13		$[a^+b^+c^+d^+e^+f^+;$ $b^+a^+d^+c^+f^+e^+]$	$p3$	e	$(\alpha)\beta\alpha\gamma\alpha\delta$	$A=\pi,$ $B=D=F=\frac{2\pi}{3}$	$\alpha)\alpha\beta\beta\gamma(\gamma$	$3D$	—	IH 7, HK-P5-10, S-22
P_5-14		$[a^+b^+c^+c^-b^-a^-;$ $a^+b^-c^+]$	pmg	$d1$	$(\alpha)\alpha\beta\beta\beta\alpha$	$A=\pi,$ $B=F=\frac{\pi}{2}, C=E$	$\alpha)\beta\gamma\gamma\beta(\alpha$	2	$D[P_4\text{-}22]$	IH 15, S-18
P_5-15		$[a^+b^+c^+c^-b^-a^-;$ $a^-c^+b^+]$	$p31m$	$d1$	$\alpha\beta\gamma(\beta)\gamma\beta$	$A=C=E=\frac{2\pi}{3},$ $B=F=\frac{\pi}{2},\ D=\pi$	$\alpha\beta(\beta\beta)\beta\alpha$	3	—	IH 16, HK-P5-8, S-23

Table III (*Continued*)

List number (1)	Net (2)	Incidence symbol (3)	Symmetry group (4)	Tile group (5)	Vertex transitivity (6)	Angle relations (7)	Edge transitivity (8)	Aspects (9)	Related tilings (10)	References (11)
*P_5-16	$[3^4.6]$	$[a^+b^+c^+d^+e^+;$ $e^+c^+b^+d^+a^+]$	$p6$	e	$\alpha\beta\gamma\beta\beta$	$A=\frac{2\pi}{3}$, $C=\frac{\pi}{3}$	$\alpha\beta\beta\gamma\alpha$	6D	$B(P_4$-24), $D(P_4$-25), $E(P_4$-23)	IH 21, S-2
*P_5-17	$[3^3.4^2]$	$[a^+b^+c^+d^+e^+;$ $a^-e^+d^-c^-b^+]$	cm	e	$\alpha\alpha\beta\beta\beta$	$A+B=\pi$	$\alpha\beta\gamma\gamma\beta$	1D, 1R	$C(P_4$-26), $D(P_4$-27)	IH 22, S-6
*P_5-18		$[a^+b^+c^+d^+e^+;$ $a^+e^+c^+d^+b^+]$	$p2$	e	$\alpha\alpha\beta\beta\beta$	$A+B=\pi$	$\alpha\beta\gamma\delta\beta$	2D	$C(P_4$-29), $D(P_4$-28)	IH 23, S-4
*P_5-19		$a^+b^+c^+d^+e^+;$ $a^-e^+c^+d^+b^+]$	pmg	e	$\alpha\alpha\beta\beta\beta$	$A+B=\pi$	$\alpha\beta\gamma\delta\beta$	2D, 2R	$C(P_4$-31), $D(P_4$-30)	IH 24, S-5
*P_5-20		$[a^+b^+c^+d^+e^+;$ $a^+e^+d^-c^-b^+]$	pgg	e	$\alpha\alpha\beta\beta\beta$	$A+B=\pi$	$\alpha\beta\gamma\gamma\beta$	2D, 2R	$C(P_4$-33), $D(P_4$-32)	IH 25, S-7
*P_5-21		$[ab^+c^+c^-b^-;$ $ab^-c^+]$	cmm	$d1$	$\alpha\alpha\beta\beta\beta$	$A=B=\frac{\pi}{2}$, $C=E$	$\alpha\beta\gamma\gamma\beta$	2	$D(P_4$-34)	IH 26, S-1
*P_5-22	$[3^2.4.3.4]$	$[a^+b^+c^+d^+e^+;$ $a^+d^-e^-b^-c^-]$	pgg	e	$\alpha\alpha\beta\alpha\beta$	$C+E=\pi$	$\alpha\beta\gamma\beta\gamma$	2D, 2R	$A(P_4$-35), $D(P_4$-36)	IH 27, S-8
*P_5-23		$[a^+b^+c^+d^+e^+;$ $a^+c^+b^+e^+d^+]$	$p4$	e	$\alpha\alpha\beta\alpha\gamma$	$C=E=\frac{\pi}{2}$	$\alpha\beta\beta\gamma\gamma$	4D	$A(P_4$-38), $D(P_4$-37)	IH, 28, HK-P5-9, S-9
*P_5-24		$[ab^+c^+c^-b^-;$ $ac^+b^+]$	$p4g$	$d1$	$\alpha\alpha\beta\alpha\beta$	$A=B$, $C=E=\frac{\pi}{2}$	$\alpha\beta\beta\beta\beta$	4	$D(P_4$-39)	IH 29, S-3

Table IV.
Isohedral tilings by hexagons.

List number (1)	Net (2)	Incidence symbol (3)	Symmetry group (4)	Tile group (5)	Vertex transitivity (6)	Angle relations (7)	Edge transitivity (8)	Aspects (9)	Related tilings (10)	References (11)
$^*P_6\text{-}1$	$[3^6]$	$[a^+b^+c^+d^+e^+f^+;$ $b^-a^-f^+e^-d^-c^+]$	pg	e	$\alpha\alpha\alpha\beta\beta\beta$	$A+B+C=D+E+F$ $=2\pi$	$\alpha\alpha\beta\gamma\gamma\beta$	$1D$, $1R$	$A(P_5\text{-}2)$, $B(P_5\text{-}1)$, $AD[P_4\text{-}19]$, $AE(P_4\text{-}1)$, $BE[P_4\text{-}18]$	IH 2
$^*P_6\text{-}2$		$a^+b^+c^+d^+e^+f^+;$ $c^-e^+a^-f^-b^+d^-]$	pg	e	$\alpha\beta\alpha\beta\alpha\beta$	$A+B+F=C+D+E$ $=2\pi$	$\alpha\beta\alpha\gamma\beta\gamma$	$1D$, $1R$	$B(P_5\text{-}3)$, $AD(P_4\text{-}2)$, $BC[P_4\text{-}20]$, $BE[P_4\text{-}19]$	IH 3
$^*P_6\text{-}3$		$[a^+b^+c^+d^+e^+f^+;$ $a^+e^+c^+d^+b^+f^+]$	$p2$	e	$\alpha\alpha\beta\beta\beta\alpha$	$A+B+F=C+D+E$ $=2\pi$	$\alpha\beta\gamma\delta\beta\varepsilon$	$2D$	$A(P_5\text{-}5)$, $B(P_5\text{-}4)$, $AD(P_4\text{-}5)$, $AE(P_4\text{-}4)$, $BC(P_4\text{-}3)$, $BE(P_4\text{-}6)$	IH 4, HK-P6-4
$^*P_6\text{-}4$		$[a^+b^+c^+d^+e^+f^+;$ $a^+e^+d^-c^-b^+f^+]$	pgg	e	$\alpha\alpha\beta\beta\beta\alpha$	$A+B+F=C+D+E$ $=2\pi$	$\alpha\beta\gamma\gamma\beta\delta$	$2D$, $2R$	$A(P_5\text{-}9)$, $B(P_5\text{-}6)$, $C(P_5\text{-}7)$, $D(P_5\text{-}9)$, $AC(P_4\text{-}8)$, $AD(P_4\text{-}7)$, $BC(P_4\text{-}11)$, $BD(P_4\text{-}10)$, $BE(P_4\text{-}9)$	IH 5
$^*P_6\text{-}5$		$[a^+b^+c^+d^+e^+f^+;$ $a^+e^-c^+f^-b^-d^-]$	pgg	e	$\alpha\alpha\beta\beta\alpha\beta$	$A+B+E=C+D+F$ $=2\pi$	$\alpha\beta\gamma\delta\beta\delta$	$2D$, $2R$	$A(P_5\text{-}11)$, $B(P_5\text{-}10)$, $E(P_5\text{-}12)$, $AC(P_4\text{-}15)$, $AD(P_4\text{-}12)$, $AF(P_4\text{-}16)$, $BC[P_4\text{-}20]$, $BF(P_4\text{-}13)$	IH 6, HK-P6-10
$^*P_6\text{-}6$		$[a^+b^+c^+d^+e^+f^+;$ $b^+a^+d^+c^+f^+e^+]$	$p3$	e	$\alpha\beta\alpha\gamma\alpha\delta$	$B=D=F=\frac{2\pi}{3}$	$\alpha\alpha\beta\beta\gamma\gamma$	$3D$	$A(P_5\text{-}13)$	IH 7, HK-P6-20
$^*P_6\text{-}7$		$[a^+b^+c^+a^+b^+c^+;$ $a^+b^+c^+]$	$p2$	$c2$	$\alpha\alpha\alpha\alpha\alpha\alpha$	$A=D$, $B=E$, $C=F$	$\alpha\beta\gamma\alpha\beta\gamma$	$1D$	$BE(P_4\text{-}17)$	IH 8
$^*P_6\text{-}8$		$[a^+b^+c^+a^+b^+c^+;$ $a^+c^-b^-]$	pgg	$c2$	$\alpha\alpha\alpha\alpha\alpha\alpha$	$A=D$, $B=E$, $C=F$	$\alpha\beta\beta\alpha\beta\beta$	$1D$, $1R$	$CF(P_4\text{-}18)$, $AD(P_4\text{-}19)$	IH 9
$^*P_6\text{-}9$		$[ab^+c^+dc^-b^-;$ $db^+c^+a]$	pmg	$d1(s)$	$\alpha\alpha\alpha\alpha\alpha\alpha$	$A=B$, $C=F$, $D=E$	$\alpha\beta\gamma\alpha\gamma\beta$	2	$CF(P_4\text{-}21)$, $DE(P_4\text{-}20)$	IH 13
$^*P_6\text{-}10$		$[a^+b^+c^+c^-b^-a^-;$ $a^+b^-c^+]$	pmg	$d1(l)$	$\alpha\alpha\beta\beta\beta\alpha$	$B=F$, $C=E$, $F+A+B=2\pi$	$\alpha\beta\gamma\gamma\beta\alpha$	2	$A(P_5\text{-}14)$, $AD[P_4\text{-}22]$	IH 15
$^*P_6\text{-}11$		$[a^+b^+c^+c^-b^-a^-;$ $a^-c^+b^+]$	$p31m$	$d1(l)$	$\alpha\beta\gamma\beta\gamma\beta$	$A=C=E=\frac{2\pi}{3}$, $B=F$	$\alpha\beta\beta\beta\beta\alpha$	3	$D(P_5\text{-}15)$	IH 16
$^*P_6\text{-}12$		$[ab^+b^-ab^+b^-;\ ab^+]$	cmm	$d2$	$\alpha\alpha\alpha\alpha\alpha\alpha$	$A=B=D=E$, $C=F$	$\alpha\beta\beta\alpha\beta\beta$	1	$CF(P_4\text{-}22)$	IH 17
$^*P_6\text{-}13$		$[aaaaaa;\ a]$	$p6m$	$d6$	$\alpha\alpha\alpha\alpha\alpha\alpha$	$A=B=C=D=E$ $=F=\frac{2\pi}{3}$	$\alpha\alpha\alpha\alpha\alpha\alpha$	1	—	IH 20

tile group (that is the subgroup of $S(\tau)$ that maps a tile onto itself). In column (6) we indicate the transitivity classes of the vertices of the tiling that belong to the boundary of a tile T. Again we follow the conventions of [15] except that here we put a symbol in parentheses if it corresponds to a vertex which is *not* a corner of T. To aid identification, in each diagram we mark a tile T and one of its vertices – that from which we begin listing the transitivity classes in a counterclockwise direction round T. In Column (7) we state the restrictions that are imposed by the particular isohedral type on the interior angles of T at each vertex on its boundary. Sometimes there are additional relations derivable from the given ones and the fact that the sum of the angles of an n-gon is $(n\text{-}2)\pi$.) The specification that an angle is π means that the corresponding vertex is not a corner of T. Angle A is at the marked vertex, and B, $C, \ldots$, follow cyclically counterclockwise.

The listing of edge transitivities in Column (8) follows lines similar to the vertex transitivities in Column (6). Here parentheses enclose two or three symbols corresponding to edges that together form a side of T. This column also gives information about restrictions on the lengths of the sides of the polygons that are imposed by the particular polygonal isohedral type. For example, the entry corresponding to P_4-33 is $\alpha(\beta\gamma)\gamma\beta$. This means that the corresponding quadrangle has its second side $(\beta\gamma)$ (counted counterclockwise from the marked vertex) equal in length to the sum of the third γ and fourth β sides. It is worth remarking that *all* equations involving edge-length arise from the transitivities in this way – unlike the case of the angles of T in which additional relations are forced by the geometry of the tile or tiling. There are, of course, other restrictions on the edge-lengths in the form of inequalities that arise trivially as consequences of the constraints on sides and angles, and convexity. For example, in the case P_3-5 the triangle inequality implies that the edge in transitivity class γ must be shorter than that in β.

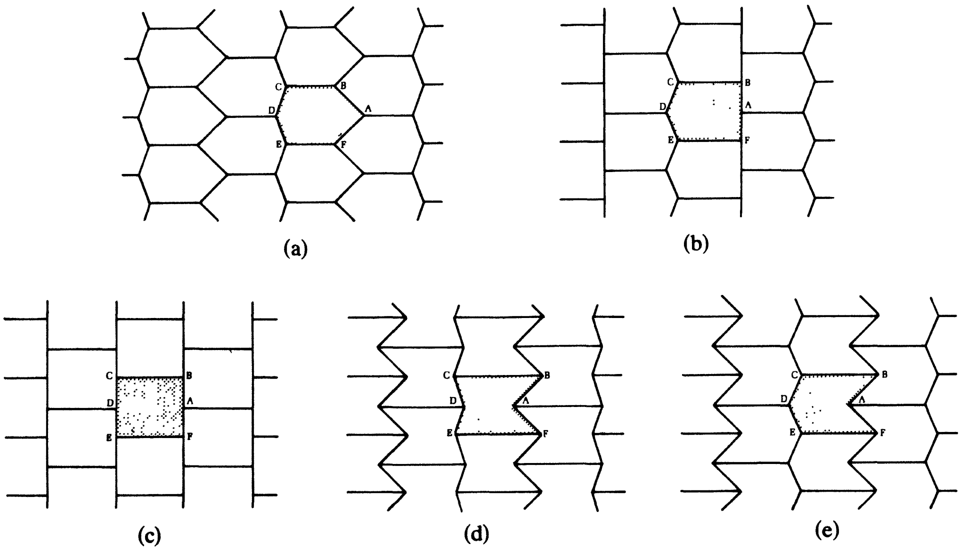
In Column (9) we list the number of different *aspects* of the tiles in the tiling. The notation follows [15] – two tiles are of the same aspect if one is a translate of the other. In certain cases these differ from the corresponding values for the isohedral types given in Column (8) of Table I of [15]. This is caused by the fact that the requirement that a vertex is not a corner of a tile may force the tile to have extra symmetries, and so reduce the number of aspects. For example, in the tiling P_3-4 the condition $D = \pi$ and the equality of two sides of the triangle force it to have symmetry group $d1$; hence the tiles occur in just 2 aspects instead of $2D$, $2R$ as in the general $IH53$ tiling. The additional symmetries are "spurious" in the sense that the symmetry group of the tile is no longer equal to the induced tile group of the tiling.

In Column (10) we list, for each type of tiling, the vertices at which the interior angles can be increased to π, thus reducing the number of corners of each tile. After each such vertex, or pair of vertices, we indicate the polygonal isohedral type of the resultant tiling. This reference is given in (round) parentheses when the new tiling is of the same isohedral type as that from which it was derived; if it is of a different isohedral type then the reference in enclosed in [square] brackets. The latter possibility occurs when making an angle equal to π increases the symmetry group and so alters the isohedral type (see Figures 4(a), (b), (c)). Besides being useful as a cross-check on the accuracy of the tables, the data in Column(10) is of interest in connection with the classification of isohedral tilings using non-convex tiles. This will be discussed in the next section.

Finally, in Column(11) we indicate the isohedral type of tiling (in the notation of [15]) as well as references to occurrences of the type in the literature. It is surprising how few such references we were able to find, and, in spite of their aesthetic appeal, it seems as if many of the tilings are displayed here for the first time.

3. Proper isohedral tilings by non-convex polygons

In a proper tiling by polygons, each edge is a straight-line segment joining two vertices of the tiling. Hence to convert arbitrary tilings into polygonal ones we may be able to use a process which we shall call *straightening*. This means that we



(a) (b)
(c) (d) (e)

Figure 4

start from a tiling τ of given isohedral type (in the sense of [15]) and then replace each edge of τ by a line segment joining the corresponding vertices if this can be done so as to form a new tiling τ_s. Two possibilities arise:

(i) τ_s is of the same isohedral type as τ; in this case we say that the type can be realized by polygons; or

(ii) whatever tiling of the type of τ is chosen, the straightened tiling τ_s is always of a different type. This situation occurs because straightening necessarily introduces new symmetries, so that $S(\tau_s) > S(\tau)$. A typical example occurs in the case of type *IH* 62 (see Figure 1(b)). Here straightening always produces the regular tiling by squares, which is of type *IH* 76.

Examination of each of the 81 isohedral types shows that the distinction between cases (i) and (ii) *is not dependent upon whether the resultant polygonal tiles are convex or not.* We cannot reproduce here full details of the proof of this assertion, but it is made plausible by the following argument. Let us, as in Figure 2 of [15], label the oriented or unoriented edges of each tile of τ and then assign the same labels to the corresponding edges of each tile in τ_s. Then the *marked tiling* τ_s, labelled in this manner, is clearly of the same isohedral type as τ. We are concerned with the question whether the type is changed by removing the labels. But it is evident that this depends *only* on the symmetries of a tile T and its relationship to its adjacents, and *not* on the convexity character of T.

Now it is easy to check, from the diagrams given in [15], that τ can always be chosen so that the tiles of τ_s are convex polygons. We deduce the following:

THEOREM 2. *Any isohedral type of tiling that can be represented by a proper tiling with polygonal tiles can necessarily be represented by one with convex (polygonal) tiles.*

COROLLARY. *Any polygonal tiling of one of the* 34 *isohedral types marked N in column* (9) *of Table I of* [15] *is necessarily not proper.*

To find all proper isohedral tilings by non-convex polygons it therefore suffices to restrict attention to the 47 types whose reference numbers are given in heavy type in Tables I to IV. In fact, if the previous definition of "polygonal isohedral type" is retained, we see now that these tables give all the information about the non-convex case as well. But it is more appropriate to adopt a slightly finer classification based on the following definition.

Two tilings τ_1 and τ_2 by (convex or non-convex) polygons are of the same *refined polygonal isohedral type* if they are of the same polygonal isohedral type (that is, satisfy conditions P.1 and P.2 of the previous section) and also:

P.3 For each vertex v lying on the boundary of a tile T of τ_1, the interior angle of T at v is less than, equal to, or greater than π according as the interior angle of $\varphi(T)$ at $\varphi(v)$ is less than, equal to, or greater than π.

This new condition may be loosely stated as asserting that the *convexity character* of each tile at a vertex on its boundary is unchanged by the mapping φ.

To determine the refined types of tiling we must therefore decide which angles of the polygonal tile can exceed π. It turns out that this is easy to do, for we already know when it is possible to increase an angle at a vertex to the value π–the possibilities are listed in Column (10) of Tables II, III and IV–and it is in precisely these cases that it is possible to increase the angle to a value greater than π, leading to a non-convex tile.

To illustrate the definitions and the process just described, see Figures 4 and 5. From the isohedral tiling by hexagons of type P_6-10 shown in Figure 4(a) we obtain a tiling by pentagons (Figure4(b)) and one by quadrangles (Figure 4(c)) by increasing angles to π. Increasing these angles still further leads to the hexagonal tilings of Figures 4(d), 4(e). These are also of type P_6-10 according to the definition of the previous section – but adopting the refined definition these tilings are of different types. A similar example involving pentagons is shown in Figure 5.

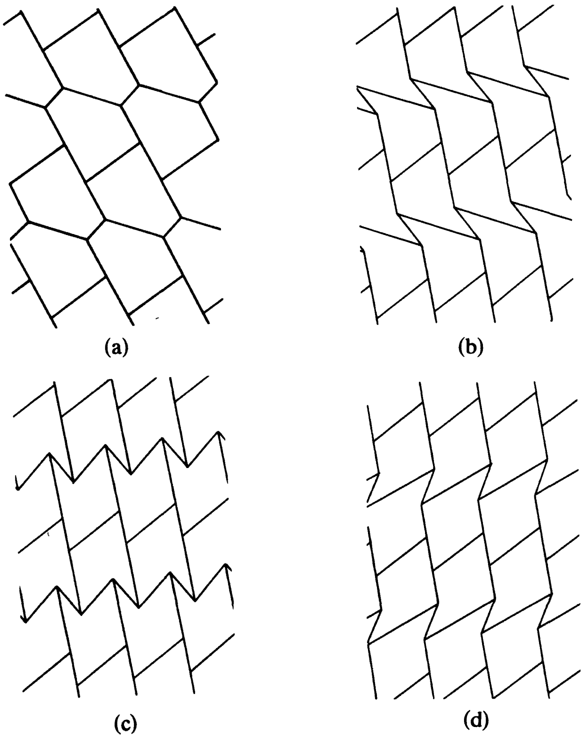
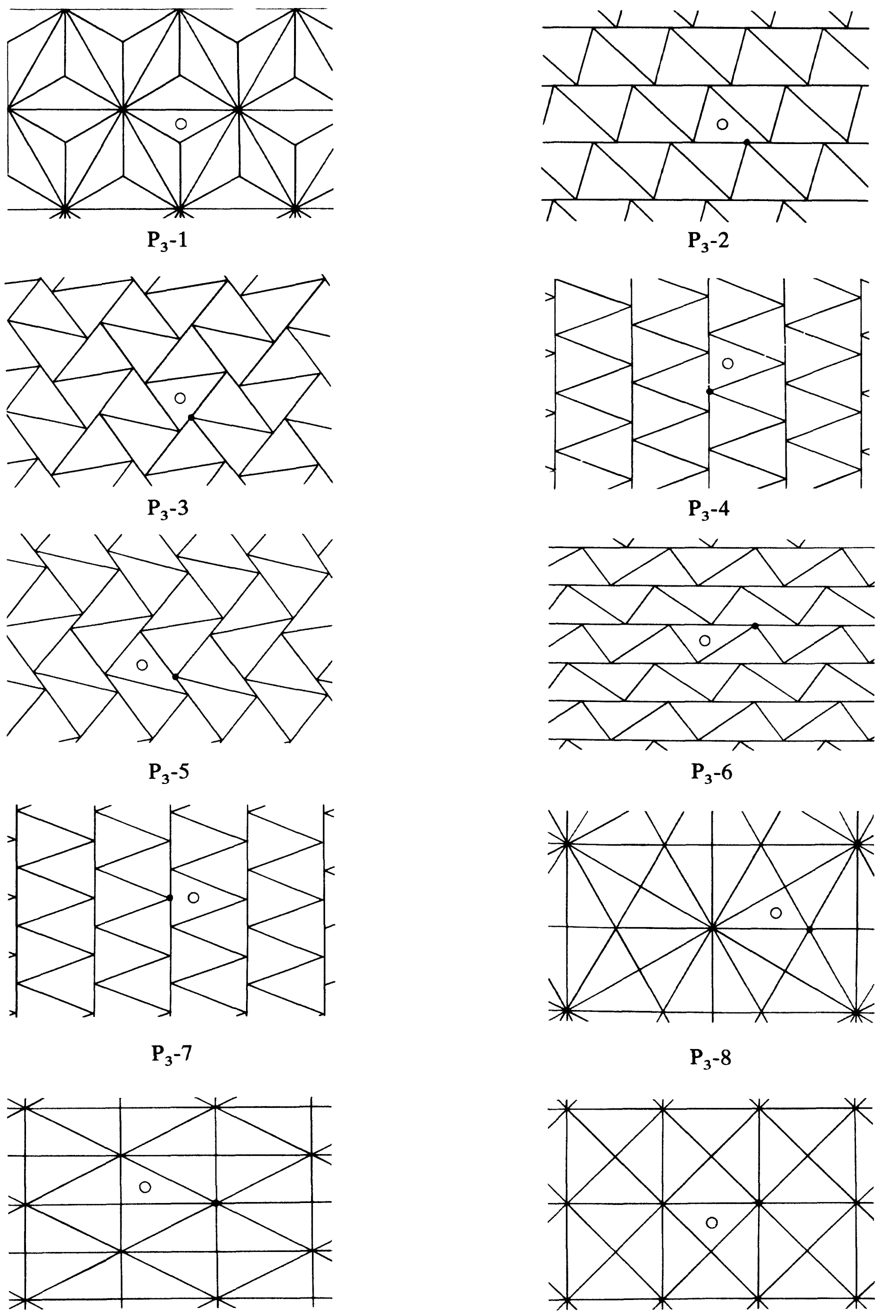

(a) (b) (c) (d)

Figure 5



P_3-1 P_3-2

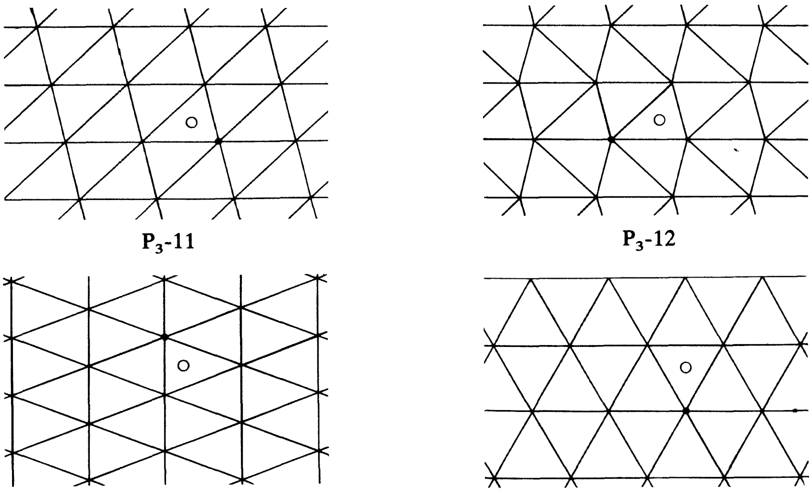
P_3-3 P_3-4

P_3-5 P_3-6

P_3-7 P_3-8

P_3-9 P_3-10

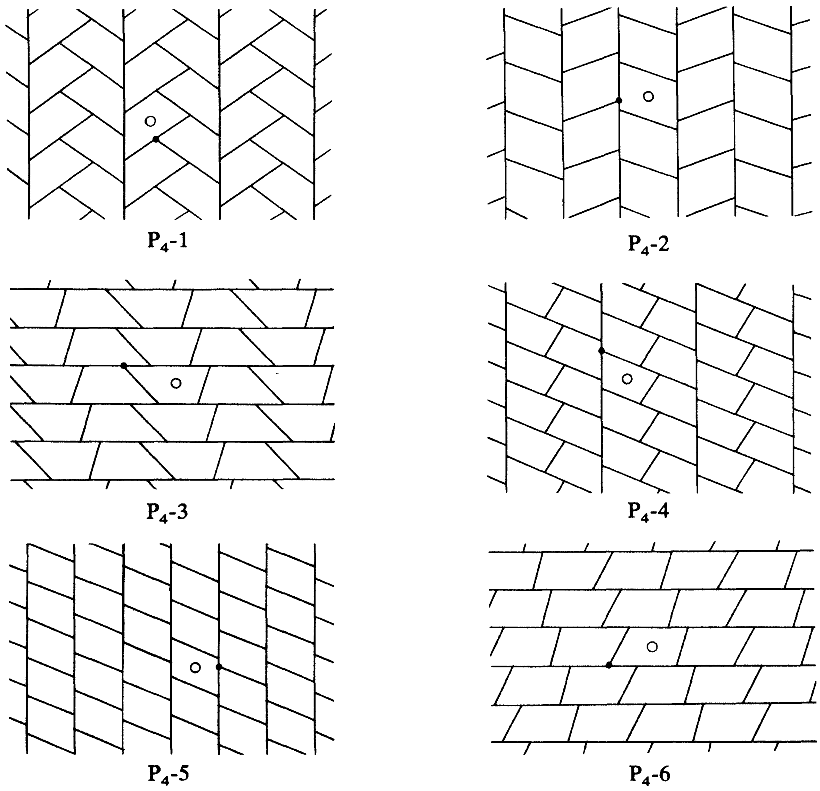
Figure 6



P_3-11 P_3-12

P_3-13 P_3-14

Figure 6 (concluded)

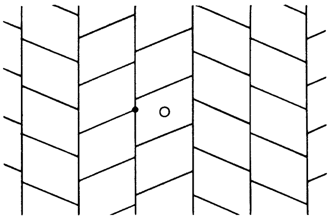


P_4-1 P_4-2

P_4-3 P_4-4

P_4-5 P_4-6

Figure 7



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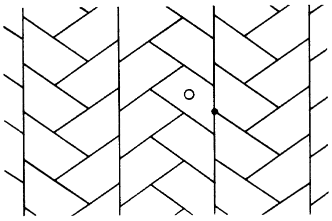

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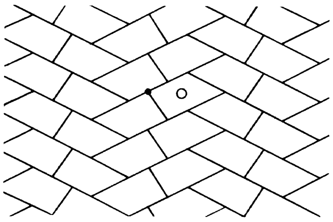

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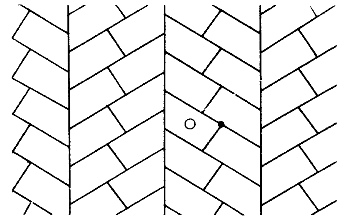

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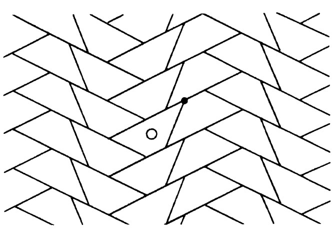

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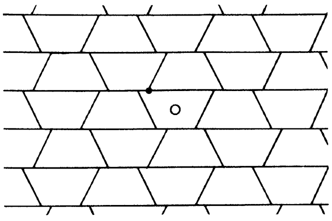

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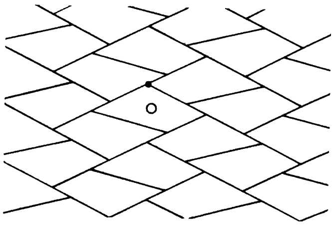

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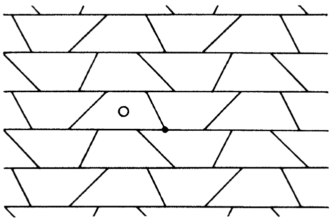

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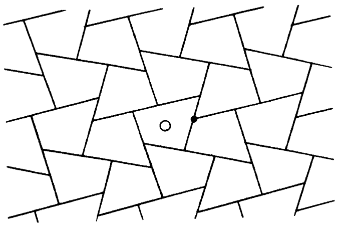

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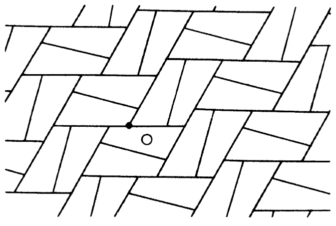

P_4-16

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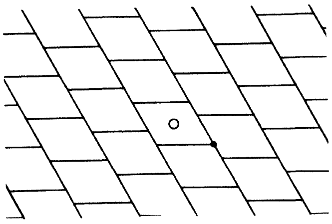

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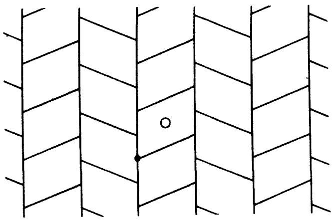

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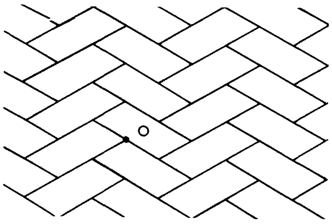

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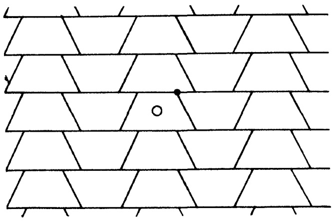

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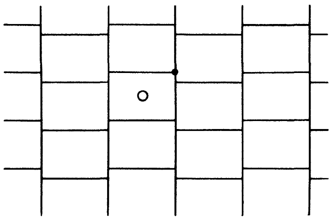

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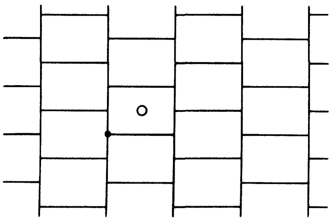

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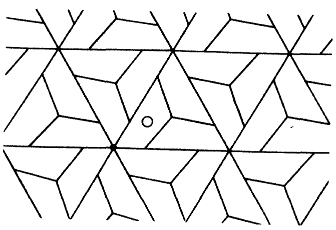

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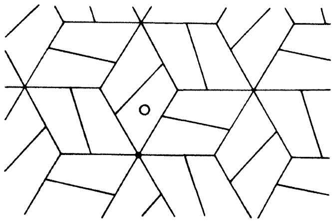

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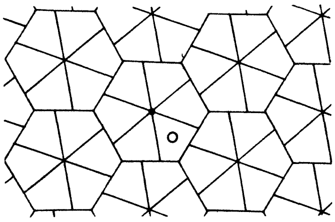

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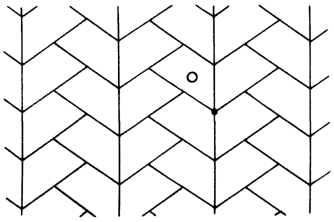

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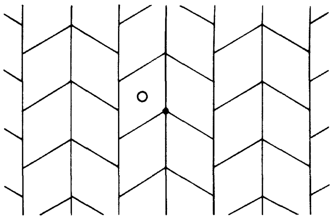

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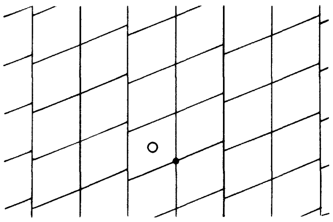

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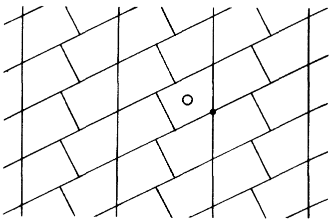

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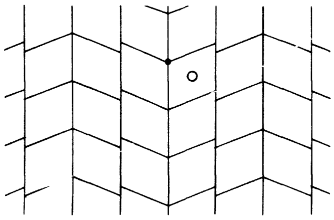

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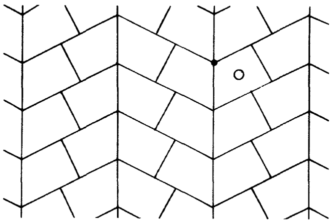

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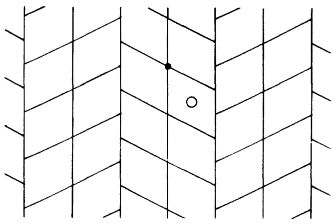

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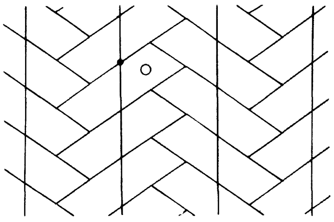

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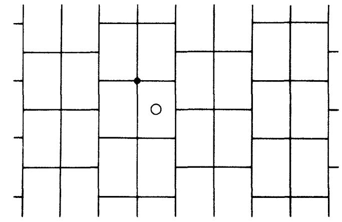

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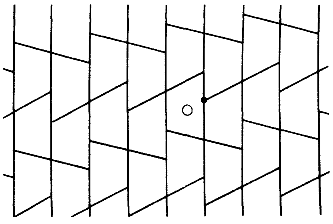

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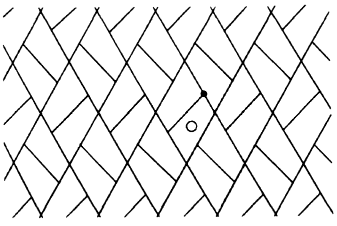

P_4-36

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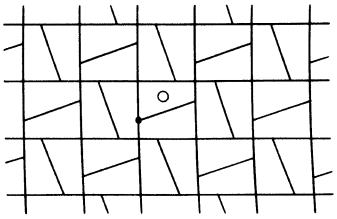

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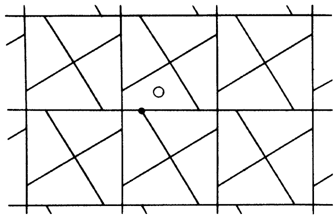

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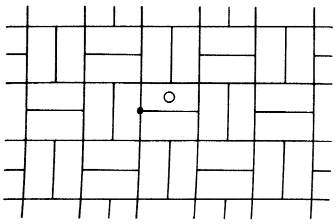

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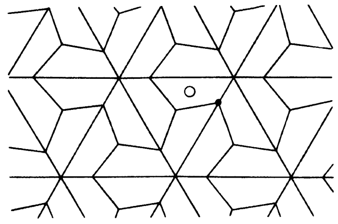

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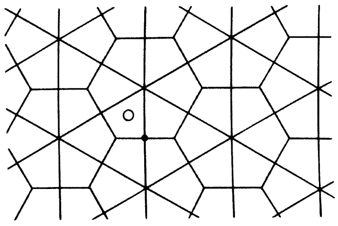

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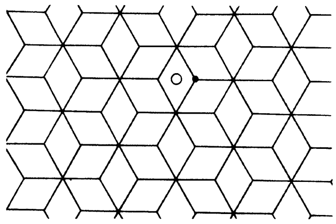

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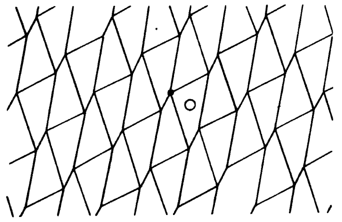

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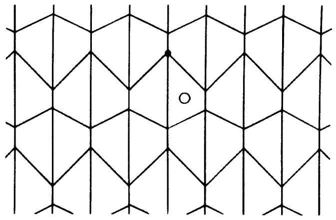

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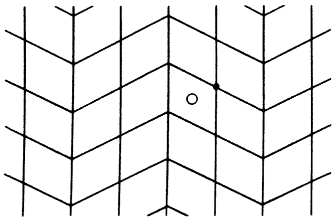

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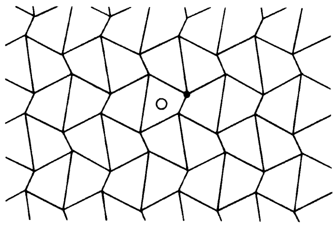

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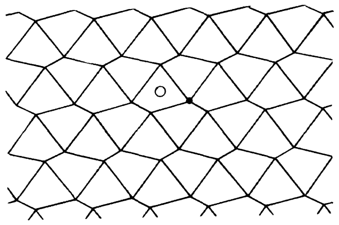

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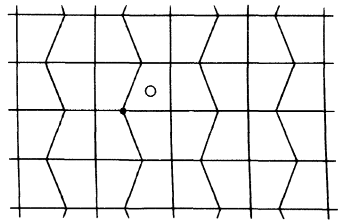

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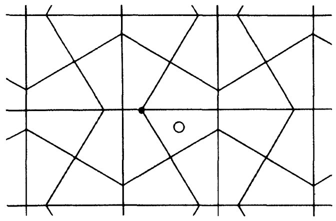

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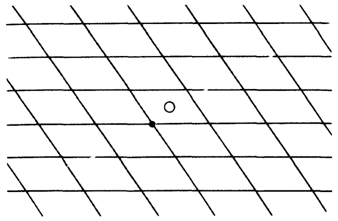

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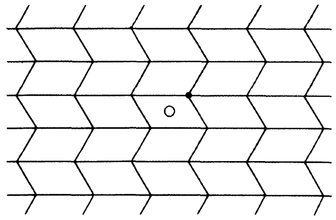

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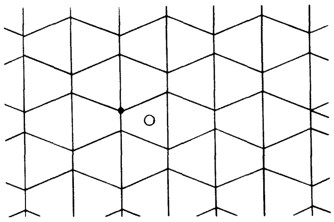

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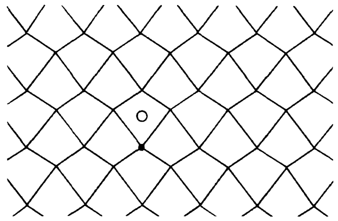

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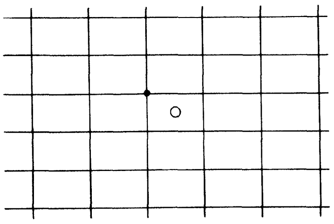

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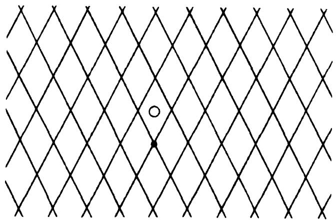

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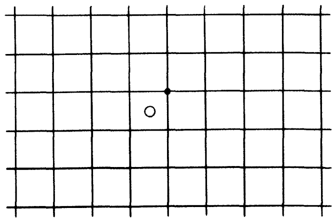

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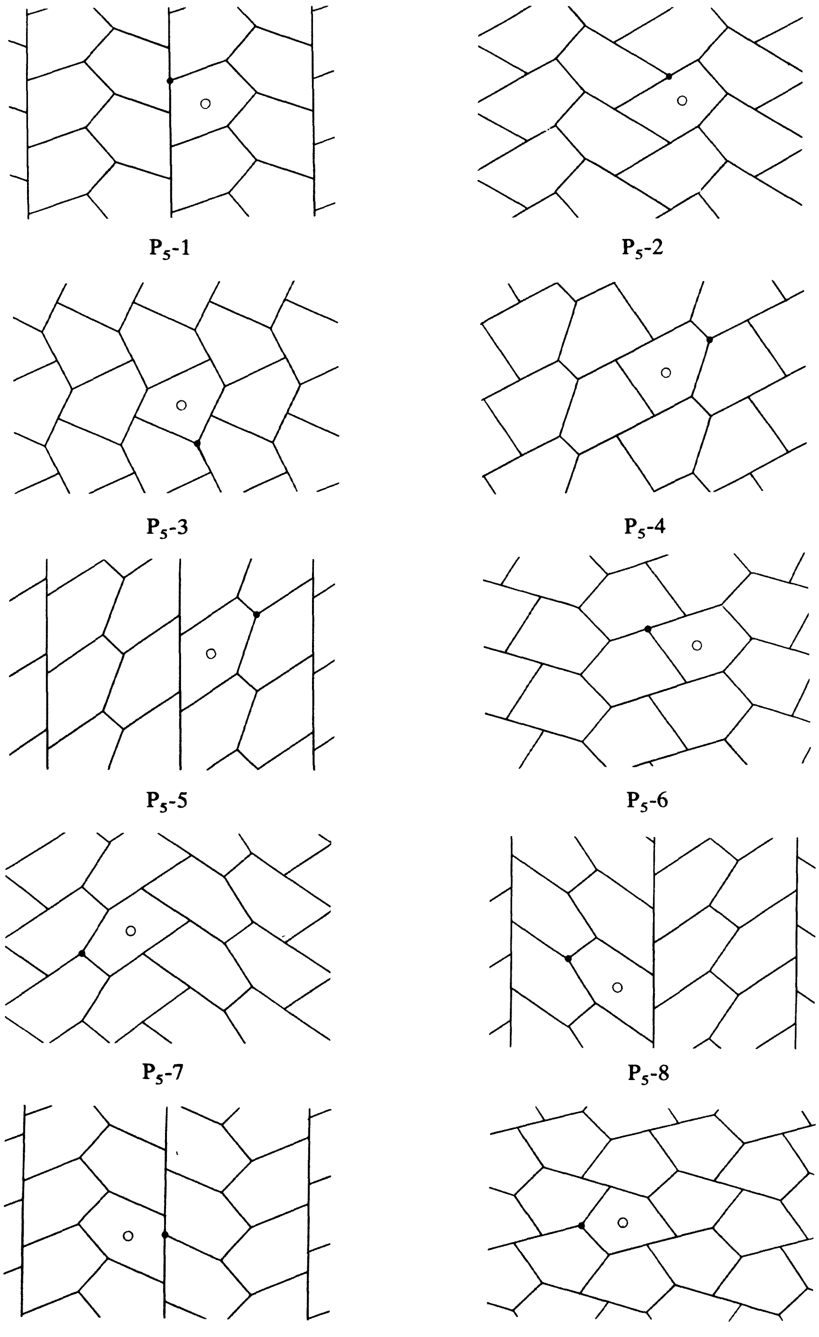


P_5-1 P_5-2

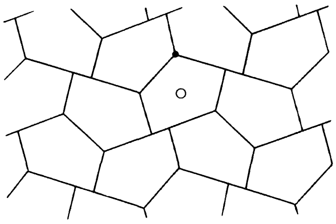
P_5-3 P_5-4

P_5-5 P_5-6

P_5-7 P_5-8

P_5-9 P_5-10

Figure 8



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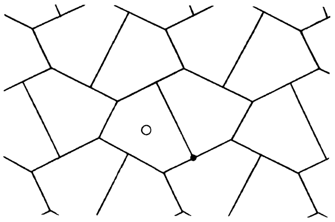

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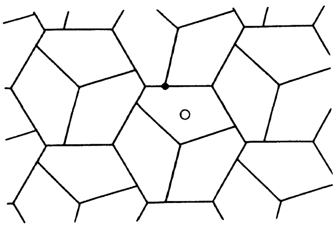

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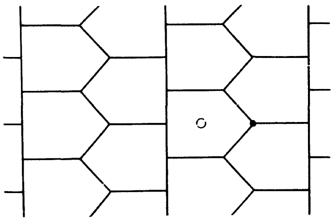

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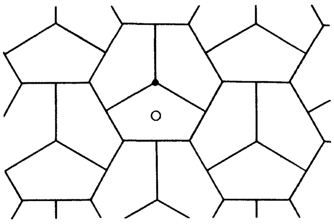

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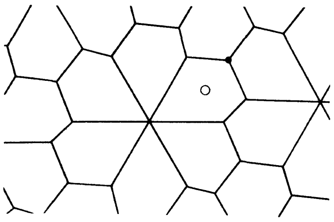

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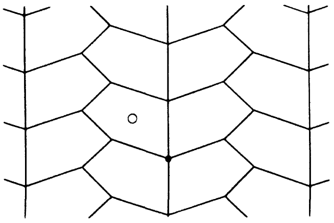
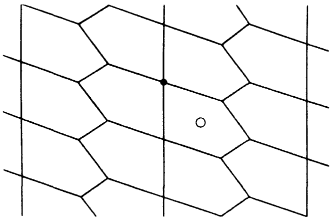
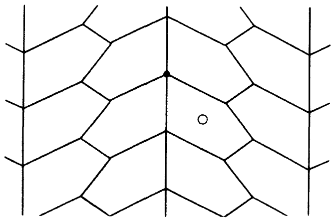


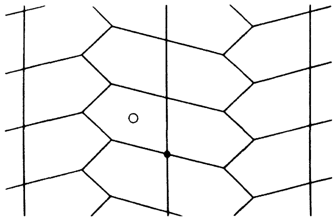
P_5-17



P_5-18

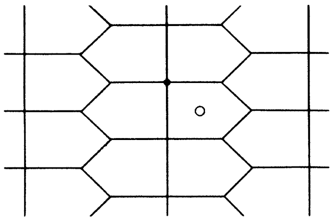


P_5-19

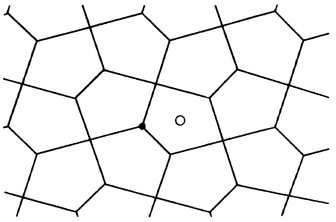


P_5-20

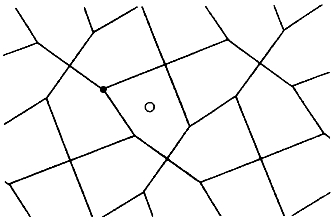
Figure 8 (continued)



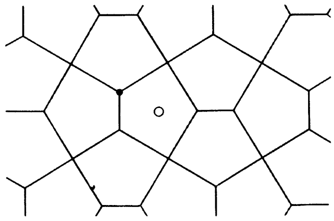
P_5-21



P_5-22

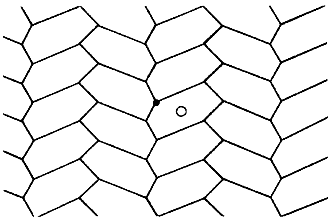


P_5-23

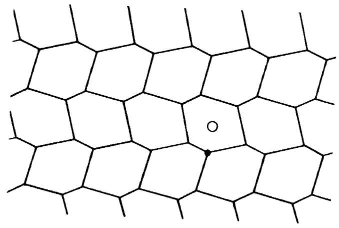


P_5-24

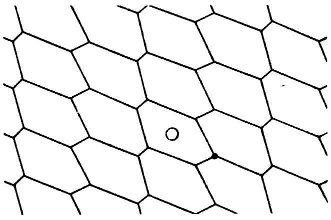
Figure 8 (concluded)



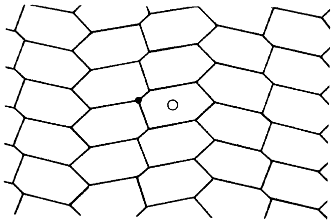
P_6-1



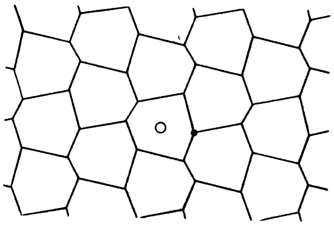
P_6-2



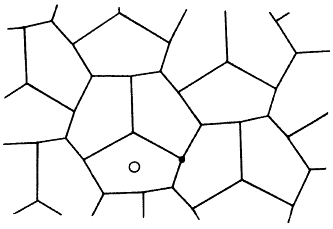
P_6-3



P_6-4

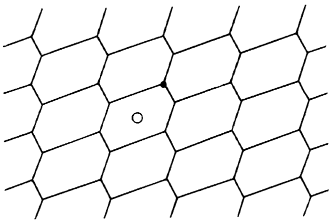


P_6-5

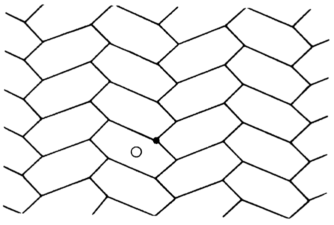


P_6-6

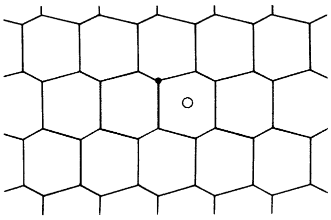
Figure 9



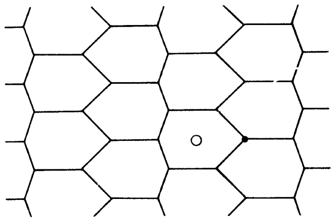
P_6-7



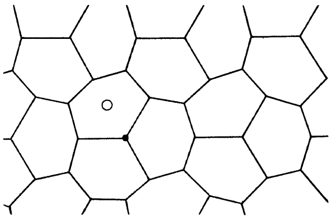
P_6-8



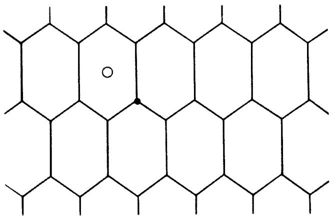
P_6-9



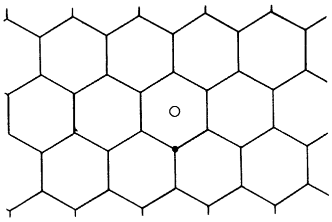
P_6-10



P_6-11



P_6-12



P_6-13

Figure 9 (concluded)

Here three isohedral tilings by non-convex pentagons can be derived, and these are distinct from the original P_5-4 according to the refined definition.

Using the refined notion of type and examining the various possibilities that occur, we arrive at the following result.

THEOREM 3. *There are* 96 *refined polygonal isohedral types of proper tilings by non-convex polygons* (6 *by quadrangles,* 48 *by pentagons and* 42 *by hexagons*).

These types are specified by their non-convex corners listed in Column (10) *of Tables II, III and IV.*

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