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THE NINETY-ONE TYPES OF ISOGONAL TILINGS IN THE PLANE[1]

BY

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ABSTRACT. A tiling of the plane by closed topological disks of *isogonal* if its symmetries act transitively on the vertices of the tiling. Two isogonal tilings are of the same *type* provided the symmetries of the tiling relate in the same way every vertex in each to its set of neighbors. Isogonal tilings were considered in 1916 by A. V. Šubnikov and by others since then, without obtaining a complete classification. The isogonal tilings are vaguely dual to the isohedral (tile transitive) tilings, but the duality is not strict. In contrast to the existence of 81 isohedral types of planar tilings we prove the following result: *There exist* 91 *types of isogonal tilings of the plane in which each tile has at least three neighbors.*

1. A plane tiling $\mathfrak{T} = \{T_i | i = 1, 2, \ldots\}$ is called *isohedral* if its symmetry group $S(\mathfrak{T})$ acts transitively on the tiles T_i of $\mathfrak{T}$, and is called *isogonal* if $S(\mathfrak{T})$ acts transitively on the vertices of $\mathfrak{T}$. In [5] we showed that there exist 81 types of isohedral tilings in the plane. The main purpose of this paper is to establish the following result.

THEOREM 1. *There exist exactly* 91 *types of normal plane isogonal tilings. Of these* 34 *are also isohedral, and* 63 *can be realized by convex tiles.*

Here "normal" means that the tiling is *bounded* (that is, the tiles are uniformly bounded in diameter and width), and the intersection of any two tiles is either empty or is an edge or a vertex of each. Thus normal tilings cannot contain digons or vertices of valence two.

The classification of isogonal tilings is based on the idea that a "type" of tiling is determined by the way that $S(\mathfrak{T})$ relates any vertex to its neighbors. To make this concept precise we shall need to define "vertex symbols" and "adjacency symbols" in an analogous way to that in which we introduced

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TABLE 1

List No.	Net	Vertex group and Symbol	Adjacency Symbol	Space group	Edge transi-tivity	Tile transitivity	Aspects	Reali-sations	References
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
IG 1.	(3^6)	E $a^+b^+c^+d^+e^+f^+$	$d^+e^+f^+a^+b^+c^+$	p1	αβγαβγ	$T_1T_2T_1T_2T_1T_2$	T_1,1D;T_2,1D	N	
IG 2.			$b^-a^-f^+e^-d^-c^+$	pg	ααβγγβ	$T_1T_1T_1T_2T_2T_2$	T_1,1D1R;T_2,1D1R	N	
IG 3.			$c^-e^+a^-f^-b^+d^-$	pg	αβγβαγ	$T_1T_2T_1T_2T_1T_2$	T_1,1D1R;T_2,1D1R	N	
IG 4.			$a^+e^+c^+d^+b^+f^+$	p2	αβγδβε	$T_1T_1T_2T_2T_2T_1$	T_1,2D;T_2,2D	C	SK34,S1
IG 5.			$a^+e^+d^-c^-b^+f^+$	pgg	αβγγβδ	$T_1T_1T_2T_2T_2T_1$	T_1,2D2R;T_2,2D2R	C	SK33
IG 6.			$a^+e^-c^+f^-b^-d^-$	pgg	αβγδβδ	$T_1T_1T_2T_2T_1T_2$	T_1,2D2R;T_2,2D2R	C	SK32
IG 7.			$b^+a^+d^+c^+f^+e^+$	p3	ααββγγ	$T_1T_2T_1T_3T_1T_4$	T_1,3D;T_2,1D;T_3,1D;T_4,1D	C	SK13
IG 8.		C_2 $a^+b^+c^+a^+b^+c^+$	$a^+b^+c^+$	p2	αβγαβγ	T T T T T T	T,2D	C,IH 84	SK35,S2
IG 9.			$a^+c^-b^-$	pgg	αββαββ	T T T T T T	T,2D2R	N,IH 86	
IG 10.		C_3 $a^+b^+a^+b^+a^+b^+$	b^+a^+	p3	αααααα	$T_1T_2T_1T_2T_1T_2$	T_1,1D;T_2,1D	N	
IG 11.		C_6 $a^+a^+a^+a^+a^+a^+$	a^+	p6	αααααα	T T T T T T	T,2D	N,IH 90	
IG 12.		D_1^s $ab^+c^+dc^-b^-$	dc^-b^-a	cm	αββαββ	T T T T T T	T,1D1R	N,IH 83	
IG 13.			db^+c^+a	pmg	αβγαγβ	T T T T T T	T,2D2R	C,IH 85	SK30
IG 14.		D_1^ℓ $a^+b^+c^+c^-b^-a^-$	$c^-b^-a^-$	cm	αβααβα	$T_1T_2T_1T_2T_1T_2$	T_1,1;T_2,1	N	
IG 15.			$a^+b^-c^+$	pmg	αβγγβα	$T_1T_1T_2T_2T_2T_1$	T_1,2;T_2,2	C	SK31
IG 16.			$a^-c^+b^+$	p31m	αββββα	$T_1T_2T_3T_2T_3T_2$	T_1,1;T_2,3; T_3,1	C	SK12

"tile symbols" and "adjacency symbols" in the classification of isohedral tilings.

At first sight this method of classification may appear to be somewhat elaborate. However, this is a natural consequence of the fact that it is a simultaneous refinement of all previously proposed classifications and also of the ones that may seem more natural–such as by the crystallographic symmetry group or by the net of the tiling. Moreover, it seems to be the simplest method of codifying one's intuitive concept of a type of tiling, and appears to be precisely what many previous authors have attempted to formulate **[1]**, **[6]**, **[8]**, **[9]**. It turns out that the use of adjacency symbols provides a convenient algorithmic method of classifying the tilings, and is one which has many extensions and variants.

The use of vertex and adjacency symbols enables us, in §2, to show that there exist exactly 93 *combinatorial* types of isogonal tilings. These are dual to the 93 combinatorial types of isohedral tilings determined in [3].

To find the isogonal tilings it is necessary to examine each of the 93 combinatorial types in turn, and see if it is possible to realize it by "shaped tiles". In spite of assertions to the contrary–see, for example, Fedorov **[2]**,

List No.	Net	Vertex group and Symbol	Adjacency Symbol	Space group	Edge transitivity	Tile transitivity	Aspects	Realisations	References
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
IG 17.	(3^6) cont.	D_2 $ab^+b^-ab^+b^-$	ab^+	cmm	αββαββ	T T T T T T	T,2	C, IH 91	SK25
IG 18.		D_3^s ababab	ba	p31m	αααααα	T T T T T T	T,1D1R	M, IH 89	
IG 19.		D_3^ℓ $a^+a^-a^+a^-a^+a^-$	a^-	p3m1	αααααα	$T_1T_2T_1T_2T_1T_2$	T_1,1; T_2,1	N	
IG 20.		D_6 aaaaaa	a	p6m	αααααα	T T T T T T	T,2	C, IH 93	SK7,S12
IG 21.	$(3^4.6)$	E $a^+b^+c^+d^+e^+$	$e^+c^+b^+d^+a^+$	p6	αββγα	$HT_1T_2T_1T_1$	H,1D;T_1,6D;T_2,2D	C	SK8,S10
IG 22.	$(3^3.4^2)$	E $a^+b^+c^+d^+e^+$	$a^-e^+a^-c^-b^+$	cm	αβγγβ	Q Q T T T	Q,1;T,1D1R	N	
IG 23.			$a^+e^+c^+d^+b^+$	p2	αβγδβ	Q Q T T T	Q,1D;T,2D	C	SK56,S1
IG 24.			$a^-e^+c^+d^+b^+$	pmg	αβγδβ	Q Q T T T	Q,2:T,2D2R	C	SK28
IG 25.			$a^+e^+d^-c^-b^+$	pgg	αβγγβ	Q Q T T T	Q,1D1R;T,2D2R	C	SK54
IG 26.		D_1 $ab^+c^+c^-b^-$	ab^-c^+	cmm	αβγγβ	Q Q T T T	Q,1;T,2	C	SK23
IG 27.	$(3^2.4.3.4)$	E $a^+b^+c^+d^+e^+$	$a^+d^-e^-b^-c^-$	pgg	αβγβγ	T T Q T Q	Q,1D1R;T,2D2R	C	SK50
IG 28.			$a^+c^+b^+e^+d^+$	p4	αββγγ	T T Q_1T Q_2	Q_1,1D;Q2,1D;T,4D	C	SK20,S5
IG 29.		D_1 $ab^+c^+c^-b^-$	ac^+b^+	p4g	αββββ	T T Q T Q	Q,1D1R;T,4	C	SK19

Heesch [6]–duality cannot be used here, even in the convex case. This is borne out by the fact that whereas there are 81 types of isohedral tilings of which 47 are convex, there are 91 types of isogonal tilings of which 63 are convex. We shall display our results in tabular form and we also show, in Figure 5, diagrams of all 91 types of isogonal tilings. For those which are isohedral we give reference to Table I of [3]. The two combinatorial types of tilings which are *not* realizable as isogonal tilings (IG 18 and IG 73 in Table I) are also isohedral, and may be represented by *marked* isohedral tilings. These are shown in Figure 6.

In the third section we briefly examine isogonal tilings which are not normal. It is easy to see that there exist no bounded isogonal tiling with vertices of valence two, but there are infinitely many types with digonal tiles. The main result of this section is a description of how all these may be constructed by a simple process from the 93 combinatorial types already determined.

The word "isogons" was originally introduced by Fedorov [1] to denote isogonal tilings by convex tiles, or *convex isogonal tilings*, as we shall call them. Several attempts have been made to enumerate the different types of these, for example, see [1], [2], [5], [7], [8]. Incomplete definitions and careless

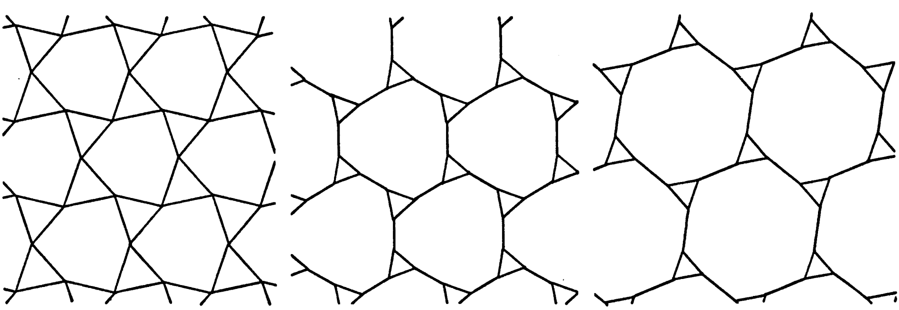
List No.	Net	Vertex group and Symbol	Adjacency Symbol	Space group	Edge transitivity	Tile transitivity	Aspects	Realisations	References
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
IG 30.	(3.4.6.4)	E $a^+b^+c^+d^+$	$a^-b^-d^+c^+$	p31m	αβγγ	Q H Q T	H,1;Q,3;T,1D1R	C	SK11
IG 31.			$b^+a^+d^+c^+$	p6	ααββ	Q H Q T	H,1D;Q,3D;T,2D	C	SK36,S10
IG 32.		D_1 $a^+a^-b^+b^-$	a^-b^-	p6m	ααββ	Q H Q T	H,1;Q,3;T,2	C	SK2
IG 33.	(3.6.3.6)	E $a^+b^+c^+d^+$	$d^+c^+b^+a^+$	p3	αββα	T_1H T_2H	H,1D;T_1,1D;T_2,1D	C	SK38,S8
IG 34.		C_2 $a^+b^+a^+b^+$	b^+a^+	p6	αααα	T H T H	H,1D;T,1D	N	
IG 35.		D_1^s $a^+b^+b^-a^-$	a^-b^-	p3m1	αββα	T_1H T_2H	H,1;T_1,1;T_2,1	C	SK10
IG 36.		D_1^ℓ $a^+a^-b^+b^-$	b^-a^-	p31m	αααα	T H T H	H,1;T,1D1R	C	Figure 1a
IG 37.		D_2 $a^+a^-a^+a^-$	a^-	p6m	αααα	T H T H	H,1;T,2	C	SK5,S13
IG 38.	(3.12.12)	E $a^+b^+c^+$	$a^-c^+b^+$	p31m	αββ	D D T	D,1;T,1D1R	C	Figure 1b
IG 39.			$a^+c^+b^+$	p6	αββ	D D T	H,1D;T,2D	C	S15, Figure 1c
IG 40.		D_1 ab^+b^-	ab^-	p6m	αββ	D D T	H,1;T,2	C	SK4
IG 41.	(4^4)	E $a^+b^+c^+d^+$	$c^+d^+a^+b^+$	p1	αβαβ	Q Q Q Q	Q,1D	N,IH 41	
IG 42.			$c^+b^-a^+d^-$	pm	αβαγ	$Q_1Q_2Q_2Q_1$	Q_1,1;Q_2,1	N	
IG 43.			$c^-d^+a^-b^+$	pg	αβαβ	Q Q Q Q	Q,1D1R	N,IH 43	
IG 44.			$b^-a^-d^-c^-$	pg	ααββ	Q Q Q Q	Q,1D1R	N,Ih 44	

exposition often make it hard to compare results, but none of these enumerations is complete. In the relatively accessible book of Šubnikov and Kopcik [9] it is asserted that there are 60 types of convex isogonal tilings. However, as we shall show, in fact there are 63 types, and representatives of the three types missed by Šubnikov [8] and Šubnikov and Kopcik (IG 36, IG 38 and IG 39 of Table I) are reproduced in Figure 1. In addition, several of the diagrams in [8] and [9] are incorrect. For example, diagram number 35 in Figure 176 of [9] should consist of scalene triangles, not isosceles ones. Also, the description of the construction of convex isogonal tilings in Chapter 7 of [9] is misleading in that it does not yield all of them as claimed.

In view of these remarks, we feel that no further justification is needed for presenting here, after the long period since their introduction, a complete and accurate list of the isogonal and convex isogonal tilings.

2. The *net* $N(\mathfrak{T})$ of a tiling $\mathfrak{T}$ is defined as the graph consisting of the nodes (vertices, that is, points belonging to at least three tiles) and arcs (edges) of $\mathfrak{T}$. The first stage in the enumeration of the normal isogonal tilings is the determination of all possible nets. To do this we use Euler's theorem, applicable since the tilings we are considering are normal, from which we

List No.	Net	Vertex group and Symbol	Adjacency Symbol	Space group	Edge transitivity	Tile transitivity	Aspects	Realisations	References
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
IG 45.	(4^4) cont.	E $a^+b^+c^+d^+$ cont.	$c^-b^-a^-d^-$	cm	αβαγ	$Q_1Q_2Q_1Q_2$	Q_1,1;Q_2,1	N	
IG 46.			$a^+b^+c^+d^+$	p2	αβγδ	Q Q Q Q	Q,2D	C,IH 46	SK59,S4
IG 47.			$c^+b^+a^+d^+$	p2	αβαγ	$Q_1Q_2Q_2Q_1$	Q_1,1D;Q_2,1D	C	SK57,S1
IG 48.			$a^-b^-c^-d^-$	pmm	αβγδ	$Q_1Q_2Q_3Q_4$	Q_1,1;Q_2,1;Q_3,1;Q_4,1	C	SK27
IG 49.			$a^-b^+c^-d^+$	pmg	αβγδ	$Q_1Q_2Q_2Q_1$	Q_1,2;Q_2,2	C	SK29
IG 50.			$c^+b^-a^+d^+$	pmg	αβαγ	$Q_1Q_2Q_2Q_1$	Q_1,1D1R;Q_2,2	C	SK47
IG 51.			$c^-b^+a^-d^+$	pgg	αβαγ	Q Q Q Q	Q,2D2R	C,IH 51	SK55
IG 52.			$c^-d^-a^-b^-$	pgg	αβαβ	$Q_1Q_2Q_1Q_2$	Q_1,1D1R;Q_2,1D1R	C	SK53
IG 53.			$b^-a^-c^+d^+$	pgg	ααβγ	Q Q Q Q	Q,2D2R	C,IH 53	SK51
IG 54.			$a^-b^-c^-d^+$	cmm	αβγδ	$Q_1Q_2Q_3Q_1$	Q_1,2;Q_2,1;Q_3,1	C	SK21
IG 55.			$b^+a^+c^+d^+$	p4	ααββ	$Q_1Q_2Q_1Q_3$	Q_1,1D;Q_2,2D;Q_3,1D	C	SK42,S5
IG 56.			$b^+a^+c^-d^-$	p4g	ααβγ	$Q_1Q_2Q_1Q_3$	Q_1,4;Q_2,1D1R;Q_3,2	C	SK18
IG 57.		C_2 $a^+b^+a^+b^+$	a^+b^+	p2	αβαβ	Q Q Q Q	Q,1D	C,IH 57	SK60,S2
IG 58.			a^-b^+	pmg	αβαβ	Q Q Q Q	Q,2	N,IH 66	
IG 59.			b^-a^-	pgg	αααα	Q Q Q Q	Q,1D1R	N,IH 59	
IG 60.			a^-b^-	cmm	αβαβ	$Q_1Q_2Q_1Q_2$	Q_1,1;Q_2,1	N	
IG 61.			b^+a^+	p4	αααα	$Q_1Q_2Q_1Q_2$	Q_1,1;Q_2,1	N	



IG 36 IG 38 IG 39

FIGURE 1
The three convex isogonal tilings
missed by Šubnikov [8] and Šubnikov and Kopcik [9]

obtain the relation

$$\sum_{n \geq 3} \frac{n-2}{n} p_n = 2,$$

List No.	Net	Vertex group and Symbol	Adjacency Symbol	Space group	Edge transitivity	Tile transitivity	Aspects	Realisations	References
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
IG 62.	(4^4) cont.	C_4 $a^+a^+a^+a^+$	a^+	p4	αααα	Q Q Q Q	Q,1D	N,IH 62	
IG 63.			a^-	p4g	αααα	Q Q Q Q	Q,2	N,IH 73	
IG 64.		D_1^s ab^+cb^-	cb^-a	pm	αβαβ	Q Q Q Q	Q,1	N,IH 64	
IG 65.			ab^-c	pmm	αβγβ	$Q_1Q_1Q_2Q_1$	Q_1,1;Q_2,1	C	SK26
IG 66.			cb^+a	pmg	αβαβ	Q Q Q Q	Q,1D1R	C,IH 58	SK48
IG 67.			ab^+c	cmm	αβγβ	Q Q Q Q	Q,2	C,IH 67	SK22
IG 68.		D_1^ℓ $a^+b^+b^-a^-$	b^-a^-	cm	αααα	Q Q Q Q	Q,1	N,IH 62	
IG 69.			a^+b^+	pmg	αββα	Q Q Q Q	Q,2	C,IH 69	SK49
IG 70.			a^-b^-	p4m	αββα	$Q_1Q_2Q_3Q_2$	Q_1,1;Q_2,2;Q_3,1	C	SK15
IG 71.			b^+a^+	p4g	αααα	$Q_1Q_2Q_1Q_2$	Q_1,2;Q_2,1D1R	C	SK40
IG 72.		D_2^s abab	ab	pmm	αβαβ	Q Q Q Q	Q,1	C,IH 72	SK24
IG 73.			ba	p4g	αααα	Q Q Q Q	Q,1D1R	M,IH 63	
IG 74.		D_2^ℓ $a^+a^-a^+a^-$	a^+	cmm	αααα	Q Q Q Q	Q,1	C,IH 74	SK45
IG 75.			a^-	p4m	αααα	$Q_1Q_2Q_1Q_2$	Q_1,1;Q_2,1	N	
IG 76.		D_4 aaaa	a	p4m	αααα	Q Q Q Q	Q,1	C,IH 76	SK17,S6
IG 77.	(4.6.12)	E $a^+b^+c^+$	$a^-b^-c^-$	p6m	αβγ	D H Q	D,1;H,2;Q,3	C	SK1

where p_n is the number of n-gonal tiles meeting at each node of $N(\mathfrak{T})$. (Here, as elsewhere, a tile will be described as an n-gon if it has n vertices and n edges. The word n-gon is not to be understood to imply that the tile is a polygon.) There exist 17 solutions of this equation in integers $p_3, p_4, \ldots$ and since the tiles round a vertex may, in general, be arranged in several different ways, we arrive at a total of 21 possibilities. But many of these can be eliminated immediately on combinatorial grounds–in other words, the particular arrangement cannot be continued in a consistent manner over the whole plane. We do not give full details of the analysis here since it is essentially the same as that described in [4]. Finally we arrive at eleven different nets one of which occurs in two enantiomorphic forms. These are familiar as the nets of the 11 "uniform" or "Archimedean" tilings [4], shown in Figure 2.

Let $\mathfrak{T}$ be an isogonal tiling, and let V be any vertex of $\mathfrak{T}$. Consider any edge incident with V, and to the end of this edge near V associate a "sense of crossing" by which we mean that we associate with the end a clockwise or counterclockwise sense of rotation about V. We shall say that we have a

List No.	Net	Vertex group and Symbol	Adjacency Symbol	Space group	Edge transitivity	Tile transitivity	Aspects	Realisations	References
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
IG 78.	(4.8^2)	E $a^+b^+c^+$	$a^+b^-c^-$	cmm	αβγ	0 0 Q	0,1;Q,1	C	SK43
IG 79.			$a^+c^+b^+$	p4	αββ	0 0 Q	0,1D;Q,1D	C	SK41,S5
IG 80.			$a^-b^-c^-$	p6m	αβγ	$0_1 0_2 Q$	0_1,1:0_2,1;Q,2	C	SK14
IG 81.			$a^-c^+b^+$	p4g	αββ	0 0 Q	0,2;Q1D1R	C	SK39
IG 82.		D_1 ab^+b^-	ab^-	p4m	αββ	0 0 Q	0,1;Q,1	C	SK16
IG 83.	(6^3)	E $a^+b^+c^+$	$b^-a^-c^-$	cm	ααβ	H H H	H,1	N,IH 12	
IG 84.			$a^+b^+c^+$	p2	αβγ	H H H	H,1D	C,IH 8	SK58,S4
IG 85.			$a^-b^+c^+$	pmg	αβγ	H H H	H,2	C,IH 13	SK46
IG 86.			$b^-a^-c^+$	pgg	ααβ	H H H	H,1D1R	C,IH 9	SK52
IG 87.			$a^-b^-c^-$	p3m1	αβγ	$H_1H_2H_3$	H_1,1;H_2,1;H_3,1	C	SK9
IG 88.			$b^+a^+c^+$	p6	ααβ	$H_1H_2H_1$	H_1,2D;H_2,1D	C	SK37,S10
IG 89.		C_3 $a^+a^+a^+$	a^-	p31m	ααα	H H H	H,1	N,IH 18	
IG 90.			a^+	p6	ααα	H H H	H,1D	N,IH 11	
IG 91.		D_1 ab^+b^-	ab^+	cmm	αββ	H H H	H,1	C,IH 17	SK44
IG 92.			ab^-	p6m	αββ	$H_1H_1H_2$	H_1,2;H_1,1	C	SK3
IG 93.		D_3 aaa	a	p6m	ααα	H H H	H,1	C,IH 20	SK6,S12

sensed end and denote it on diagrams by inserting a small arrow whose shaft crosses the edge near to V. The terminology "directed" or "oriented" is avoided here because these words are usually reserved for the assignation of a direction *along* an edge, rather than *across* the end of an edge as required here.

To one of the sensed ends at V let us assign a symbol, say a, and then apply the operations of $S(\mathfrak{T})$ to yield a corresponding assignation of the same symbol to at least one sensed end at every other vertex of $\mathfrak{T}$. Not only may two or more sensed ends at a given vertex be assigned the same symbol, but it may also happen (if the sense of an end is reversed by an operation of $S(\mathfrak{T})$) that the same symbol is assigned a second time to the same end, but with the opposite sense. In this case we shall consider the symbol as attached to an *unsensed* end. If there still remain ends in the tiling to which no symbol is attached, then we assign a new symbol, say b, to one of the free ends, and proceed in this way until a symbol has been assigned to every end of every edge in $\mathfrak{T}$. The *vertex symbol* is obtained by reading off the symbols attached to the ends at V, in cyclic order. If we are reading counterclockwise, then a superscript + or − is used to indicate counterclockwise or clockwise sensing

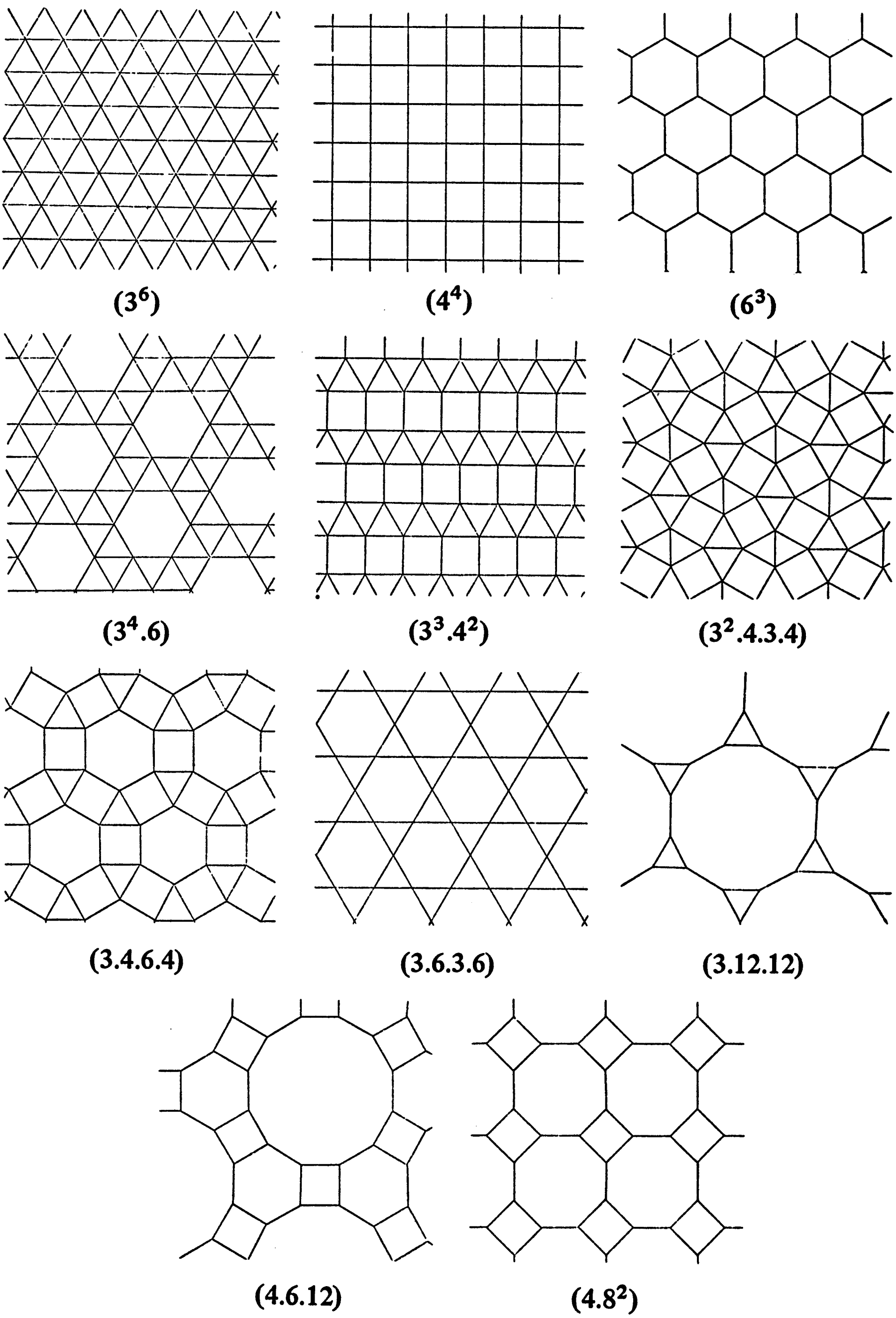


FIGURE 2
The eleven uniform (Archimedean) tilings

of the ends, and no superscript is used if the end is unsensed. An example of an isogonal tiling is given in Figure 3, together with the labelling of the ends at the vertex V. Here the vertex symbol is $a^+b^+c^+c^-b^-a^-$. We do not, of course, distinguish between vertex symbols which can be obtained from one another by cyclic permutations or reversals.

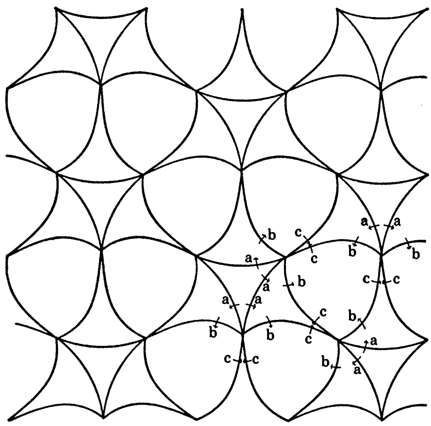


FIGURE 3
Example of an isogonal tiling with sensed ends of edges labelled

We now define the *adjacency symbol* of $\mathfrak{T}$ in the following manner. Consider, in order, the edges incident with V whose ends are labelled a, b, $c, \ldots$. Suppose that the edge with end labelled a bears a symbol x at its other end. If end a is unsensed, then necessarily so is x, and then x is the first component of the adjacency symbol. If a is sensed, however, then *either* x is sensed in the *same* direction as a (by which we mean that their senses are both clockwise or counterclockwise about their respective vertices) in which case the first component of the adjacency symbol is x^+, *or* x is sensed in the *opposite* direction to a, in which case the first component is x^-. For the second, third, . . . , components of the adjacency symbol we define symbols in the same manner corresponding to $b, c, \ldots$, until all the distinct letters in the vertex symbol are exhausted. For example, the adjacency symbol of the tiling of Figure 3 is $a^-c^+b^+$.

The adjacency symbol clearly indicates the relationship between V and its neighboring vertices. Hence we make the following definition.

DEFINITION. Two isogonal tilings with the same combinatorial type of net are *of the same type* if they have the same vertex and adjacency symbols.

Of course, adjacency symbols are not considered distinct if they differ from each other trivially, that is by change of notation, cyclic permutation, or reversal of order.

To enumerate the normal isogonal tilings we must therefore take each of the 11 possible nets, and then determine all the vertex and adjacency symbols that can be associated with each. This is a purely combinatorial problem, and, in fact, it turns out to be very simple. The reason is that the list of symbols is the same as that determined in [3] for the isohedral tilings. This is suggested by the fact that the construction of the vertex and adjacency symbols described above closely parallels that of the tile and adjacency symbols as described in [3]. This relationship may be make precise in the following way. For our isogonal tiling $\mathfrak{T}$ construct a *dual tiling* by selecting a point in the interior of each tile, and joining two such points by an arc (straight or not) if and only if the tiles in which they lie have an edge in common. These arcs must, of course, be chosen to be disjoint except at their endpoints. In this way we obtain a "combinatorial" dual tiling $\mathfrak{T}^*$, each tile of which contains a vertex of $\mathfrak{T}$. The tiling $\mathfrak{T}^*$ is combinatorially isohedral in that its isomorphism group is transitive on its tiles, a fact which follows from the isogonality of $\mathfrak{T}$. We may assign to each tile of $\mathfrak{T}^*$ a symbol coinciding with the vertex symbol of the vertex of $\mathfrak{T}$ which it contains, and moreover we may do this in a canonical way as indicated in Figure 4. It is now clear that the problem of finding all vertex and adjacency symbols for $\mathfrak{T}$ is precisely the same as that of finding all tile and adjacency symbols for $\mathfrak{T}^*$, thus justifying the assertion made above.

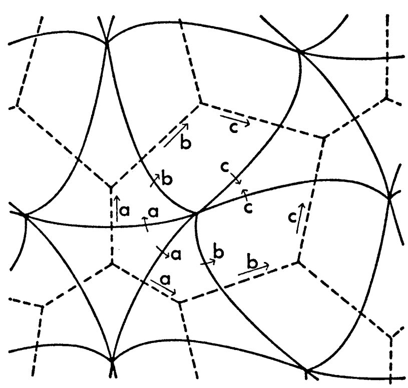


FIGURE 4
The correspondence between the vertex symbol of an isogonal tiling and the tile symbol of the dual tiling

Just as the 93 entries in Table I of [3] correspond to the 93 combinatorial types of normal isohedral tilings, so the same symbols in Table I of this paper correspond to the combinatorial types of normal isogonal tilings. This completes the first stage in the enumeration.

We must now see which of these types can be represented by an isogonal tiling as originally defined. In other words we must consider the shapes of the tiles only, and not think of them as having the ends of their edges marked by symbols and sensed. To do this we examine each type in turn. For each we may, if we wish, first distort the corresponding uniform net so as not to affect the crystallographic group of the tiling, but to make edges in different transitivity classes of different lengths. Then replace each edge by an arc according to the following scheme:

(a) If the two ends of an edge bear the same letter and are sensed in the same direction, then replace the edge by a *C*-curve, that is, any centrally symmetric curve whose center of symmetry is the midpoint of the edge.

(b) If the two ends of an edge bear the same letter and are sensed in opposite directions, then replace the edge by a *D*-curve, that is, any arc which possesses the perpendicular bisector of the edge as a line of reflective symmetry.

(c) If the two ends of an edge bear different letters, then replace it by any arc (an *E*-curve).

In all other cases we must leave the edge as a straight line segment. In (a), (b) and (c) the only restriction on the arcs is that in the final tiling they must be disjoint except at their endpoints, and that congruent arcs correspond to equally marked edges.

TABLE I. *The* 93 *adjacency symbols*. Column (1) gives the list number, and column (2) indicates the net. Here $(p.q. \dots)$ means that a p-gon, a q-gon, ... meet at each vertex, exponents being used to abbreviate in the usual manner. Column (3) indicates the possible vertex symbols and also the *vertex group*, that is, the restriction of $S(\mathfrak{T})$ to the neighborhood of one vertex. The same notation is used for the groups as in [3]. Column (4) shows all possible adjacency symbols, column (5) indicates the crystallographic group of the tiling, and column (6) shows the transitivity classes of edges at each vertex in the same notation as that of Table I of [3]. The transitivity classes of the tiles incident with each vertex are shown in column (7) and are denoted in a similar manner, except that $T_1, T_2, \dots$ denote different classes of triangles; $Q_1, Q_2, \dots$ denote different classes of quadrangles, and so on. H stands for hexagons, O for octagons, and D for dodecagons.

Column (8) indicates the number of different aspects of the tiles in each of the transitivity classes. Thus T_1, $2D$; T_2, $1D$ $1R$; Q, 3 means that the triangles of class T_1 occur in two direct aspects; triangles of class T_2 occur in one direct and one reflected aspects; and quadrangles occur in three aspects, and each quadrangle coincides with its mirror image. The distinction between direct and reflected aspects is made whenever a tile does not coincide with its mirror image.

Column (9) shows the possible realizations, with abbreviations as follows:

C: Convex isogonal tiling exists.

N: Isogonal tiling exists, but it cannot be convex.

M: A marked isogonal tiling exists, but a normal isogonal tiling does not.

IH: The tiling is isohedral. The number of the corresponding tiling in Table I of [3] is indicated.

Column (10) gives reference to the literature. SK means Šubnikov and Kopcik [9] and S means Sauer [7]. Where convex isogonal tilings exist which have been missed by Šubnikov and Kopcik, a reference is made to Figure 1 of this paper, where each such tiling is shown.

Carrying out this process we find that in 91 cases isogonal tilings can be constructed. The two cases that fail are IG 18 and IG 73. In the case of IG 18 the specification implies that it coincides with the regular tiling by triangles, and so must be of type IG 20; similarly the specification of IG 73 implies that it coincides with type IG 76. The result of this process is to establish the first statement of Theorem 1. Going through the list of Table I again we see that of these 91 types, exactly 34 are transitive on the tiles, and so are also isohedral tilings. Finally we examine each type to see if it is realizable by convex tiles. It turns out that this is possible in 63 cases, and so the theorem is proved.

In Figure 5 we reproduce diagrams of all 91 types of normal isogonal tilings. For convex realizations, the reader is referred to Figures 174 to 178 in the book by Šubnikov and Kopcik [9] and to the three diagrams of Figure 1. The two combinatorial types (IG 18 and IG 73) which are not realizable by tilings can be exemplified by "marked" tilings in a similar manner to that described in [3]. Diagrams of these two types appear in Figure 6.

3. In this section we briefly consider bounded isogonal tilings $\mathcal{T}$ which contain digons. To begin with we determine possible nets, as at the beginning of the preceding section. First we notice that if we put $n = 2$ in the equation derived from Euler's theorem, then the coefficient of the term in p_2 is zero, thus indicating that infinitely many nets are possible. In fact, we can go further, and construct bounded isogonal tilings with arbitrarily many digons at each vertex by using the following procedure.

Consider an arc α of a normal isogonal tiling, and let its endpoints be denoted by P, Q. Replace α by a set D of digons, all of whose vertices lie at P and Q, and also replace the images of α under the operations of $S(\mathcal{T})$ by the images of D. Then if the set D has the same symmetries as α, the new tiling will be isogonal. For example, if α is a C-curve, then we may replace it by a set D of digons which is centrally symmetric in the midpoint of α. It should be noted that this operation can be carried out on any combinatorial type of isogonal tiling, whether such a type can be realized by an actual normal tiling or by a marked tiling.

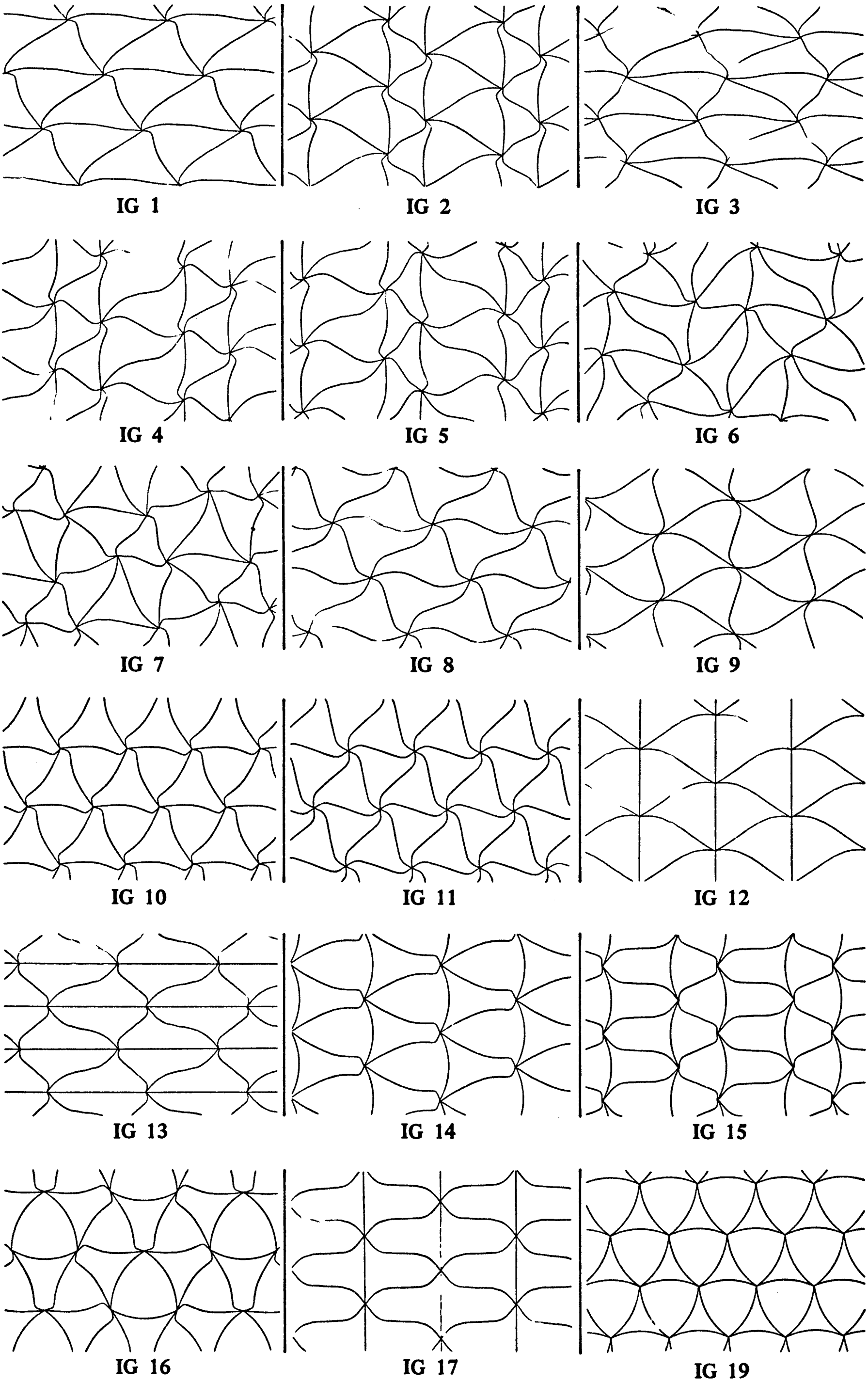


FIGURE 5. (Part (i)) The ninety-one types of isogonal tilings

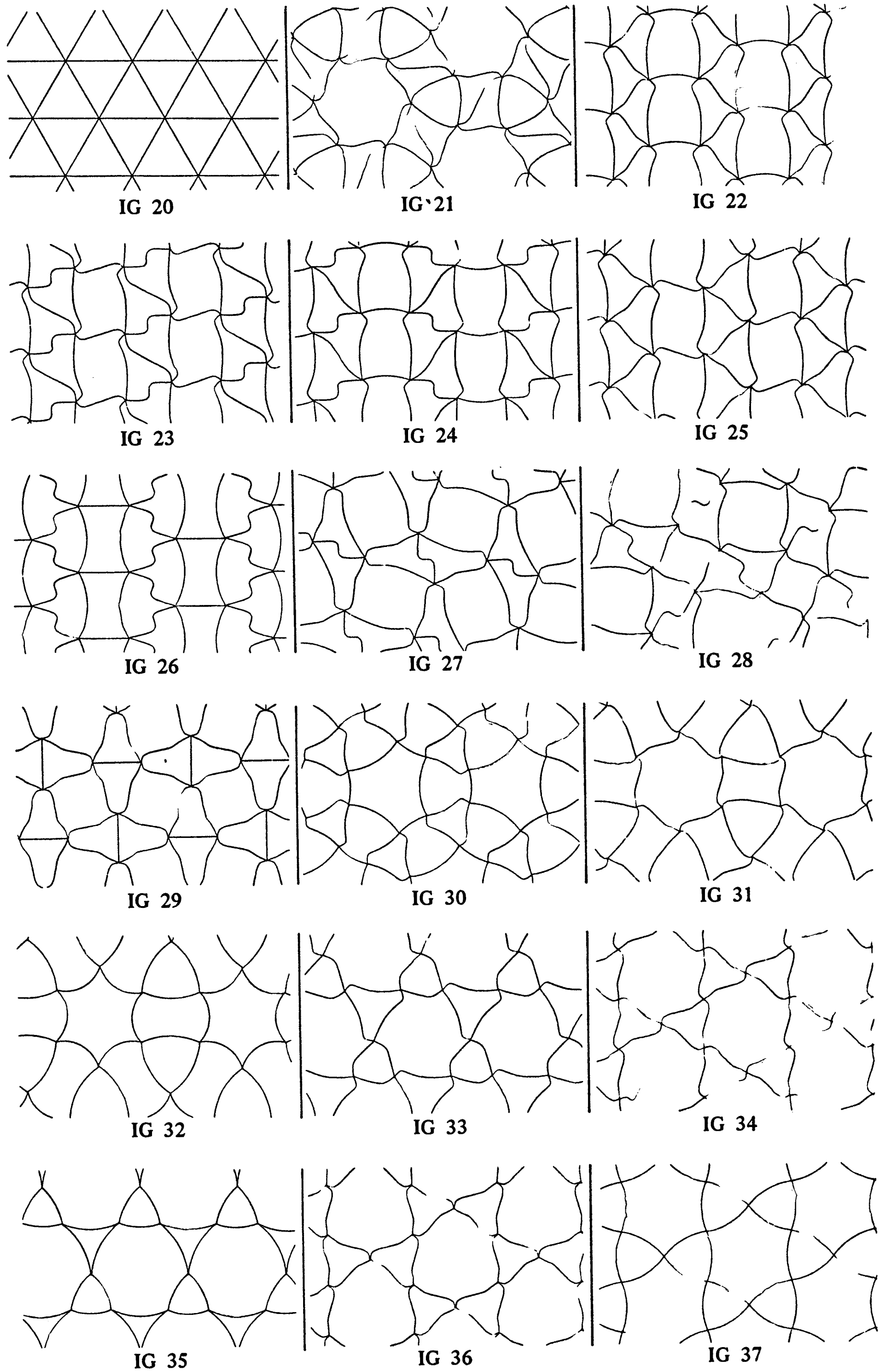
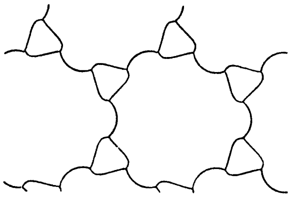


FIGURE 5. (Part (ii))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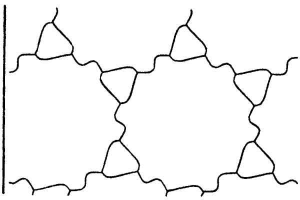
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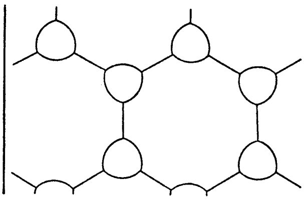
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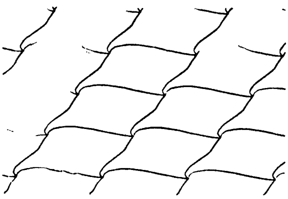
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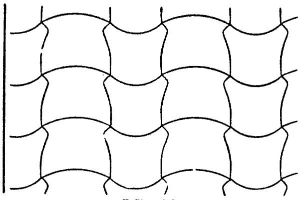
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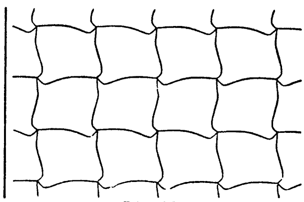
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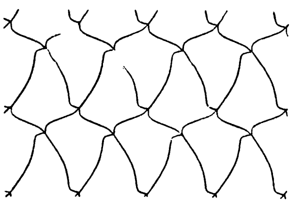
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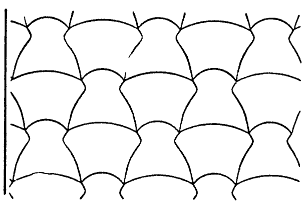
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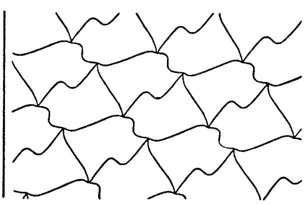
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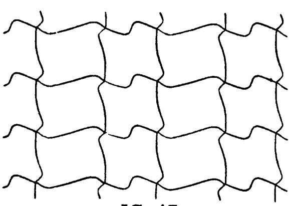
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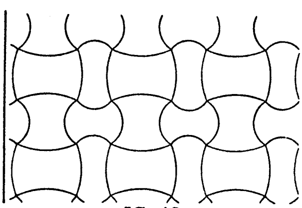
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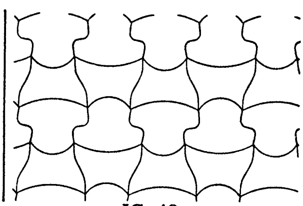
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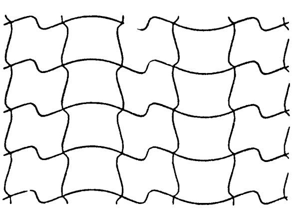
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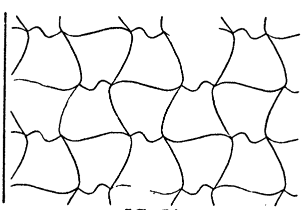
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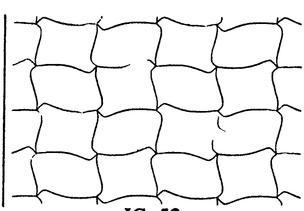
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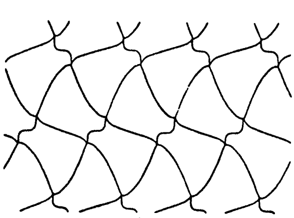
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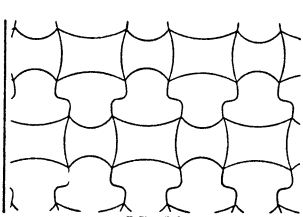
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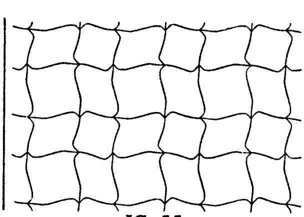
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FIGURE 5. (Part (iii))

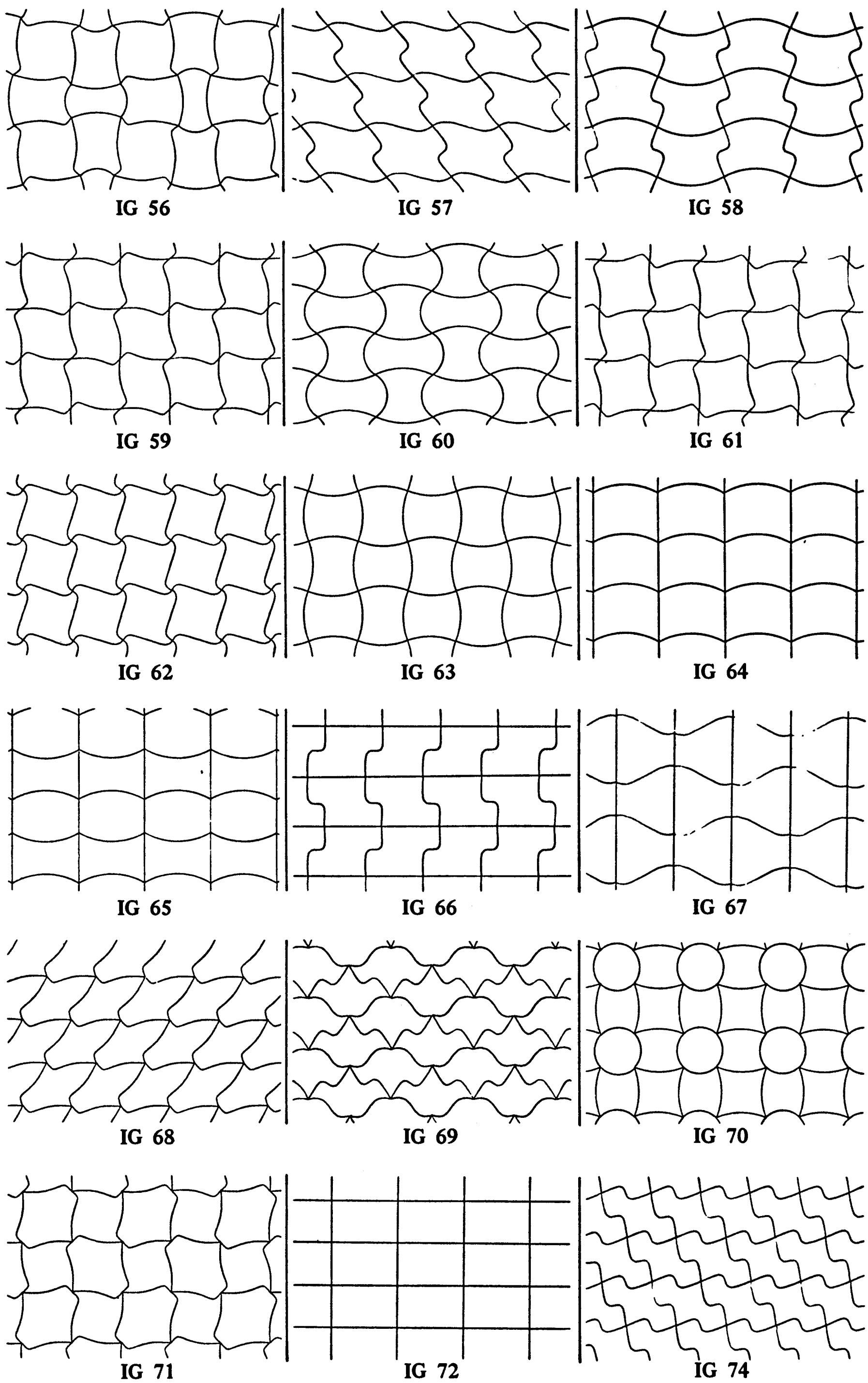
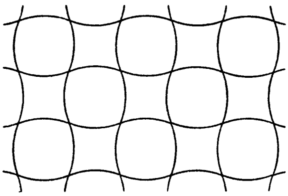


FIGURE 5. (Part (iv))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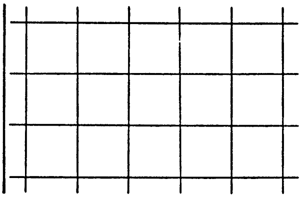
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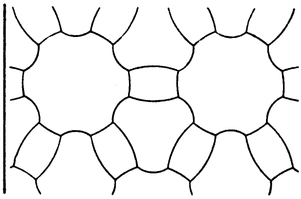
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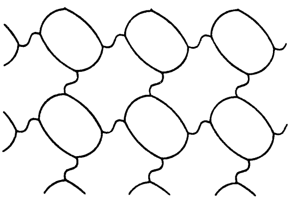
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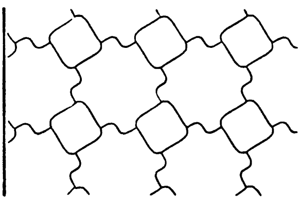
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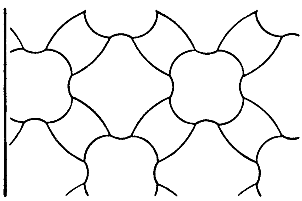
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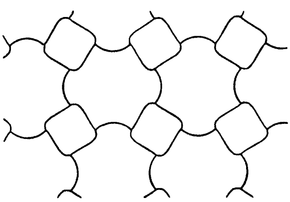
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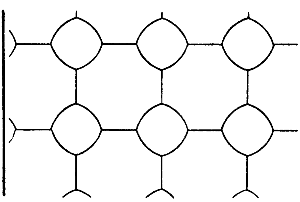
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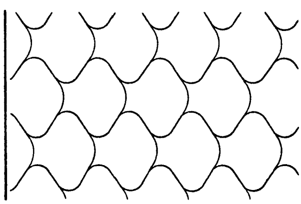
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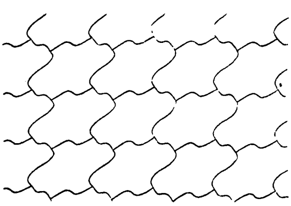
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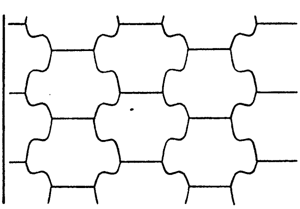
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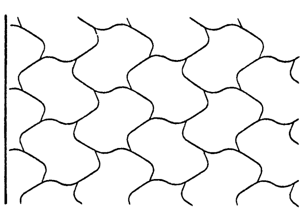
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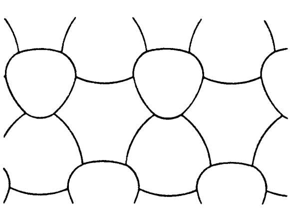
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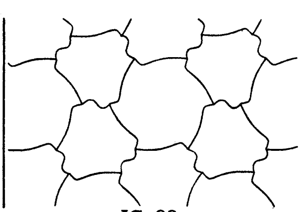
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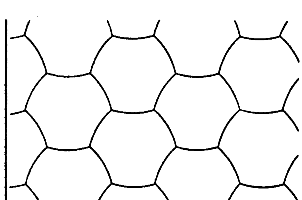
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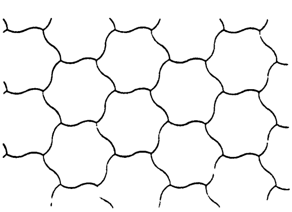
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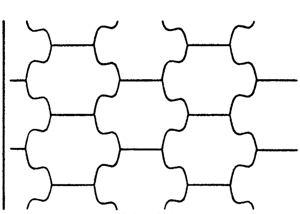
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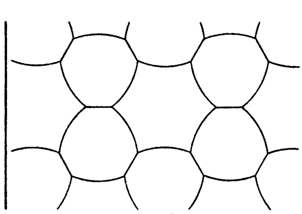
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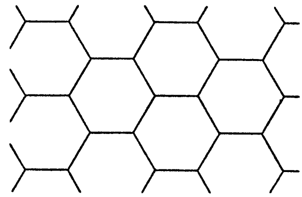
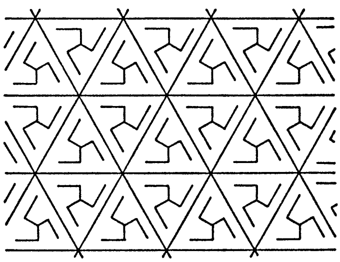
FIGURE 5. (Part (v))



IG 93

FIGURE 5. (Part (vi))



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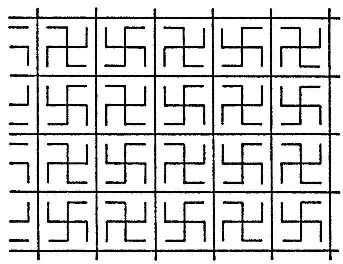
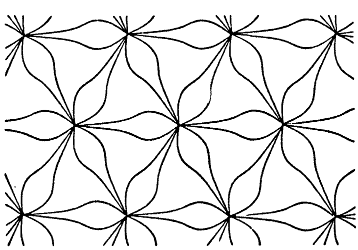


IG 73

FIGURE 6
The two combinatorial types of isogonal tilings that can be realized by marked tilings only

In Figure 7 we show examples of tilings produced by this operation from normal tilings.



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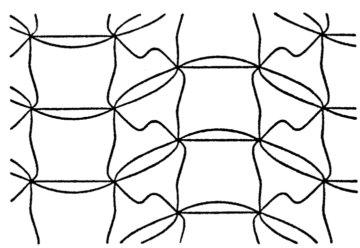

IG 24

FIGURE 7
Two examples of isogonal tilings containing digons

Our main result here is the following.

THEOREM 2. *Every bounded isogonal tiling is either normal or can be constructed from one of the* 93 *combinatorial types of normal isogonal tilings by applying the operation described above to one or more transitivity classes of arcs (edges).*

The proof of this is immediate from the observation that the operation described above has an inverse. In other words, if we start from any bounded isogonal tiling, then we may replace any set of digons which share the same two points P, Q as vertices by a suitable arc with endpoints P, Q. Moreover, if this is done to the whole transitivity class of digons, then the tiling remains isogonal. Hence all the digons in a given tiling can be eliminated and the resulting isogonal tiling is normal.

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