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REGULARITY OF GRAPHS, COMPLEXES AND DESIGNS (*)

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Résumé. — On définit certaines structures partiellement ordonnées appelées *polystromes*. Les polystromes constituent un cadre commode pour l'étude de nombreux problèmes sur les graphes, cartes, complexes et configurations, concernant en particulier leurs propriétés de régularité.

Abstract. — A type of partially ordered structures called *polystromas* is defined. Polystromas provide a convenient framework for the consideration of many questions about graphs, maps, complexes and designs, and, in particular, their regularity properties.

The family of simplicial complexes has the following remarkable structure : There are « building blocks » called simplices; putting together sets of k-simplices, where $k \leqslant n$, we obtain n-complexes. A special way of doing so yields a particular n-complex called the minimally triangulated n-sphere $\mathcal{S}^n$. Adjoining to $\mathcal{S}^n$ a single $(n + 1)$-dimensional element we obtain an $(n + 1)$-simplex.

A completely analogous structure is exhibited by the so-called cubical complexes, and by geometric cell complexes the elements of which are convex polytopes. Very similar descriptions may be given to the structure of the collection of flats (subspaces) of a projective or affine space (finite or not), and in many other instances.

These examples serve as motivations and models for the definition of structures we shall call *polystromas* ($\sigma\tau\rho\omega\mu\alpha$ is Greek for layer or stratum). Various properties of polystromas will be seen to lead to several remarkable classes of objects, and to very interesting types of unsolved problems.

I hope that the following pages will manage to convey to the reader my conviction that intuitive geometry is still the source of meaningful new combinatorial problems and constructions. Most of the terminology is patterned after that of the examples mentioned above and the theory of convex polytopes.

A *polystroma* $\mathcal{C}$ is any partially ordered set with a single least element $0 = 0_{\mathcal{C}}$, or any collection of objects, with some relation among them, that is isomorphic to such a partially ordered set. In the latter case we shall say that the collection of objects is a *realization* of the partially ordered set.

We shall use the equality sign $=$ to denote isomorphism of polystromas. A polystroma $\mathcal{C}^*$ is a *dual* of a polystroma $\mathcal{C}$ if there exists a one-to-one order-reversing correspondence between the non-zero elements of $\mathcal{C}$ and the non-zero elements of $\mathcal{C}^*$. As all duals of $\mathcal{C}$ are clearly isomorphic, we may speak about *the* dual of $\mathcal{C}$; obviously $(\mathcal{C}^*)^* = \mathcal{C}$. We shall denote by $\hat{\mathcal{C}}$ the polystroma obtained from the polystroma $\mathcal{C}$ by adjoining to it a single element $1 = 1_{\mathcal{C}}$ that majorizes all the elements of $\mathcal{C}$.

In many of the examples of polystromas $\mathcal{C}$ we shall encounter either $\mathcal{C}$ is a lattice, or at least $\hat{\mathcal{C}}$ is a lattice.

An *atom* or *vertex* of a polystroma $\mathcal{C}$ is any element of $\mathcal{C}$ that is minimal in $\mathcal{C} \setminus \{ 0 \}$. A *facet* of $\mathcal{C}$ is any maximal element of $\mathcal{C}$. If V is a vertex of $\mathcal{C}$, the *vertex-figure* $\mathcal{V}(V, \mathcal{C})$ of at V is the polystroma consisting of V and of all the elements of $\mathcal{C}$ that majorize V; hence $V = 0_{\mathcal{V}(V,\mathcal{C})}$. If F is a facet of $\mathcal{C}$ the *facet-figure* $\mathcal{F}(F, \mathcal{C})$ of $\mathcal{C}$ at F is the polystroma consisting of F and of all the elements of $\mathcal{C}$ that are majorized by F; hence F is the only facet of $\mathcal{F}(F, \mathcal{C})$. Vertices and facets of dual polystromas clearly correspond to each other, and their vertex-figures and facet-figures are duals of each other.

A *0-polystroma* is a polystroma in which each atom is a facet. It is obvious that for each natural $k \geqslant 1$ there is precisely one 0-polystroma $\mathcal{T}_k$ with k atoms. It may be realized by any k-tuple of objects. Each finite 0-polystroma is isomorphic to $\mathcal{T}_k$ for some k.

Let $\{ \mathcal{C}_i \mid i \subset I \}$ be any family of polystromas. If $\mathcal{C}$ is a polystroma with facets F_i, $i \in I$, such that $\mathcal{F}(F_i, \mathcal{C}) = \hat{\mathcal{C}}_i$ for each $i \in I$, and if each $\mathcal{C}_i$ is an n_i-polystroma for some $n_i \leqslant n$, with equality for at least one i, then $\mathcal{C}$ is an $(n + 1)$-polystroma.

If $\mathcal{C}$ and $\mathcal{A}$ are polystromas we say that $\mathcal{C}$ is *unifaceted* with $\mathcal{A}$ provided $\mathcal{F}(F, \mathcal{C}) = \hat{\mathcal{A}}$ for every facet F of $\mathcal{C}$.

Using these definitions we see that graphs may be identified as 1-polystromas unifaceted with $\mathcal{T}_2$. Each simplicial n-complex is an n-polystroma. The mini-

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mally triangulated n-sphere S^n (and every triangulated n-manifold) is unifaceted with $\mathcal{S}^{(n-1)}$. The n-simplex T^n is the only facet in the n-polystroma $\widehat{\mathcal{S}^{(n-1)}}$. It is easily verified that for every vertex V of S^n we have $\mathcal{V}(V, \mathcal{S}^n) = \mathcal{S}^{n-1}$, and that $(\mathcal{S}^n)^* = \mathcal{S}^n$ is selfdual.

If a graph $\mathcal{G}$ is the union of a family $\mathcal{G}_i$ of subgraphs, we may choose to consider $\mathcal{G}$ as a 2-polystroma in which the facet-figures are the polystromas $\mathcal{G}_i$. A well-known example of this procedure is the Mac Lane characterization of planar graphs by their face-circuits. For a different example, let $\mathcal{T}_3$ be a path of length 3; see figure 1 in which the 3-path $\mathcal{T}_3$ is

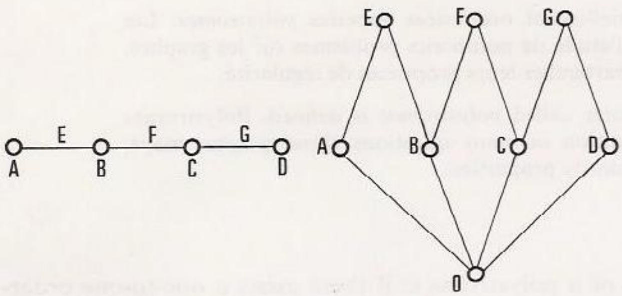


FIG. 1. — The 3-path $\mathcal{T}_3$.

shown together with the Hasse diagram of an isomorphic partially ordered set. It is not hard to show that each 2-connected 3-valent graph G may be interpreted as a 2-polystroma $\mathcal{G}$ unifaceted with $\mathcal{T}_3$ in several ways :

(i) So that the intersection of any two facet-figures of $\mathcal{G}$ is either empty, or a 0-polystroma ; several examples are shown in figure 2. Each vertex-figure is isomorphic to the 1-polystroma $\mathfrak{Z}_1$ indicated in figure 2e. Such an interpretation of G is equivalent to the existence of a 1-factor of G, hence it is not always possible if the 3-valent graph G is only connected (*).

(ii) So that each edge of the graph G is in two facet-figures ; see the examples in figure 3. The representation may be chosen so that the vertex-figure at each vertex is isomorphic to the 1-polystroma $\mathfrak{Z}_2 = (\mathcal{T}_3)^*$ shown in figure 3e.

(iii) If G is planar and the map corresponding to G is 4-colorable, then each edge of G may be taken to belong to three facet-figures (see examples in figure 4), with the vertex-figure at each vertex isomorphic to the 1-polystroma $\mathfrak{Z}_3$ shown in figure 4c. Conversely, it is easily seen that if a 3-valent 2-connected planar graph is unifaceted with $\mathcal{T}_3$ and all vertex figures are isomorphic to $\mathfrak{Z}_3$, then the map that corresponds to G is 4-colorable.

The examples just considered make reasonable the following definition : If $\mathcal{A}$ and $\mathcal{B}$ are polystromas

(*) If the 3-valent, 2-connected graph G is, moreover, planar, then it is possible to choose its facets as 3-paths that are parts of Petrie-polygons of G (P. Kleinschmidt, private communication).

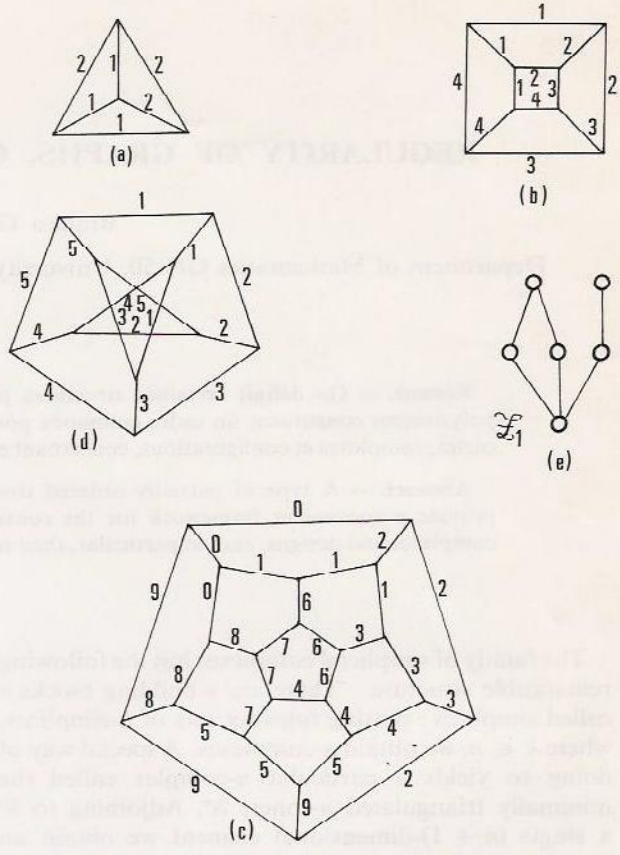


FIG. 2. — Some graphs represented as 2-polystromas unifaceted with $\mathcal{T}_3$. Each vertex-figure is a 1-polystroma isomorphic to $\mathfrak{Z}_1$, the Hasse diagram of which is shown in (e). In (a) to (d) the different facets are indicated by the labels.

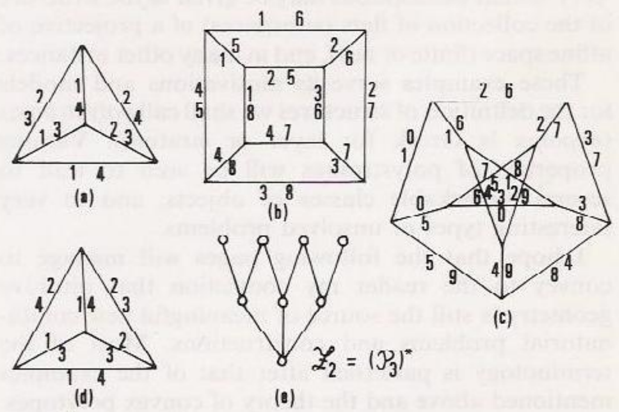


FIG. 3. — Some graphs represented as 2-polystromas unifaceted with $\mathcal{T}_3$. Each vertex-figure in (a), (b), (c) is isomorphic to $\mathfrak{Z}_2$ indicated in (e).

we shall denote by $\langle \mathcal{A}, \mathcal{B} \rangle$ the collection (possibly empty) of polystromas $\mathcal{C}$ with the following properties : $\mathcal{C}$ is unifaceted with $\mathcal{A}$ and each vertex figure of $\mathcal{C}$ is isomorphic to $\mathcal{B}$. Clearly, $\mathcal{C} \in \langle \mathcal{A}, \mathcal{B} \rangle$ if and only if $\mathcal{C}^* \in \langle \mathcal{B}^*, \mathcal{A}^* \rangle$.

With this definition, the 2 polystromas just discussed belong to the following collections : Those

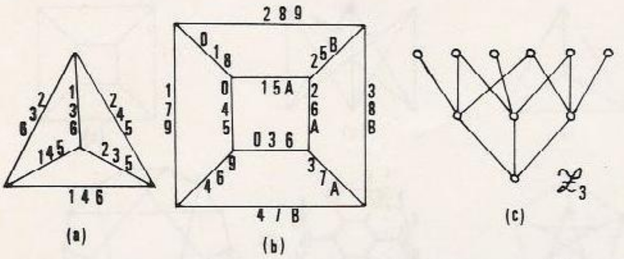


FIG. 4. — Two graphs represented as 2-polystromas unifaceted with $\mathcal{T}_3$. Each vertex-figure is isomorphic to the 1-polystroma $\mathfrak{Z}_3$, the Hasse diagram of which is shown in (c).

in (i) to $\langle \mathcal{T}_3, \mathfrak{Z}_1 \rangle$, those in (ii) to $\langle \mathcal{T}_3, \mathfrak{Z}_2 \rangle$, and those in (iii) to $\langle \mathcal{T}_3, \mathfrak{Z}_3 \rangle$.

The question for which $\mathcal{A}$, $\mathcal{B}$ is $\langle \mathcal{A}, \mathcal{B} \rangle$ non-empty, or contains members realizable by various interesting classes of objects, appears to be very hard, by even partial solutions will probably be rather rewarding.

For a different type of examples we consider the Kirkman-Steiner triple systems $\mathcal{KS}(n)$, where, as is well known, $n \equiv 1$ or 3 mod 6. Each $\mathcal{KS}(n)$ may be considered as a 1-polystroma unifaceted with triplets $\mathcal{C}_3$. If we denote the atoms of $\mathcal{KS}(7)$ by 0, 1, 2, 3, 4, 5, 6, then the facets may be chosen as the triplets 0, 1, 3 ; 1, 2, 4 ; 2, 3, 5 ; 3, 4, 6 ; 4, 5, 0 ; 5, 6, 1 ; 6, 0, 2. Analogously we shall interpret the structure of each 1-polystroma isomorphic to $\mathcal{KS}(7)$. As is easily verified, $\mathcal{KS}(7)$ belongs to $\langle \mathcal{C}_3, \mathcal{C}_3 \rangle$ and, more generally, each $\mathcal{KS}(6n+1) \in \langle \mathcal{C}_3, \mathcal{C}_{3n} \rangle$, while each

$$\mathcal{KS}(6n+3) \in \langle \mathcal{C}_3, \mathcal{C}_{3n+1} \rangle .$$

The finite geometry $PG(3, 2)$ may be interpreted as a 2-polystroma unifaceted with $\mathcal{KS}(7) - PG(2, 2)$; more precisely, it belongs to $\langle \mathcal{KS}(7), \mathcal{KS}(7) \rangle$. However, it appears not to be known whether $\langle \mathcal{KS}(n), \mathcal{KS}(n) \rangle$ is non-empty for any Kirkman-Steiner system $\mathcal{KS}(n)$ with $n > 7$.

On the other hand, an interesting 2-polystroma $\mathcal{E}$ may be derived from a construction of J. E. Edmonds. The 28 atoms of $\mathcal{E}$ may be denoted by unordered pairs from the set { 0, 1, 2, 3, 4, 5, 6, 7 } ; $\mathcal{E}$ is unifaceted with $\mathcal{KS}(7)$, formed with the following 8 sets of atoms :

F_0 :	01,	02,	03,	04,	05,	06,	07
F_1 :	10,	16,	14,	13,	17,	12,	15
F_2 :	20,	27,	25,	24,	21,	23,	26
F_3 :	30,	31,	36,	35,	32,	34,	37
F_4 :	40,	42,	47,	46,	43,	45,	41
F_5 :	50,	53,	51,	57,	54,	56,	52
F_6 :	60,	64,	62,	61,	65,	67,	63
F_7 :	70,	75,	73,	72,	76,	71,	74.

Then it is easily verified that each vertex-figure is isomorphic to the 1-polystroma $\mathfrak{Z}_4$ shown in figure 5 ; hence $\mathcal{E} \in \langle \mathcal{KS}(7), \mathfrak{Z}_4 \rangle$. It may be conjectured that

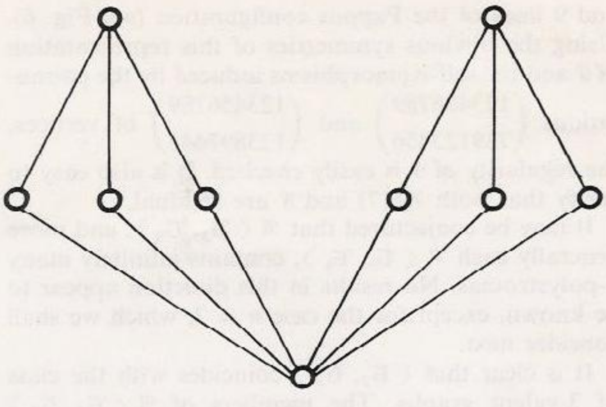


FIG. 5. — Hasse diagram of the 1-polystroma $\mathfrak{Z}_4$.

similar 2-polystromas can be constructed with other Kirkman-Steiner systems.

A *flag* in a polystroma $\mathcal{C}$ is any totally ordered subset of $\mathcal{C}$ that is maximal, i.e. not properly contained in any other such chain. A very useful and versatile definition of regularity of a polystroma, much more restrictive than the belonging to a family $\langle \mathcal{A}, \mathcal{B} \rangle$, is the following : A polystroma $\mathcal{C}$ is *regular* if the group of self-isomorphisms of $\mathcal{C}$ acts transitively on the flags of $\mathcal{C}$. The set of all regular polystromas in $\langle \mathcal{A}, \mathcal{B} \rangle$ shall be denoted by $\mathcal{R} \langle \mathcal{A}, \mathcal{B} \rangle$.

For example, it is easily seen that

$$\mathcal{KS}(7) \in \mathcal{R} \langle \mathcal{C}_3, \mathcal{C}_3 \rangle .$$

Another member of $\mathcal{R} \langle \mathcal{C}_3 \mathcal{C}_3 \rangle$ is the 1-polystroma $\mathcal{T}$ with atoms 1, 2, 3, 4, 5, 6, 7, 8, 9 and facets { 1, 4, 7 }, { 1, 5, 9 }, { 1, 6, 8 }, {2, 4, 9 }, { 2, 5, 8 }, { 2, 6, 7 }, { 3, 4, 8 }, {3, 6, 9 }, { 3, 5, 7 }. In order to verify its regularity it is convenient to realize $\mathcal{T}$ by the 9 points

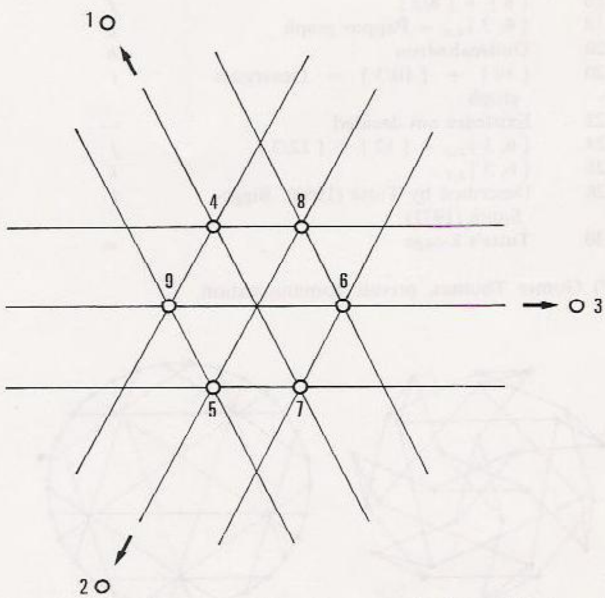


FIG. 6. — The Pappus configuration as realization of the 1-polystroma $\mathcal{T}$.

and 9 lines of the Pappus configuration (see Fig. 6). Using the obvious symmetries of this representation of $\mathcal{T}$ and the self-isomorphisms induced by the permutations $\begin{pmatrix}123456789\\789123456\end{pmatrix}$ and $\begin{pmatrix}123456789\\123897645\end{pmatrix}$ of vertices, the regularity of $\mathcal{T}$ is easily checked. It is also easy to verify that both $\mathcal{KS}(7)$ and $\mathcal{T}$ are selfdual.

It may be conjectured that $\mathcal{R}\langle \mathcal{T}_3, \mathcal{T}_3 \rangle$, and more generally each $\mathcal{R}\langle \mathcal{T}_n, \mathcal{T}_k \rangle$, contains infinitely many 1-polystromas. No results in this direction appear to be known, except for the case $n = 2$, which we shall consider next.

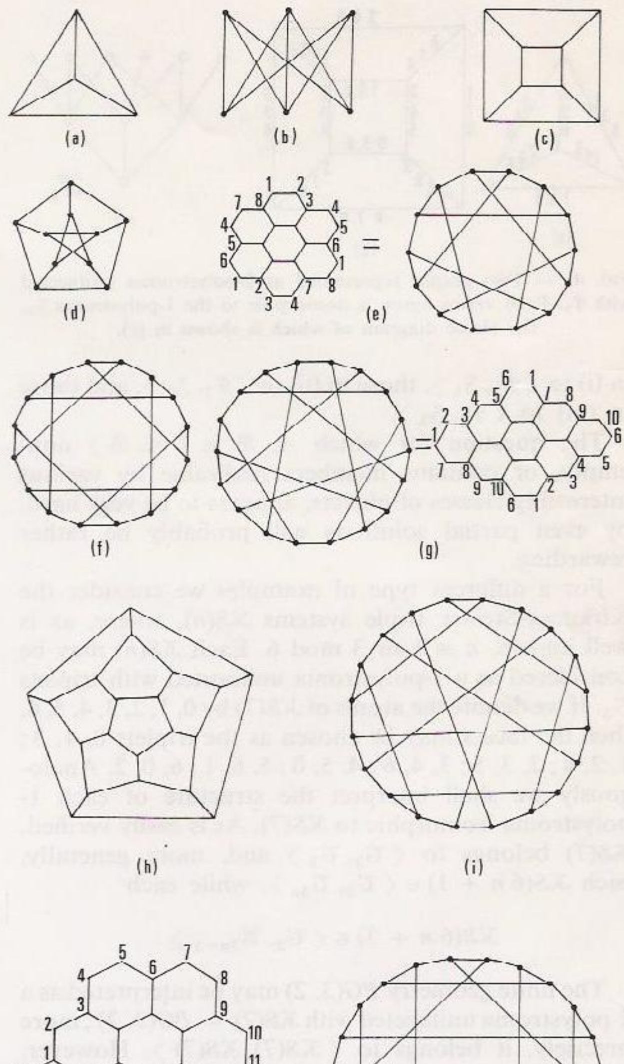
It is clear that $\langle \mathcal{T}_2, \mathcal{T}_3 \rangle$ coincides with the class of 3-valent graphs. The members of $\mathcal{R}\langle \mathcal{T}_2, \mathcal{T}_3 \rangle$ (the reader should beware of the unfortunate habit of some authors to call graphs in $\langle \mathcal{T}_2, \mathcal{T}_k \rangle$ « regular ») exhibit very remarkable symmetries. Indeed, in table 1 we bring all known graphs in $\mathcal{R}\langle \mathcal{T}_2, \mathcal{T}_3 \rangle$ with at most 30 vertices. The list reads like a « Who is who » of graph theory; the graphs are shown in figure 7. Infinitely many members of $\mathcal{R}\langle \mathcal{T}_2, \mathcal{T}_3 \rangle$ may be obtained as the graphs of the toroidal maps denoted

TABLE 1

The known graphs (1-polystromas) in $\mathcal{R}\langle \mathcal{T}_2, \mathcal{T}_3 \rangle$ with at most 30 vertices

Number of vertices	Description of the graph	Part of figure 7 in which the graph is shown
4	K_4 = tetrahedron	*a*
6	Utilities graph = Thomsen graph = Kuratowski graph = $K_{3,3}$	*b*
8	Cube	*c*
10	Petersen graph	*d*
12	Does not exist (*)	—
14	$\{6, 3\}_{2,1}$ = 6-cage	*e*
16	$\{8\} + \{8/3\}$	*f*
18	$\{6, 3\}_{3,0}$ = Pappus graph	*g*
20	Dodecahedron	*h*
20	$\{10\} + \{10/3\}$ = Desargues graph	*i*
22	Existence not decided	—
24	$\{6, 3\}_{2,2} - \{12\} + \{12/3\}$	*j*
26	$\{6, 3\}_{3,1}$	*k*
28	Described by Tutte (1960), Biggs-Smith (1971)	*l*
30	Tutte's 8-cage	*m*

(*) Gomer Thomas, private communication.



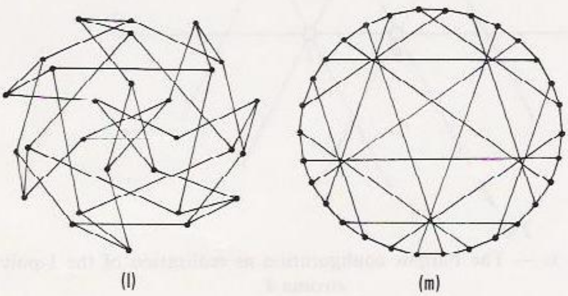


FIG. 7. — The known graphs in $\mathcal{R}\langle \mathcal{T}_2, \mathcal{T}_3 \rangle$ with at most 30 vertices.

$\{6, 3\}_{b,c}$ by Coxeter, with b, c non-negative integers; those graphs have $2(b^2 + bc + c^2)$ vertices. However, the numbers of vertices for which there exist graphs in $\mathcal{R}\langle \mathcal{T}_2, \mathcal{T}_3 \rangle$ have not been characterized; possibly every sufficiently large even number has that property.

Similar results are known concerning $\langle \mathcal{T}_2, \mathcal{T}_4 \rangle$ and $\mathcal{R}\langle \mathcal{T}_2, \mathcal{T}_4 \rangle$.

Turning to a different type of questions, we note that the traditional « regular » polyhedra — the five Platonic or convex ones, and the four Kepler-Poinsot or star-polyhedra — may best be described as special geometric realizations of 2-polystromas that belong to $\mathcal{R}\langle \mathcal{C}_p, \mathcal{C}_q \rangle$ for suitable p and q. Here $\mathcal{C}_n$ is the 1-polystroma we usually call « n-circuit »; clearly $\mathcal{C}_n \in \mathcal{R}\langle \mathcal{T}_2, \mathcal{T}_2 \rangle$ and it is the only connected member with n vertices, $n \geqslant 3$. This point of view at once leads to the question whether any other elements of suitable $\mathcal{R}\langle \mathcal{C}_p, \mathcal{C}_q \rangle$ may have geometric realizations with appropriate symmetries. The answer — very surprising, because it was overlooked so long — is affirmative! Details are set out in a forthcoming paper (« Regular polyhedra, old and new », to appear in Aequationes Mathematicae), but the crucial point is the following : A polygon with n sides is a realization of $\mathcal{C}_n$; when representing or imagining realizations of « regular polyhedra » people usually think of their facets $\hat{\mathcal{C}}_n$ as « patches » of a surface, bounded by the n-circuit $\mathcal{C}_n$. While this may be easily accepted in case the realization of $\mathcal{C}_n$ is by the boundary of a convex polygon, it becomes much less natural in case the realization of $\mathcal{C}_n$ is by a star n-gon, or by a skew n-gon, and it becomes quite meaningless if the polygon is an infinite regular polygon. In the interpretation of regular polygons as regular realizations of 1-polystromas the tendency (or temptation) to span the facets of the regular 2-polystromas by « membranes » disappears, and hence the psychological block that used to obscure the vision is not effective any more. We note in passing that the completeness of the list of « new » regular polyhedra has not been established, and that the possibilities of regular realizations of 2-polystromas in Euclidean 4-space (or higher spaces) has not been investigated either.

Similarly deserving of study are various geometrically symmetric realizations of members of $\mathcal{R}\langle \mathcal{C}_n, \mathcal{C} \rangle$, where $\mathcal{C}$ is a suitable regular graph (1-polystroma) other than a circuit. For example, the squares of unit side with vertices at the points of the integral lattice in 3-space realize a member of $\mathcal{R}\langle \mathcal{C}_4, \mathcal{O}_0 \rangle$, where $\mathcal{O}_0 \in \mathcal{R}\langle \mathcal{T}_2, \mathcal{T}_4 \rangle$ is the graph of the octahedron. Such structures have recently attracted the attention of architects and chemists, but their mathematical treatment is still at its beginning.

The final topic I would like to discuss deals with problems related to the question which first brought polystromas to my attention. As mentioned at the beginning, simplices of various dimensions are related by the fact that $(n + 1)$-dimensional ones are unifaceted with n-dimensional ones. Similarly for cubes of various dimensions. The sequences

point, segment, triangle, tetrahedron, 4-simplex, ..., and

point, segment, square, cube, 4-cube, ...

suggest the quest for sequences

point, segment, pentagon, platonic dodecahedron, regular 120-cell, ..., and

point, segment, hexagon, ...

etc.

However, the last two examples are usually thought of as not extending beyond the terms just written. The question that puzzled me is whether those sequences can be extended in any sensible and meaningful way. The answer is again affirmative — at least in some cases of this general nature. Since the question is rather imprecise there are different possible interpretations; many of them seem to lead to interesting problems in geometry, topology, and combinatorics.

For example, we have already mentioned the toroidal maps denoted $\{6, 3\}_{b,c}$. They are 2-polystromas, and it is easily seen that they belong to $\langle \mathcal{C}_6, \mathcal{C}_3 \rangle$. Moreover, if $c = 0$ or $c = b$, then

$$\{6, 3\}_{b,c} \in \mathcal{R}\langle \mathcal{C}_6, \mathcal{C}_3 \rangle .$$

Similarly, there exist toroidal maps

$$\{4, 4\}_{b,c} \in \langle \mathcal{C}_4, \mathcal{C}_4 \rangle \quad \text{and} \quad \{3, 6\}_{b,c} \in \langle \mathcal{C}_3, \mathcal{C}_6 \rangle$$

for all non-negative integers b, c; they are regular if $c = 0$ or if $c = b$. Maps on 2-manifolds that realize 2-polystromas from $\langle \mathcal{C}_p, \mathcal{C}_q \rangle$ have been investigated for almost a century, and it is well known that $\mathcal{R}\langle \mathcal{C}_p, \mathcal{C}_q \rangle$ contains, for each p, q with

$$\frac{1}{p} + \frac{1}{q} < \frac{1}{2},$$

a member realizable as an infinite regular tiling of the real hyperbolic plane. It may be conjectured that $\mathcal{R}\langle \mathcal{C}_p, \mathcal{C}_q \rangle$ contains infinitely many finite members for each such pair p, q. However, it is not known even whether every $\langle \mathcal{C}_p, \mathcal{C}_q \rangle$ contains some finite members. Investigations of the question which maps on 2-manifolds realize finite members of $\langle \mathcal{C}_p, \mathcal{C}_q \rangle$ were started by G. Brunel in 1891; later work on this topic was done by White, Errera, Brahana, Threlfall, Kagno and many others. Very considerable advances concerning the regular members of those families were obtained in the just completed thesis of S. E. Wilson (« New techniques for the construction of regular maps », University of Washington, 1976).

The toroidal maps realizing the 2-polystromas $\{6, 3\}_{b,c}$ may clearly be interpreted as the continuations of the sequence « point, edge, hexagon ».

But another step in this direction is possible. There exist finite 3-polystromas in the family

$$\langle \{6, 3\}_{b,c}, \{3, 3\} \rangle,$$

at least if b and c satisfy $3 \leqslant b + c \leqslant 4$. Some data on the known examples are given in table 2 ; details are available on request. All the information available at present is consistent with the following :

Conjecture. — For every toroidal map

$$\{6, 3\}_{b,c}, \quad (b, c) \neq (1, 1), \quad \langle \{6, 3\}_{b,c}, \{3, 3\} \rangle$$

contains finite 3-polystromas. In particular, the « naturally generated » 3-polystroma $\mathcal{H}_{b,c}$ is finite.

Here « naturally generated » means that the 3-polystroma $\mathcal{H}_{b,c}$ is built up, step by step, from disjoint copies of $\{6, 3\}_{b,c}$, identifying elements only as dictated by the vertex figures.

The situation is similar in case of 3-polystromas in the families $\langle \{4, 4\}_{b,c}, \{4, 3\} \rangle$. The known cases are summarized in table 3. The 3-polystroma denoted there by $\mathcal{L}_{3,0}$ (with 30 vertices and 20 facets) was independently discovered by H. S. M. Coxeter and G. C. Shephard, who also found (in a forthcoming paper in the Canadian J. Math.) a very symmetric realization of $\mathcal{L}_{3,0}$ in the Euclidean 4-space.

At least in some cases, still another step in continuing the sequence is possible. For example

$$\langle \mathcal{H}_{2,0}, \{3, 3, 3\} \rangle$$

contains a 4-polystroma with 6 facets (for each of which the facet-figure is $\mathcal{H}_{2,0}$) and 12 vertices, while

TABLE 2

The known examples of 3-polystromas in the family $\langle \{6, 3\}_{b,c}, \{3, 3\} \rangle$

3-polystroma	Number of facets, 2-faces, edges, vertices	Type of facet	Number of 2-faces, edges, vertices of each facet	Remarks
$\mathcal{H}_{2,0}$	5, 10, 20, 10	$\{6, 3\}_{2,0}$	4, 12, 8	(a), (b)
$\mathcal{H}_{3,0}$	12, 54, 108, 54	$\{6, 3\}_{3,0}$	9, 27, 18	(a)
$\mathcal{H}_{2,1}$	8, 28, 56, 28	$\{6, 3\}_{2,1}$	7, 21, 14	(c), (d), (e)
$\mathcal{H}_{4,0}$	80, 640, 1 280, 640	$\{6, 3\}_{4,0}$	16, 48, 32	(a)
$\mathcal{H}_{4,0}/2$	40, 320, 640, 320	$\{6, 3\}_{4,0}$	16, 48, 32	(a)
$\mathcal{H}_{4,0}/4$	20, 160, 320, 160	$\{6, 3\}_{4,0}$	16, 48, 32	(a)
$\mathcal{H}_{3,1}$	28, 182, 364, 182	$\{6, 3\}_{3,1}$	13, 39, 26	
$\mathcal{H}_{3,1}/2$	14, 91, 182, 81	$\{6, 3\}_{3,1}$	13, 39, 26	
$\mathcal{H}_{2,2}$	20, 120, 240, 120	$\{6, 3\}_{2,2}$	12, 36, 24	(a)

(a) Regular 3-polystroma.
(b) The graph is bipartite, the automorphisms of $\mathcal{H}_{2,0}$ that preserve the colors form the icosahedral group $LF(2, 5)$ of order 60.
(c) If each facet is realized by a solid torus, the set of the 3-polystroma is the 3-sphere.
(d) The dual $\mathcal{H}^*_{2,1}$ of $\mathcal{H}_{2,1}$ was independently discovered by A. Altshuler (private communication).
(e) The graph is bipartite, the automorphisms of $\mathcal{H}_{2,1}$ that preserve the colors form the simple group $LF(2, 7)$ of order 168.

TABLE 3

The known examples of 3-polystromas in the family $\langle \{4, 4\}_{b,c}, \{4, 3\} \rangle$.

3-polystroma	Number of facets, 2-faces, edges, vertices	Type of facet	Number of 2-faces, edges, vertices of each facet	Remarks
$\mathcal{L}_{2,0}$	6, 12, 16, 4	$\{4, 4\}_{2,0}$	4, 8, 4	
$\mathcal{L}_{2,1}$	6, 15, 20, 5	$\{4, 4\}_{2,1}$	5, 10, 5	
$\mathcal{L}_{2,2}$	12, 48, 64, 16	$\{4, 4\}_{2,2}$	8, 16, 8	
$\mathcal{L}_{3,0}$	20, 90, 120, 30	$\{4, 4\}_{3,0}$	9, 18, 9	(a), (b), (c)
$\mathcal{L}_{3,0}/2$	10, 45, 60, 15	$\{4, 4\}_{3,0}$	9, 18, 9	(a)
$\mathcal{L}_{3,1}$	18, 90, 120, 30	$\{4, 4\}_{3,1}$	10, 20, 10	
$\mathcal{L}_{3,2}$	42, 273, 364, 91	$\{4, 4\}_{3,2}$	13, 26, 13	

(a) Regular 3-polystroma.
(b) If each facet is realized by a solid torus, the set of the 3-polystroma is the 3-sphere.
(c) Very symmetric realizations in Euclidean 4-space and 5-space were obtained by H. S. M. Coxeter and G. C. Shephard.

$\langle \mathcal{K}_{3,0}, \{3, 3, 3\} \rangle$ contains a 4-polystroma with 5 facets and 54 vertices, and another with 15 facets and 162 vertices. It is not known whether other such 4-polystromas exist, and whether any of them may serve as the only type of facet of some 5-polystroma with all vertex-figures $\{3, 3, 3, 3\}$, etc. Probably both questions have affirmative answers.

Not only toroidal maps can serve as facets for 3-polystromas. If $\mathcal{E}_1$ denotes the map on M_2, the orientable manifold of genus 2, described by Errera (1922) and Grek (1963), that belongs to $\mathcal{R} \langle \mathcal{C}_6, \mathcal{C}_4 \rangle$ (and has 6 vertices, 12 edges, and 4 hexagonal faces) then it may be verified that $\mathcal{R} \langle \mathcal{E}_1, \{4, 3\} \rangle$ contains at least two members — one a 3-polystroma with 6 facets and 12 vertices, the other with 3 facets and 6 vertices. Another example may be constructed from the map $\mathcal{E}_2 \in \mathcal{R} \langle \mathcal{C}_8, \mathcal{C}_3 \rangle$ on M_2 also described by Errera and Grek; $\mathcal{E}_2$ has 16 vertices, 24 edges, and 6 octagonal faces. $\mathcal{R} \langle \mathcal{E}_2, \{3, 3\} \rangle$ contains a 3-polystroma with 4 facets and 16 vertices, and another with 8 facets and 32 vertices. Denoting by $\mathcal{M}$ the regular map $\{8, 3\}_6$ (in Coxeter's notation) on the orientable manifold M_3 of genus 3 ($\mathcal{M}$ has 32 vertices, 48 edges and 12 octagonal faces) it may be shown that $\langle \mathcal{M}, \{3, 3\} \rangle$ contains a 3-polystroma with 8 facets and 64 vertices, and another with 16 facets and 128 vertices. It is probable that many more such examples exist.

Two other, rather remarkable, 3-polystromas $\mathcal{F}_1$ and $\mathcal{F}_2$ shall be briefly described.

The hemidodecahedron $\tilde{\mathcal{D}} = \{5, 3\}/2$ is a map on the real projective plane that realizes the member of $\mathcal{R} \langle \mathcal{C}_5, \mathcal{C}_3 \rangle$ obtained by identifying antipodal points on the Platonic dodecahedron. Denoting by $\mathcal{O} \in \mathcal{R} \langle \mathcal{C}_3, \mathcal{C}_4 \rangle$ the octahedron, the 3-polystroma $\mathcal{F}_1$ is an element of $\mathcal{R} \langle \tilde{\mathcal{D}}, \mathcal{O} \rangle$; it has 32 facets and 40 vertices. If $\tilde{\mathcal{I}} = \{3, 5\}/2 \in \mathcal{R} \langle \mathcal{C}_3, \mathcal{C}_5 \rangle$ is the hemiicosahedron, obtained by identifying the antipodal points on the regular icosahedron, then the 3-polystroma $\mathcal{F}_2$ is a member of $\mathcal{R} \langle \tilde{\mathcal{I}}, \tilde{\mathcal{D}} \rangle$, with 11 facets and 11 vertices. The graph of $\mathcal{F}_2$ is the complete graph K_{11}, its automorphism group is $LF(2, 11)$ of order 660, and $\mathcal{F}_2$ is selfdual.

In the preceding pages we could only hint at many aspects of the theory of polystromas. I hope that this was sufficient to convince the reader that members of the various families $\langle \mathcal{A}, \mathcal{B} \rangle$ and $\mathcal{R} \langle \mathcal{A}, \mathcal{B} \rangle$ exhibit remarkable properties. Further study of their combinatorial, geometric, algebraic and topological properties will certainly lead to many new insights.