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PERFECT COLORINGS OF TRANSITIVE TILINGS AND PATTERNS IN THE PLANE*

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A k-coloring of a tiling is a partition of the set of tiles into k subsets (colors). A coloring is called perfect if each symmetry of the tiling induces a permutation of the colors. The checkerboard is a familiar example of a perfect 2-coloring of the square tiling. The set of values k for which there exists a perfect k-coloring is determined for each of the three regular tilings of the plane (by squares, by regular hexagons, or by equilateral triangles). It is also shown that the set of such k is infinite for every tile-transitive tiling of the plane.

1. Introduction

If the regular tiling of the plane by squares is colored in the familiar checkerboard pattern, then it is easy to verify that it has the following property. Every symmetry s of the tiling by (uncolored) squares can be turned into a "colored symmetry" by associating with s a suitable permutation of the colors. In other words, s will map the checkerboard onto itself if, at the same time, we either interchange the two colors, or leave them unchanged. A similar situation holds in the case of the coloring of the regular tiling by hexagons with three colors shown in Fig. 1. With the symmetry which may be described as "a translation to the right by one tile" is associated the permutation (123) of the colors, with a counterclockwise rotation by 120° about the marked vertex is associated the permutation (132), and so on. It is easy to see that with *every* symmetry a permutation of colors is associated in a similar manner.

More generally, let $\mathcal{T}$ be any tiling of the plane with the property that its symmetry group is transitive on the tiles. If $\mathcal{T}$ is colored in such a way that every symmetry of $\mathcal{T}$ can be extended, in this manner, to a "colored symmetry", then the coloring is called *perfect*.

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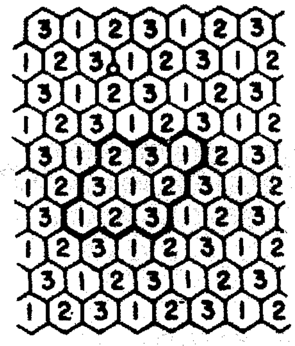


Fig. 1. A perfect 3-coloring of the regular tiling by hexagons.

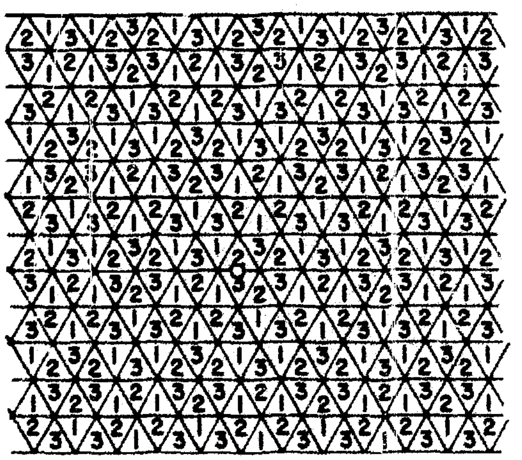


Fig. 2. A 3-coloring of the regular tiling by triangles that is very symmetric but not perfect.

Most of the well-known drawings by Escher [3, 7] that represent transitive tilings colored with $k \geqslant 2$ colors are perfectly colored. Perfect colorings can also be found in the textile ornaments of the pre-Inca cultures (see, for example [5]), and in many other art forms. This is probably a consequence of the aesthetic appeal of such colorings. Also, many of the tilings and patterns used to illustrate the crystallographic color groups [6, 9, 10, 12] are perfectly colored. On the other hand, the 3-coloring of the regular triangular tiling shown in Fig. 2 (adapted from [6]) is not perfect, since a 180° rotation about the marked midpoint of an edge may not be extended to a color symmetry by associating with it any permutation of the colors.

The aims of the present paper are as follows:

(1) to determine *all* perfect colorings of the three regular tilings of the plane, and

(2) to prove the existence of a denumerable infinity of perfect colorings for *any* transitive tiling of the plane, and, more generally, for any transitive planar pattern.

These results show that restrictions on the number of colors (as are usually imposed in the crystallographic literature; see, for example [6, 10]) are arbitrary and lead to the exclusion of many interesting examples.

In Section 2 we shall explain the necessary terminology, and formulate precisely the results to be proved. The proofs will appear in Sections 3 and 4, while Section 5 will be devoted to a number of remarks and problems related to the results of this paper.

It may be noted that the concept of "color symmetry" used in the definition of perfect colorings has already been introduced by van der Waerden and Burckhardt as early as 1961 [11, 1] but only recently (see, for example [8, 9]) has it started receiving the attention of other crystallographers and mathematicians.

2. Definitions and results

A *tiling* of the plane is a collection $\mathcal{T} = \{T_i : i \in I = \{1,2,3,\ldots\}\}$ of closed topological disks (*tiles*) which covers the Euclidean plane $\mathbf{E}^2$ and is such that the interiors of the tiles are disjoint. A tiling $\mathcal{T}$ is called *transitive* if the *symmetry group* $S(\mathcal{T})$ (that is, the group of isometries which leave $\mathcal{T}$ invariant) acts transitively on the tiles. Such tilings are sometimes called "tile-transitive" or "isohedral", but the simpler terminology is adequate here. In [4] it was shown that there exist precisely 81 *types* of transitive tilings. Two transitive tilings $\mathcal{T}_1$ and $\mathcal{T}_2$ are said to be of the same "type" if there exists a (combinatorial) isomorphism between them such that the induced one-one correspondence between the tiles commutes with every element of $S(\mathcal{T}_1) = S(\mathcal{T}_2)$. Thus two tilings of the same type are (combinatorially) indistinguishable.

A *k-coloring* of a tiling $\mathcal{T}$ is a partition of $\mathcal{T}$ into k *color-classes* $\mathcal{T}_j$, $j = 1,2,\ldots,k$, where $\mathcal{T}_j = \{T_i : i \in I_j\}$ and $I_1,\ldots,I_k$ is a partition of the index set I into non-empty sets. In accordance with the obvious interpretation, we shall say that each tile of $\mathcal{T}_j$ has *color j*.

Let $\mathcal{T}$ be a transitive tiling and $s \in S(\mathcal{T})$ be a symmetry of $\mathcal{T}$. A k-coloring of $\mathcal{T}$ is said to be *compatible* with s if s preserves the partition of $\mathcal{T}$ into the color-classes $\mathcal{T}_1,\ldots,\mathcal{T}_k$. In other words, the k-coloring is compatible if there exists a permutation σ of the colors $1,2,\ldots,k$ such that s maps each tile of color j into a tile of color σj. A k-coloring of a transitive tiling $\mathcal{T}$ is called *perfect* if it is compatible with *every* symmetry $s \in S(\mathcal{T})$.

A *pattern* in the plane consists of a *motif* M together with all the images of M under the operations of one of the plane crystallographic groups. The only restrictions we must impose are that M be a connected set, and that M and all its images be pairwise disjoint. In [4] it was proved that there are precisely 93 types of patterns in the plane, and that each type can be represented by a "marked tiling", that is to say, by a transitive tiling, each tile of which bears a "marking" or "motif". Clearly coloring, k-coloring and perfect coloring can be defined for patterns in complete analogy with the definitions for transitive tilings.

The terminology we have introduced enables us to formulate our results in a precise manner:

Theorem 2.1. *The regular square tiling of the plane admits a perfect k-coloring if and* *only if $k = n^2$ or $k = 2n^2$ for some positive integer n. The regular hexagonal tiling admits a perfect k-coloring if and only if $k = n^2$ or $k = 3n^2$. The regular triangular tiling admits a perfect k-coloring if and only if $k = 2n^2$, $k = 6n^2$, $k = (3n-2)^2$ or $k = (3n-1)^2$. In each case, for a given k, the perfect k-coloring is unique.*

Theorem 2.2. *Every transitive tiling and every pattern in the plane may be perfectly k-colored for infinitely many integers k.*

3. Proof of Theorem 2.1.

For each of the three regular tilings the arguments are similar, so we shall present the proof in detail only for the regular square tiling $\mathcal{T}$. We begin by showing that if a k-coloring of $\mathcal{T}$ is perfect then k has one of the two stated forms.

For a given perfect coloring of $\mathcal{T}$ we can see that each of the color-classes $\mathcal{T}_i$ must be congruent to every other color-class $\mathcal{T}_j$ of $\mathcal{T}$ in the sense that $\mathcal{T}_i$ can be brought into coincidence with $\mathcal{T}_j$ by a symmetry of $\mathcal{T}$. Let m be the smallest positive integer such that a horizontal translation by m squares brings a square of color 1 into coincidence with another square of color 1. By the congruence of the color-classes we can deduce the following: *every* square obtainable from a square of color j by a horizontal translation of m squares also has color j. Further, by considering rotation about the center of a tile through 90° we see that exactly the same assertion follows for vertical translations also. Thus, for each j, color j is assigned to, at least, all tiles that form a square lattice of "mesh" m.

Consider now the color-class $\mathcal{T}_1$. Two possibilities arise (see Fig. 3):

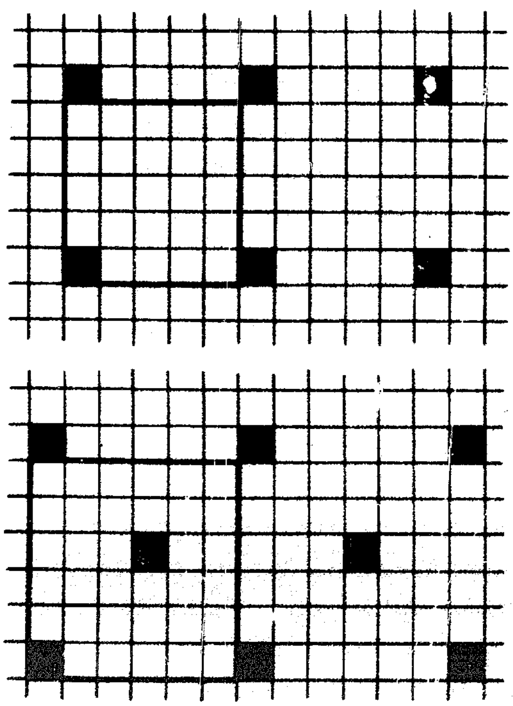


Fig. 3.

(i) either all the tiles of color 1 lie on a lattice of mesh m, or

(ii) some other tile has color 1.

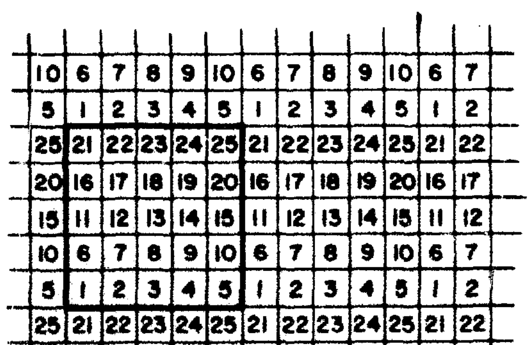
In case (i) we see that all m^2 tiles in the mesh must have different colors, and that these same colors are repeated in the same way in every mesh. Hence $k = m^2$. In Fig. 3(a), $m = 5$ and only tiles of one color are indicated.

In case (ii) consider reflections in the horizontal and vertical lines that pass through the centers of the tiles of color 1. Due to the minimality of m, the only possible positions for the extra squares of color 1 are at the centers of the meshes, and so this situation can arise only if $m = 2n$ is even. Hence each mesh of $(2n)^2 = 4n^2$ tiles contains two tiles of color 1, and the number of colors is $k = \frac{4}{2}n^2 = 2n^2$. Fig. 3(b) illustrates the case $m = 6$; again only the tiles of one color are indicated.

The conserve statements to the above also hold. If we color an m by m square patch of tiles with $k = m^2$ different colors, and cover the whole plane by translates of this patch, then the resulting k-coloring is perfect. Fig. 4(a) illustrates the case $m = 5$. Similarly, if a $2n$ by $2n$ square patch is colored by $2n^2$ colors as described above, and the plane is covered by translates of this patch, then a perfect $2n^2$-coloring is obtained. Fig. 4(b) shows the case $n = 3$.

This completes the proof for the square tiling.

(a)



(b)

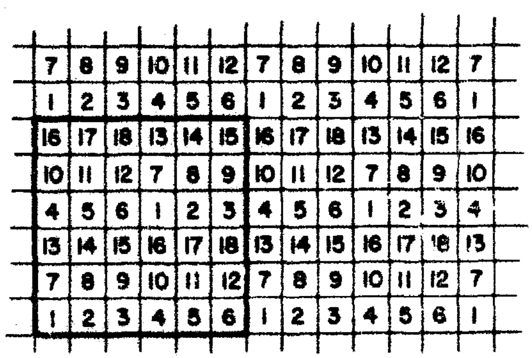
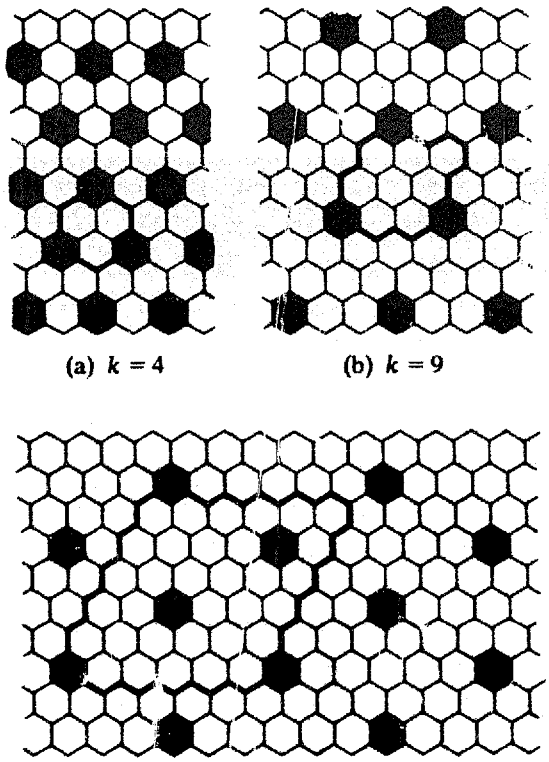


Fig. 4.

(a) A perfect 25-coloring of the regular tiling by squares.

(b) A perfect 18-coloring of the regular tiling by squares.



(a) $k = 4$ (b) $k = 9$

(c) $k = 12$

Fig. 5. Perfect k-colorings of the tiling by regular hexagons.

For the hexagonal tiling we proceed in an analogous manner. Orienting the tiling as in Fig. 5 (so that one third of the edges are vertical) we again let m be the smallest positive integer such that a horizontal translation by m tiles brings a hexagon of color 1 into coincidence with another hexagon of color 1. Then we can deduce, as before, that the tiles of color 1 form, at least, a triangular mesh of side m. A fundamental region for the corresponding group of translations is rhomb-like (indicated in Fig. 5 by thickened lines) and hence, if there are no further tiles of color 1, we have $k = m^2$ (see Figs. 5(a) and 5(b) for the cases $n = 2$ and $n = 3$). On the other hand, if another tile has color 1, then, by considering reflections in the altitudes of the triangles forming the meshes, we see that the only position it can occupy is the center of one of the "triangles" of the mesh (see Fig. 5(c)). This case only arises if $m = 3n$ for some positive integer n, and then $k = 3n^2$. For example, Fig. 1 represents the case $n = 1$, while Fig. 5(c) illustrates the arrangement of tiles for the case $n = 2$. In Fig. 5 we have indicated only tiles of one color, but it is evident how, in each case, the tiling can be completed in a perfect manner by k colors when k has the stated forms.

For the tiling by equilateral triangles the situation is slightly more complicated. Considering only tiles of one particular *aspect* (that is, translations of one another) we again define m to be the smallest positive integer such that a horizontal translation through a distance mb (where b is the length of the side of a tile) brings a triangle of color 1 into coincidence with another tile of color 1. Again we deduce that the tiles of one aspect of color 1 form, at least, a triangular mesh of side m. A fundamental region for the corresponding group of translations is a rhomb (indicated in Fig. 6 by thickened lines) and hence, if there are no further tiles of color 1, we have $k = 2m^2$ (see Figs. 6(a) and 6(b) for the cases $m = 2$ and $m = 3$). If there is another tile of color 1 then the only position it can occupy is the center of one of the triangles of the mesh. Three cases arise:

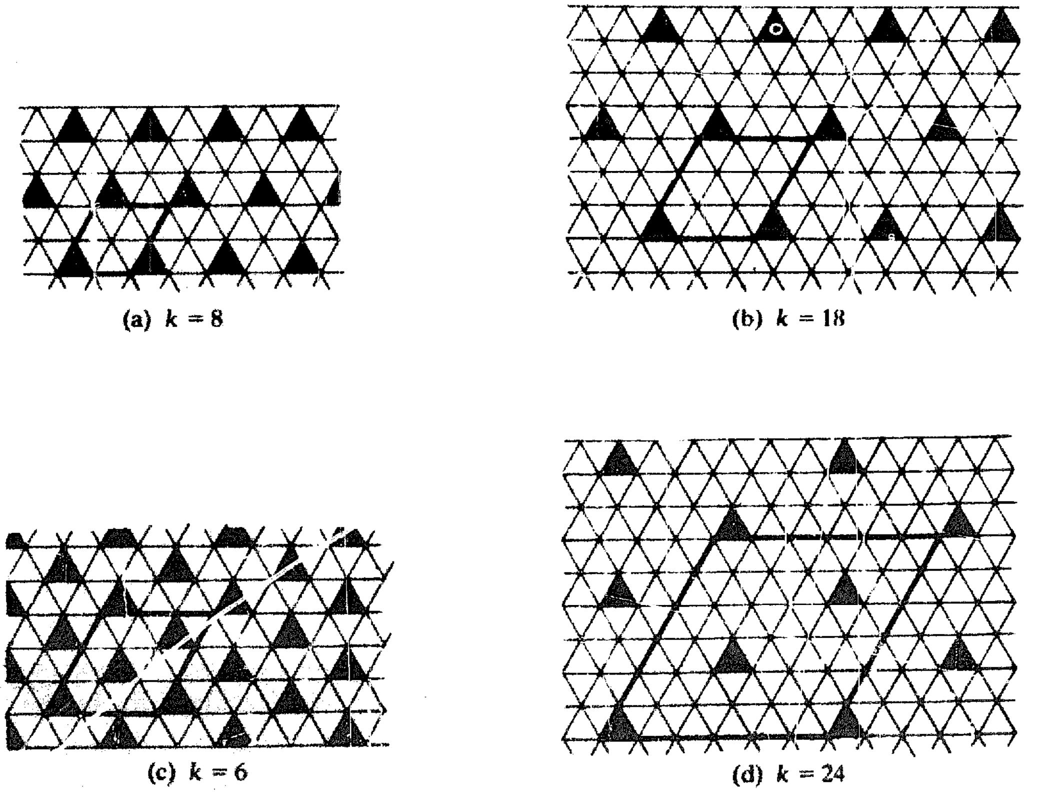
(i) If $m = 3n$ then the triangle in the left half of the rhomb has a central tile of the same aspect as the tiles forming the mesh. Hence we have $k = 6n^2$ (see Figs. 6(c) and 6(d) for the cases $n = 1$ and $n = 2$).

(ii) If $m = 3n - 2$ then the triangle in the left half of the *rhomb* has a central tile of the opposite aspect to the tiles forming the mesh. Hence $k = (3n - 2)^2$ (see Figs. 6(e) and 6(f) for the cases $n = 2$ and $n = 3$).

(iii) Finally, if $m = 3n - 1$, then the triangle in the right half of the rhomb has a central tile of the opposite aspect to the tiles forming the mesh. Hence $k = (3n - 1)^2$ (see Figs. 6(g) and 6(h) for the cases $n = 1$ and $n = 2$).

In Fig. 6 we have indicated only the tiles of one color, but it is evident how, in each case, the coloring can be completed in a perfect manner by k colors, where k has the stated form.

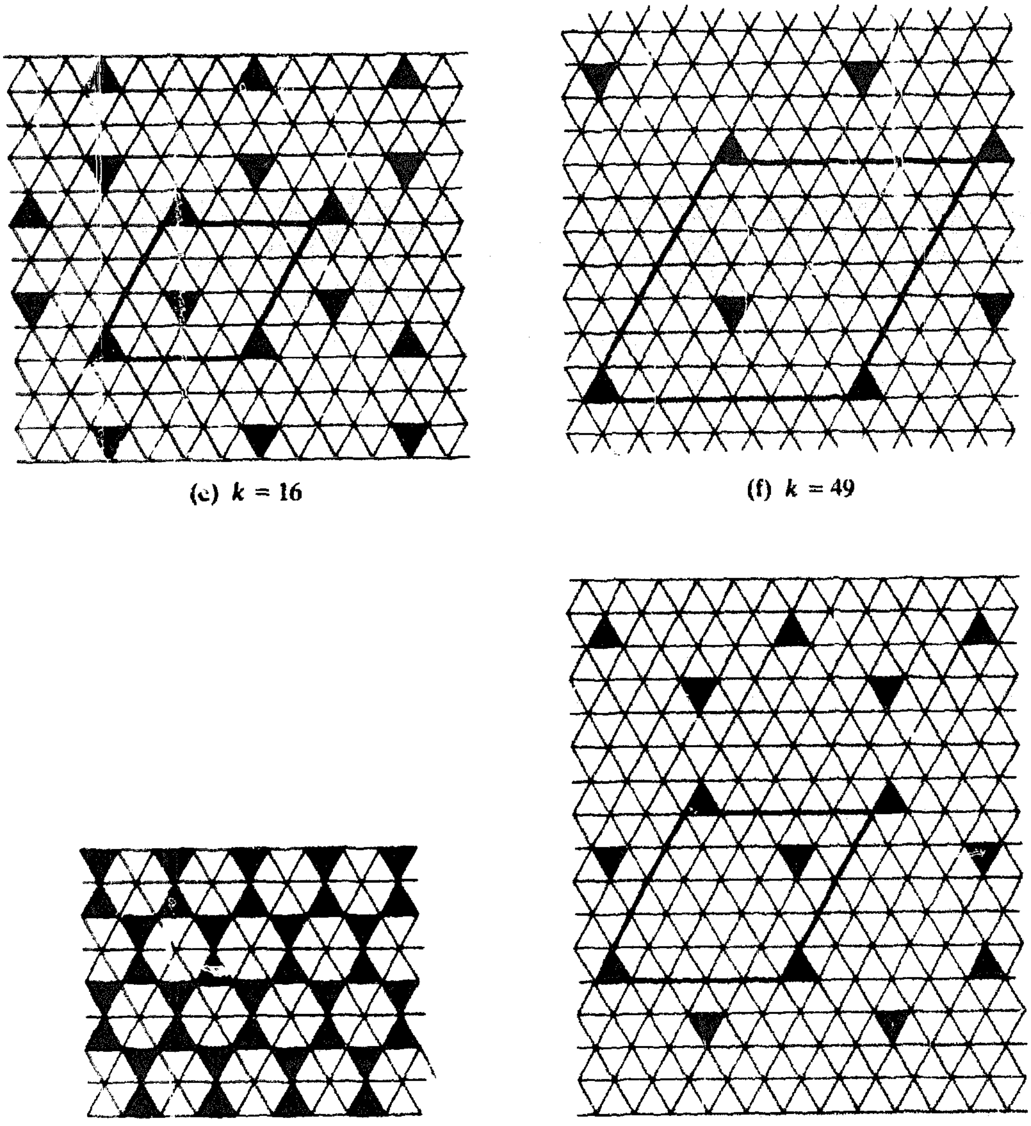
This concludes the proof for the triangular tiling, and also the proof of Theorem 2.1.



(a) $k = 8$ (b) $k = 18$

(c) $k = 6$ (d) $k = 24$

Fig. 6. Perfect k-colorings of the regular tiling by triangles.



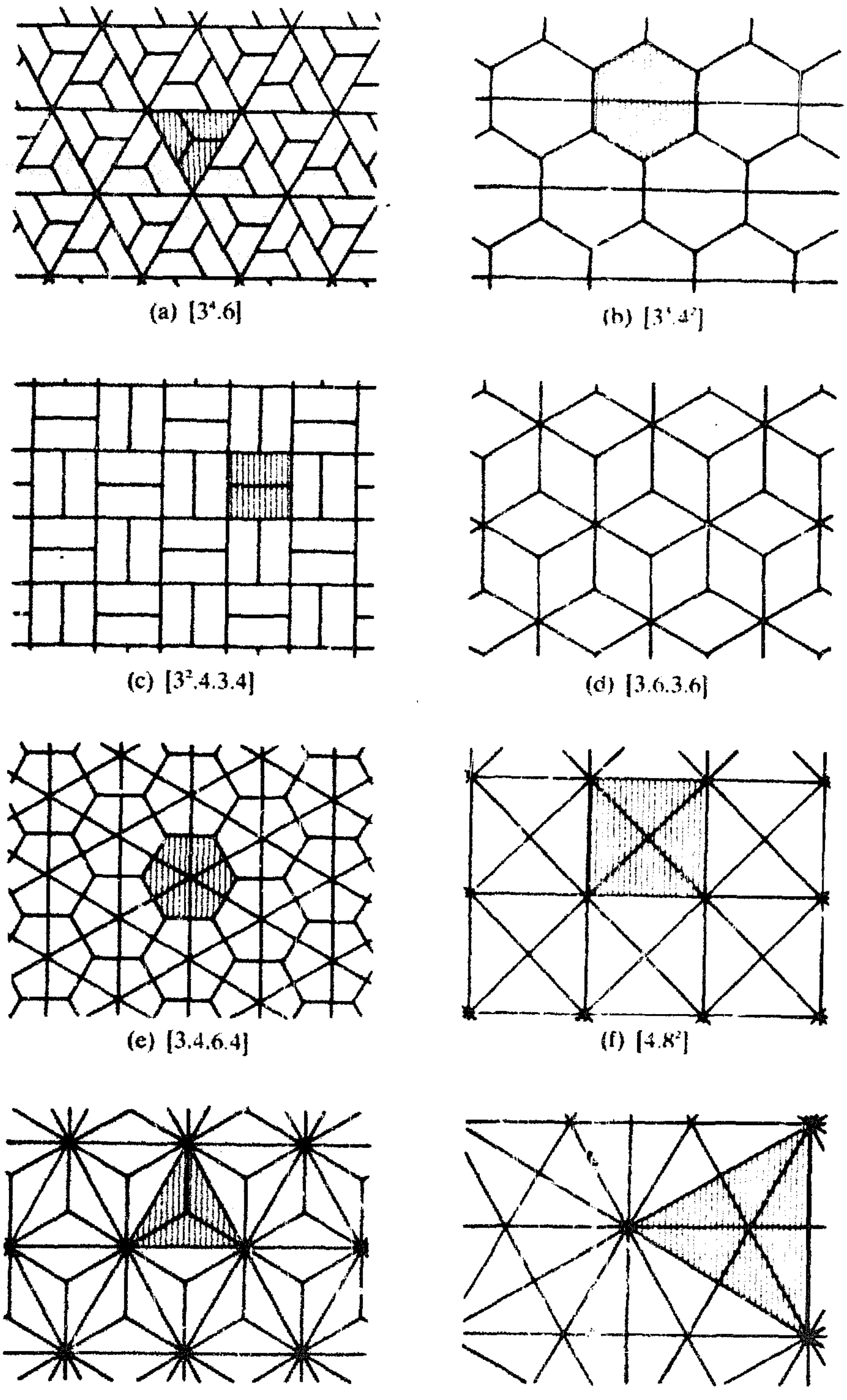
(e) $k = 16$ (f) $k = 49$

(g) $k = 4$ (h) $k = 25$

Fig. 6. Continued.

4. Proof of Theorem 2.2

It was shown in [4] that the combinatorial structure of each type of transitive tiling or pattern in the plane could be represented by taking 11 tilings (which we shall call the *semiregular* tilings), one corresponding to each of the Laves nets, and then marking the tiles with a suitable motif. This shows that if $\mathcal{T}$ is a semiregular tiling, then any other transitive tiling $\mathcal{T}'$ with the same net as $\mathcal{T}$, must have a symmetry group $S(\mathcal{T}')$ which is a subgroup of $S(\mathcal{T})$. This, in turn, implies that any perfect coloring of $\mathcal{T}$ is necessarily also a perfect coloring of $\mathcal{T}'$. Hence for a proof of Theorem 2.2 it suffices to produce an infinite number of perfect colorings for each of the semiregular tilings.



(a) $[3^4.6]$ (b) $[3^3.4^2]$ (c) $[3^2.4.3.4]$ (d) $[3.6.3.6]$ (e) $[3.4.6.4]$ (f) $[4.8^2]$ (g) $[3.12^2]$ (h) $[4.6.12]$

Fig. 7. The eight semiregular tilings that are not regular.

Three of the semiregular tilings are regular (with the Laves nets $[4^4]$, $[3^6]$, and $[6^3]$, respectively) and the existence of infinitely many perfect colorings of these has already been established in Theorem 2.1. The remaining eight semiregular tilings are shown in Fig. 7. It will be noticed that some of these have been taken in a slightly different form from those illustrated in [4]. In fact, there is considerable arbitrariness in the shapes of the various tiles, and the reason for choosing the particular representatives of Fig. 7 will become apparent in the following discussion.

Let us consider, to begin with, the tiling with Laves net $[3^4.6]$, see Fig. 7(a). The unions of sets of three tiles (one such triple is shaded in the figure) clearly leads to the regular tilings by equilateral triangles. Hence a perfect coloring of this net can be obtained by taking any perfect coloring of the triangular tiling and either assigning the same color to each of the three tiles that form a triangle, or giving them, systematically, three distinct colors. We deduce from Theorem 2.1 that this construction will lead to perfect colorings of any tiling with net $[3^4.6]$ by $2n^2$, $6n^2$, $(3n-2)^2$, $(3n-1)^2$, $3(3n-2)^2$ or $3(3n-1)^2$ colors. We are not asserting, of course, that perfect colorings by other numbers of colors are impossible.

The same technique can be applied to six of the other seven nets. For $[3.12^2]$ and $[4.6.12]$ we consider unions of three or six tiles, as illustrated in Figs. 7(g) and 7(h), and deduce in this way perfect colorings from those of the regular tiling by triangles. For $[3^2.4.3.4]$ and $[4.8^2]$ we consider unions of two or four tiles as in Figs. 7(c) and 7(f), and perfect colorings can then be deduced from those of the regular tiling by squares. For $[3^3.4^2]$ and $[3.4.6.4]$ we consider unions of two, or six, tiles, as in Figs. 7(b) and 7(e), and perfect colorings can then be deduced from those of the regular tiling by hexagons.

This leaves only the net $[3.6.3.6]$ to be considered. Here a simple construction (for which we are indebted to Stephen Wilson) shows that perfect colorings by $k = 3n$ colors are possible, for all positive integers n. The rhombs of the semiregular tiling lie in three different aspects, and all the tiles of one aspect lie in a sequence of parallel rows touching vertex to vertex. Assigning the same color to all rhombs in a row, and using n colors periodically repeating in each of the three directions of rows, leads to a coloring by $3n$ colors. It is easy to verify that each such coloring is perfect.

This concludes the proof of Theorem 2.2. It is interesting to note that in this last case the number of colors is proportional to n, whereas in the other seven cases it is proportional to n^2. Whether this shows that the colorability of tilings with net $[3.6.3.6]$ is rather special, or whether it is just a consequence of the constructions we have used, is an open problem.

5. Remarks and open problems

It would be of interest to determine, for each of the 93 types of transitive tiling or pattern in the plane, the set of all k for which a perfect k-coloring of the tiling is possible. In Theorem 2.1 we have done this for the three regular tilings, and for some other types it is also easily possible. For example, tilings of type IH 62, according to the classification of [4] (see Fig. 8), have a perfect k-coloring if and

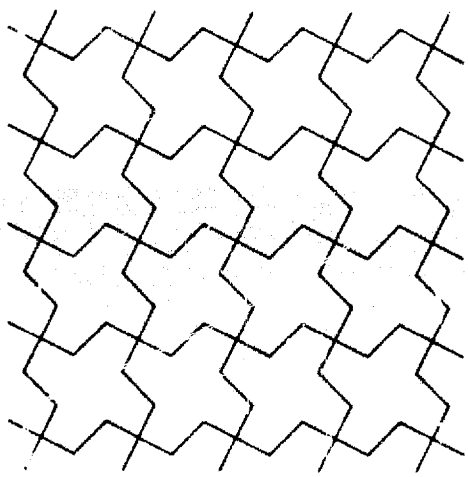


Fig. 8. A transitive tiling of type IH 62.

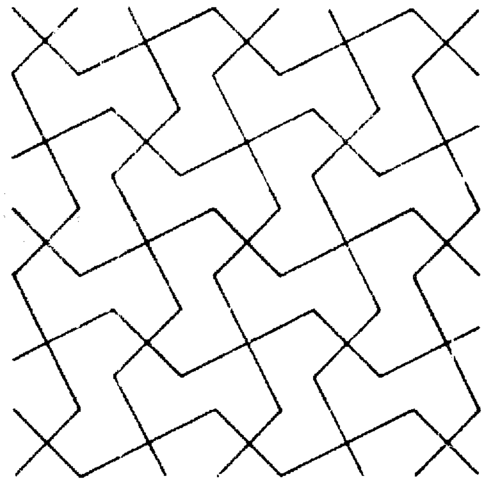


Fig. 9. A transitive tiling of type IH 61.

only if $k = a^2 + b^2$, where a and b are integers satisfying $a > 0$ and $a \geqslant b \geqslant 0$. The coloring is uniquely determined by a and b if $b = 0$ or if $b = a$, but there are two (enantiomorphic) colorings in all other cases. On the other hand, tilings of type IH 61 (see Fig. 9) admit perfect k-colorings for all even values of k as well as for k of the form $(2n + 1)^2$ (and possibly for some other values also). Tilings of types IH 61 and IH 62 have the same crystallographic symmetry group p4, and the same net $[4^4]$. We deduce from these examples that the set of values of k for which perfect k-colorings exist depends on the actual type of tilings, as defined in [4], and not only on its symmetry group and net.

In Fig. 10 we show a tiling of type IH 28 by Escher [7]. This type of tiling also admits perfect colorings by $k = a^2 + b^2$ colors, and in Fig. 10 we have indicated a perfect 5-coloring. In [7] Escher shows a perfect 4-coloring of the same tiling.

It is obvious that for a given tiling $\mathcal{T}$ and a fixed k the number of perfect k-colorings of $\mathcal{T}$ is finite. The remark made above concerning tilings of type IH 62 shows that for some k certain tilings may have distinct perfect k-colorings. It is not known what types of tilings or patterns share with the regular tilings the property of having at most one perfect k-coloring for each k.

Every perfect coloring of a regular tiling $\mathcal{T}$ leads to a *regular map* $\mathcal{M}$ on the torus (that is, a map which is regular and reflexive in the terminology of [2]). $\mathcal{T}$ serves as a universal covering surface of $\mathcal{M}$, and each face of $\mathcal{M}$ corresponds to a color in $\mathcal{T}$ (except in the third and fourth kinds of perfect colorings of the tiling by triangles, where each color in $\mathcal{T}$ corresponds to a pair of "antipodal" faces in $\mathcal{M}$). It would be interesting to investigate the relations between possible types of transitive tilings of the torus, and perfect colorings of transitive tilings of the plane.

Another unexplored direction is the investigation of the analogues of our results for perfect colorings of transitive tilings of 3- or higher-dimensional spaces, or of maps on compact 2-manifolds. Similar questions may be raised also for tilings in which the tiles form $2, 3\ldots$ transitivity classes with respect to the group of

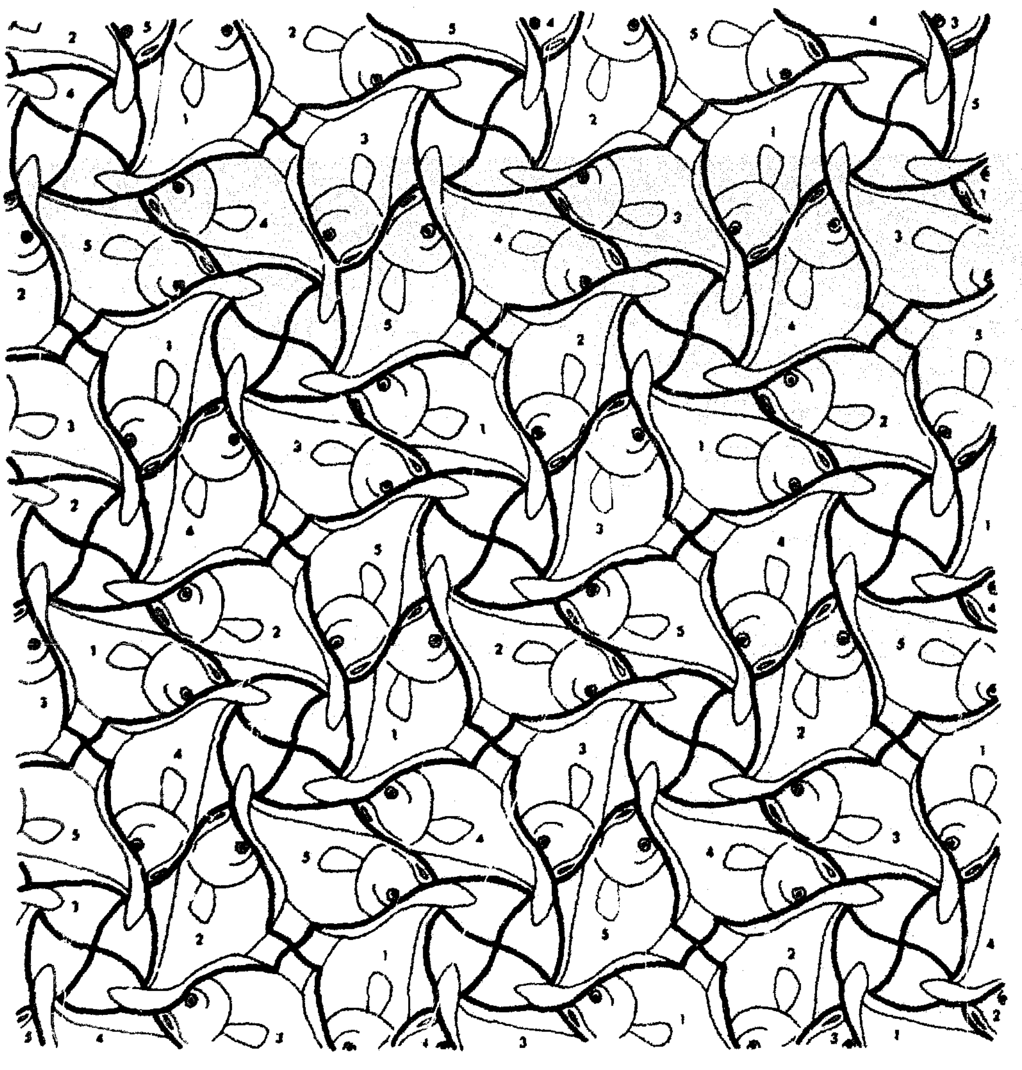


Fig. 10. A perfect 5-coloring of a transitive tiling of M.C. Escher.

symmetries of the tiling. In such variants it would probably be advantageous to strengthen the definition of perfect coloring by the requirement that all color-classes be mutually congruent.

References

[1] J.J. Burckhardt, Die Bewegungsgruppen der Kristallographie (Birkhäuser, Basel, 1966).

[2] H.S.M. Coxeter and W.O.J. Moser, Generators and Relations for Discrete Groups, 3rd ed. (Springer, Berlin, 1972).

[3] M.C. Escher, The Graphic Work of M.C. Escher (Hawthorne, New York, 1960); The World of M.C. Escher (Abrams, New York, 1971).

[4] B. Grünbaum and G.C. Shephard, The eighty-one types of isohedral tilings of the plane, Math. Proc. Cambridge Philos. Soc. 82 (1977) 177–196.

[5] S. Izumi, The Costume and Textile Ornament of the Pre-Inca Cultures (San-Ichi Shobo, Tokyo, 1964).

[6] A.L. Loeb, Color and Symmetry (Wiley-Interscience, New York, 1971).

[7] C.H. Macgillavry, Symmetry Aspects of M.C. Escher's Periodic Drawings (Oosthoek, Utrecht, 1965).

[8] M. Senechal, Point groups and color symmetry, Z. Kristallographie 142 (1975) 1–23.

[9] A.V. Shubnikov and V.A. Koptsik, Symmetry in Science and Art (Plenum Press, New York, 1974).

[10] A.V. Shubnikov, N.V. Belov and others, Colored Symmetry (Pergamon, Oxford, 1964).

[11] B.L. van der Waerden and J.J. Burckhardt, Farbgruppen, Z. Kristallographie 115 (1961) 231–234.

[12] L. Weber, Die Symmetrie homogener ebener Punktsysteme. Z. Kristallographie 70 (1929) 309–327.