Tilings by Regular Polygons

Patterns in the plane from Kepler to the present, including recent results and unsolved problems.

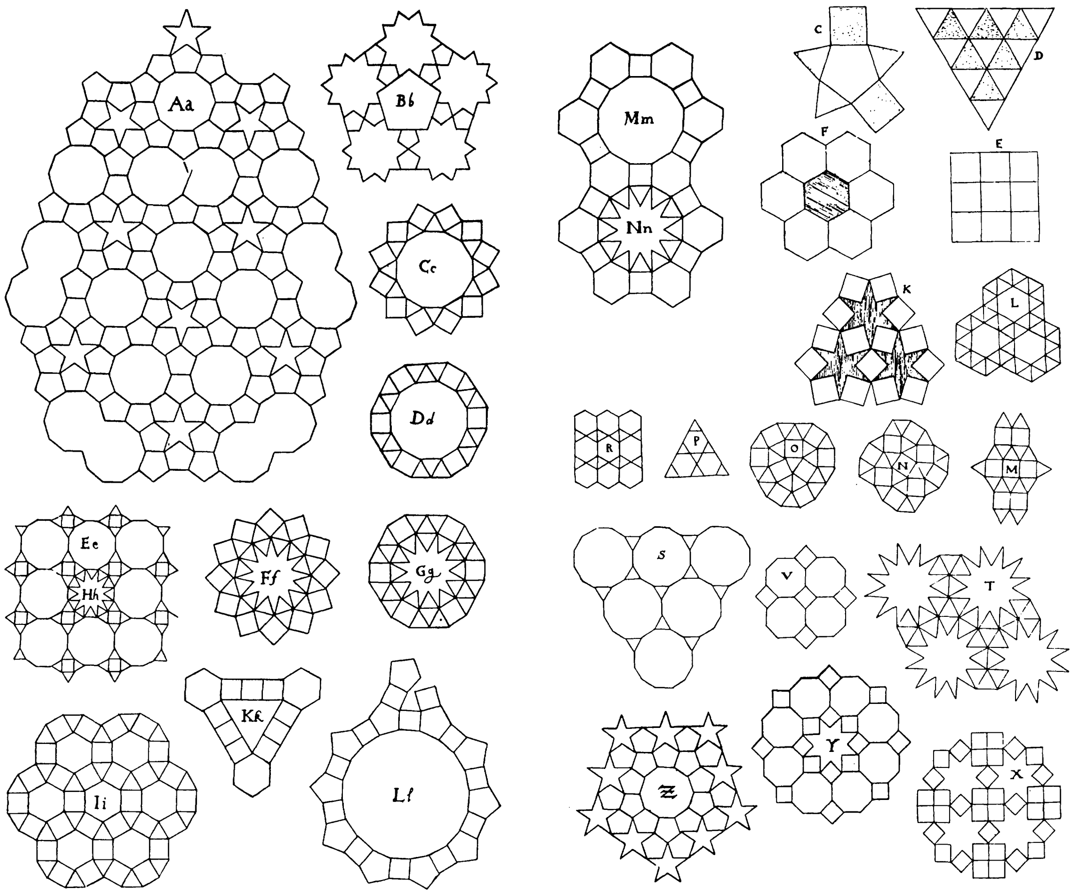
BRANKO GRÜNBAUM
University of Washington
Seattle, WA 98195

GEOFFREY C. SHEPHARD
University of British Columbia
Vancouver, British Columbia, V6T 1W5

A **tiling** of the plane is a family of sets — called **tiles** — that cover the plane without gaps or overlaps. ("Without overlaps" means that the intersection of any two of the sets has measure (area) zero.) Tilings are also known as **tessellations, pavings,** or **mosaics;** they have appeared in human activities since prehistoric times. Their mathematical theory is mostly elementary, but nevertheless it contains a rich supply of interesting and sometimes surprising facts as well as many challenging problems at various levels. The same is true for the special class of tilings that will be discussed here — more or less regular tilings by regular polygons. These types were chosen because they are accessible without any need for lengthy introductions, and also because they were the first to be the subject of mathematical research. The pioneering investigation was done by Johannes Kepler, more than three and a half centuries ago. Additional historical data will be given later (in Section 6) but as an introduction we reproduce in FIGURE 1 certain drawings from Kepler [1619]. We shall see that these drawings contain (at least in embryonic form) many aspects of tilings by regular polygons which even at present are not completely developed.

As is the case with many other notions, the concept of "more or less regular" tilings by regular polygons developed through the centuries in response to the interests of various investigators; it is still changing, and no single point of view can claim absolute superiority over all others. Our presentation reflects our preferences, although many other definitions and directions are possible; some of these will be briefly indicated in Sections 4, 5 and 7. For most of our assertions we provide only hints which we hope will be sufficient for interested readers to construct complete proofs.

Initially we shall use only regular convex polygons as tiles: if such a polygon has n edges (or sides) we shall call it an ***n*-gon,** and use for it the symbol $\{n\}$. Thus $\{3\}$ denotes an equilateral triangle, while $\{4\}$, $\{5\}$, $\{6\}$ denote a square, a (regular) pentagon, and a (regular) hexagon, respectively. All the polygons are understood to be closed sets, that is, to include their edges and vertices.



Various more or less regular tilings of the plane by regular polygons, reproduced from J. Kepler's book "Harmonices Mundi", published in 1619.

FIGURE 1

Except in Section 4 we shall restrict attention to tilings that are **edge-to-edge;** by this we mean that as far as the mutual relation of any two tiles is concerned there are just three possibilities:

(i) they are disjoint (have no point in common);

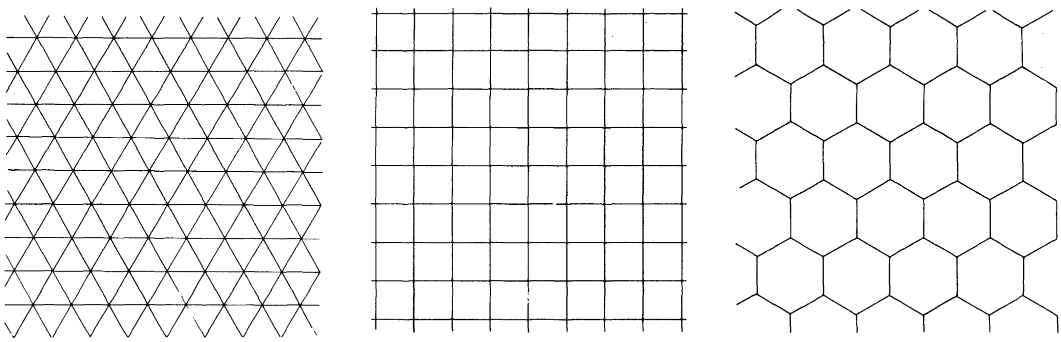
(ii) they have precisely one common point which is a vertex of each of the polygons; or

(iii) they share a segment that is an edge of each of the two polygons.

Hence a point of the plane that is a vertex of one of the polygons in an edge-to-edge tiling is also a vertex of every other polygon to which it belongs; we shall say that it is a **vertex of the tiling.** Similarly, each edge of one of the polygons is an edge of precisely one other polygon and we call it an **edge of the tiling.**

1. Regular and uniform tilings

The question about the possibilities of tiling the plane by (congruent) copies of a single regular polygon has the following simple and rather obvious answer, the origin of which is lost in antiquity. *The only possible edge-to-edge tilings of the plane by mutually congruent regular convex polygons are the three regular tilings by equilateral triangles, by squares, or by regular hexagons.* A portion of each of these three tilings is illustrated in FIGURE 2.



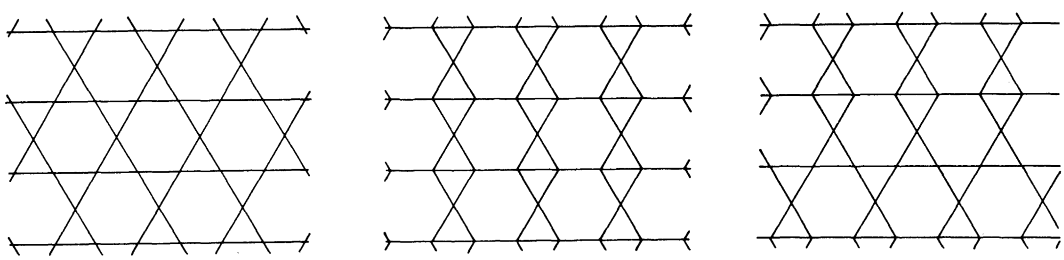
(3^6) (4^4) (6^3)

The three regular tilings of the plane.

FIGURE 2

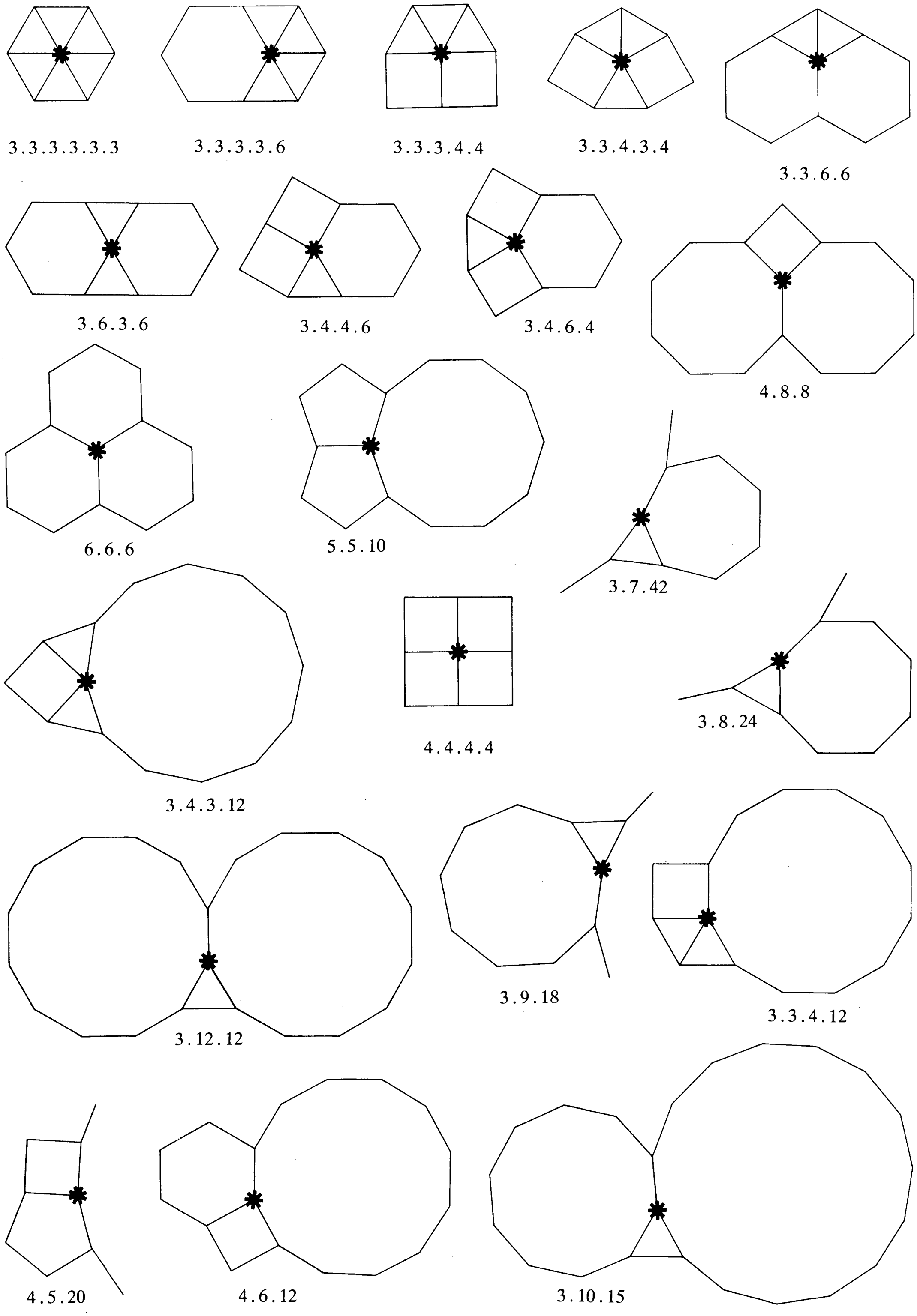
If we inquire about the possibility of edge-to-edge tilings of the plane that use as tiles regular polygons of several kinds, then the situation immediately becomes much more interesting. The angle at each vertex of $\{n\}$ is $(n-2)\pi/n$ so it is easy to check by simple arithmetic that only 17 choices of polygons can be fitted around a single vertex so as to cover a neighborhood of the vertex without gaps or overlaps. We call each such choice the **species** of the vertex, and list in TABLE 1 the 17 possible species. In four of the species there are two distinct ways in which the polygons in question may be arranged around a vertex; the mere reversal of cyclic order is not counted as distinct. Hence there are 21 possible **types** of vertices; they too are listed in TABLE 1 and also illustrated in FIGURE 3. We denote the type of a vertex around which there are, in cyclic order, an a-gon $\{a\}$, a b-gon $\{b\}$, a c-gon $\{c\}$, etc., by $a.b.c.\ldots\ldots$ Thus the three regular tilings have vertices of types 3.3.3.3.3.3, 4.4.4.4, and 6.6.6. For brevity we shall write these symbols as 3^6, 4^4 and 6^3, and we shall use similar abbreviations in other cases. In order to obtain a unique symbol for each type of vertex we shall always choose that which is lexicographically first among all possible expressions.

Contrary to frequently made assertions (see Section 6), if we require of an edge-to-edge tiling only that it be composed of regular polygons and that all its vertices be of the same species, then there are infinitely many distinct types of tilings. For example (see FIGURE 4), if at each vertex there are two triangles and two hexagons, it is possible to place each "horizontal" strip in two non-equivalent



By sliding horizontal strips independently of each other, an uncountable infinity of distinct tilings may be obtained, all vertices of which are of species 5.

FIGURE 4



The 21 possible types of vertices.

FIGURE 3

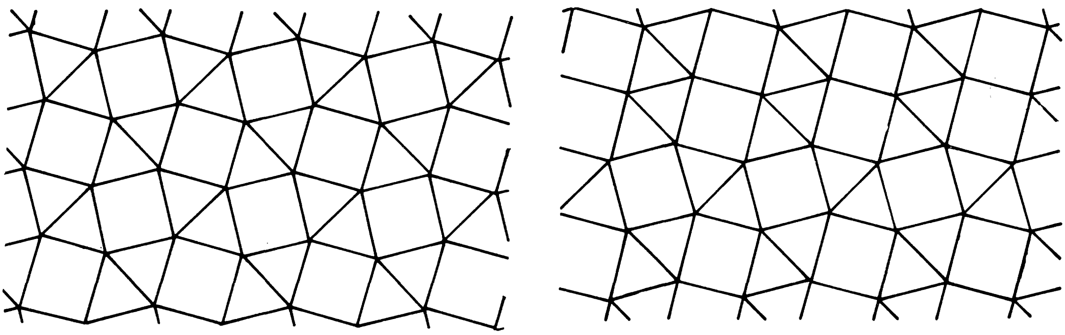
positions. Since there are infinitely many such strips, there will be uncountably many distinct tilings. The situation in Figure 5 in which 3 triangles meet 2 squares at each vertex is similar. In Figure 6 we allow three kinds of polygons; this permits each "disc" of an infinite family to be put in two positions, again leading to uncountably many tilings, with each vertex of species 6.

In view of the above remarks it is reasonable to restrict attention to tilings in which only a single **type** of vertex is allowed. If that type is $a.b.c.\ldots$, we shall denote the tiling by $(a.b.c.\ldots)$, using superscripts to shorten the expression when possible. This restriction indeed changes the situation completely and we have the following result: *There exist precisely* 11 *distinct types of edge-to-edge tilings by regular polygons such that all vertices of the tiling are of the same type.* These 11 types of

Species number	$n=3$	4	5	6	7	8	9	10	12	15	18	20	24	42	Type of vertex	Type of tiling
1	6														3.3.3.3.3.3	*A*
2	4			1											3.3.3.3.6	*A*
3	3	2													3.3.3.4.4 3.3.4.3.4	*A* *A*
4	2	1							1						3.3.4.12 3.4.3.12	
5	2			2											3.3.6.6 3.6.3.6	 *A*
6	1	2		1											3.4.4.6 3.4.6.4	 *A*
7	1				1									1	3.7.42	
8	1					1							1		3.8.24	
9	1						1				1				3.9.18	
10	1							1		1					3.10.15	
11	1								2						3.12.12	*A*
12		4													4.4.4.4	*A*
13		1	1									1			4.5.20	
14		1		1					1						4.6.12	*A*
15		1				2									4.8.8	*A*
16			2					1							5.5.10	
17				3											6.6.6	*A*

POSSIBLE SPECIES AND TYPES of vertices for edge-to-edge tilings by regular polygons. Entries in the table indicate the number of *n*-gons that meet at a vertex. Types that lead to Archimedean tilings are labelled with an "A" in the final column.

Table 1

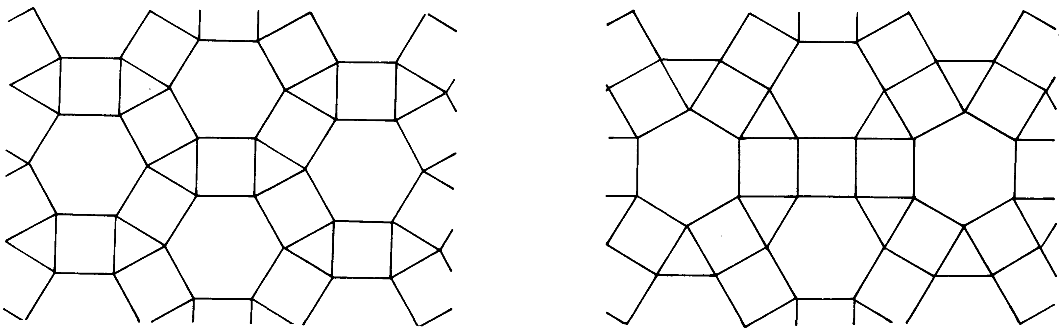


Infinitely many distinct tilings that have only vertices of species 3 may be obtained by changing the relative positions of horizontal zigzag strips in the tiling at the left.

FIGURE 5

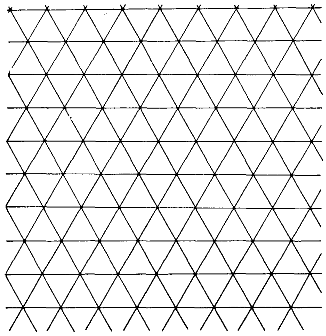
tilings, illustrated in FIGURE 7, are usually called **Archimedean** tilings (although some authors call them homogeneous, or semiregular, or uniform). They clearly include the three types of regular tilings.

Two not entirely trivial steps are required in order to prove that there are precisely 11 types of Archimedean tilings. In the first place, it must be shown that for ten of the 21 types of vertices listed in TABLE 1 it is not possible to extend a tiling from the neighborhood of a starting vertex to an Archimedean tiling of the whole plane. In fact, in each case one has to go only around one of the n-gons with odd n to show the impossibility. (For each of the six species numbered 7, 8, 9, 10 13 and 16 there is no edge-to-edge tiling of the plane by regular polygons that includes even a single vertex of the species.) In the second place, it must be established that the remaining 11 types of vertices do actually lead to Archimedean tilings. This may be deemed obvious and trivial in view of FIGURE 7, but it is just this "obviousness" that is dangerous. In FIGURE 8, adapted from a children's coloring book, we show a tiling that appears to consist of regular n-gons with $n = 4, 5, 6, 7, 8$. Actually, this visual "proof" is a fraud, since it is easy to check that the polygons in such a tiling cannot be regular. Thus there is a real need to show that the 11 Archimedean tilings do exist. It is easy to give direct proofs of existence for (4^4) and for (3^6) by considering two or three suitable families of equidistant parallel lines. The existence of the other Archimedean tilings can be deduced (with just a little thought) from these two.

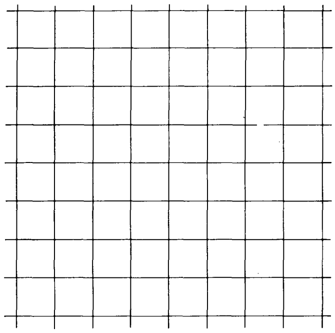


By turning "discs" in the tiling at the left infinitely many different tilings with all vertices of species 6 may be obtained.

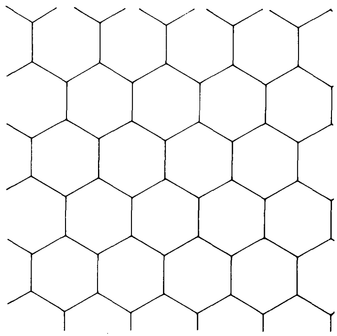
FIGURE 6



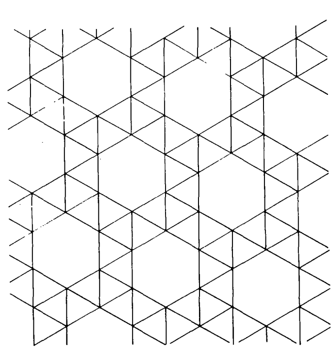
(3^6)



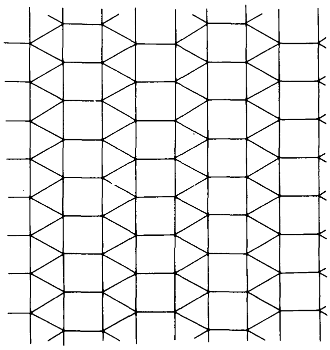
(4^4)



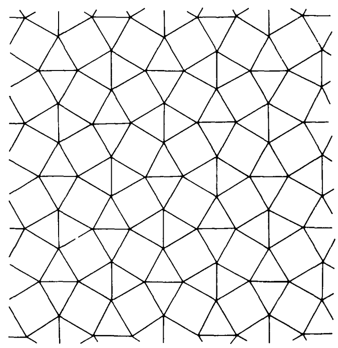
(6^3)



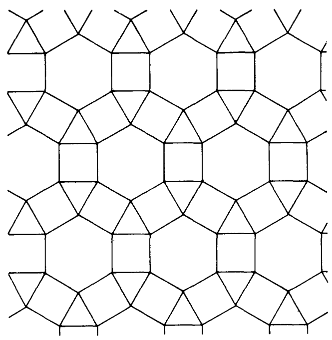
$(3^4.6)$



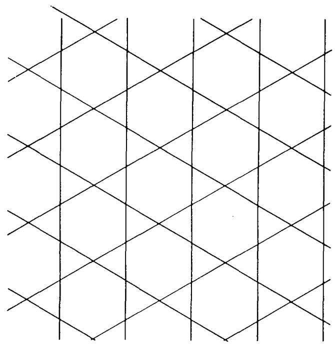
$(3^3.4^2)$



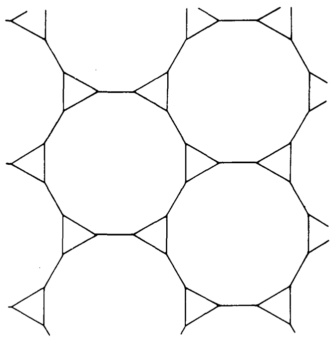
$(3^2.4.3.4)$



(3.4.6.4)



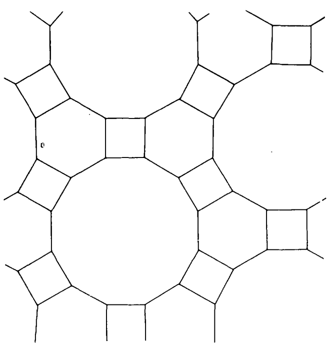
(3.6.3.6)



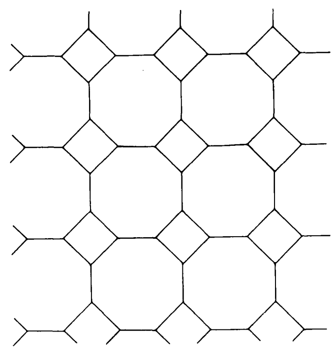
(3.12^2)

The 11 distinct types of Archimedean tilings of the plane. The tiling of type $(3^4.6)$ exists in two mirror-symmetric (enantiomorphic) forms.

FIGURE 7



(4.6.12)



(4.8^2)

It should be noted that *a priori* it is not obvious that the limitation to a single type of vertex should lead to a single type of tiling. It happens to turn out that way, but just barely so: it is only because we find it convenient not to distinguish between tilings that are congruent but not directly congruent. Indeed, the tilings of type $(3^4.6)$ are of two mirror-symmetric (enantiomorphic) forms that are counted as distinct by some authors.

Another accidental but very important feature of the Archimedean tilings is the fact that each is **vertex-transitive.** By this we mean that all vertices are equivalent under the symmetries of the tiling. Put more simply, for each pair of vertices A and B it is possible to find a motion of the plane, or a motion combined with a reflection in a line, that carries the tiling onto itself and maps A onto B. A verification of the vertex-transitivity of the Archimedean tilings is a very useful exercise. A psychologically very convincing (although logically not completely conclusive) verification of the transitivity may be obtained by tracing the tiling on a transparent sheet that may be moved over the original, and turned over. (Note that a tiling may be vertex-transitive even if its tiles are not regular polygons. Some examples of such tilings will be found in FIGURES 14 and 16.) In view of the transitivity of Archimedean tilings we shall from now on also call them **uniform** tilings. The distinction between the two words is that "Archimedean" refers only to the fact that the immediate neighborhoods of any two vertices "look the same", while the term "uniform" implies the much stronger property of equivalence of vertices under symmetries of the whole tiling.

Returning to the question of tiling with a single species of vertex we mention without proof that non-uniform tilings are possible only in case of species 3, 5 and 6. In the last two of those cases *all* tilings can be obtained from the uniform ones, (3.6.3.6) and (3.4.6.4), by the method explained above. However, in case of species 3 there are other possibilities as well and a complete description of all such tilings is still not known.

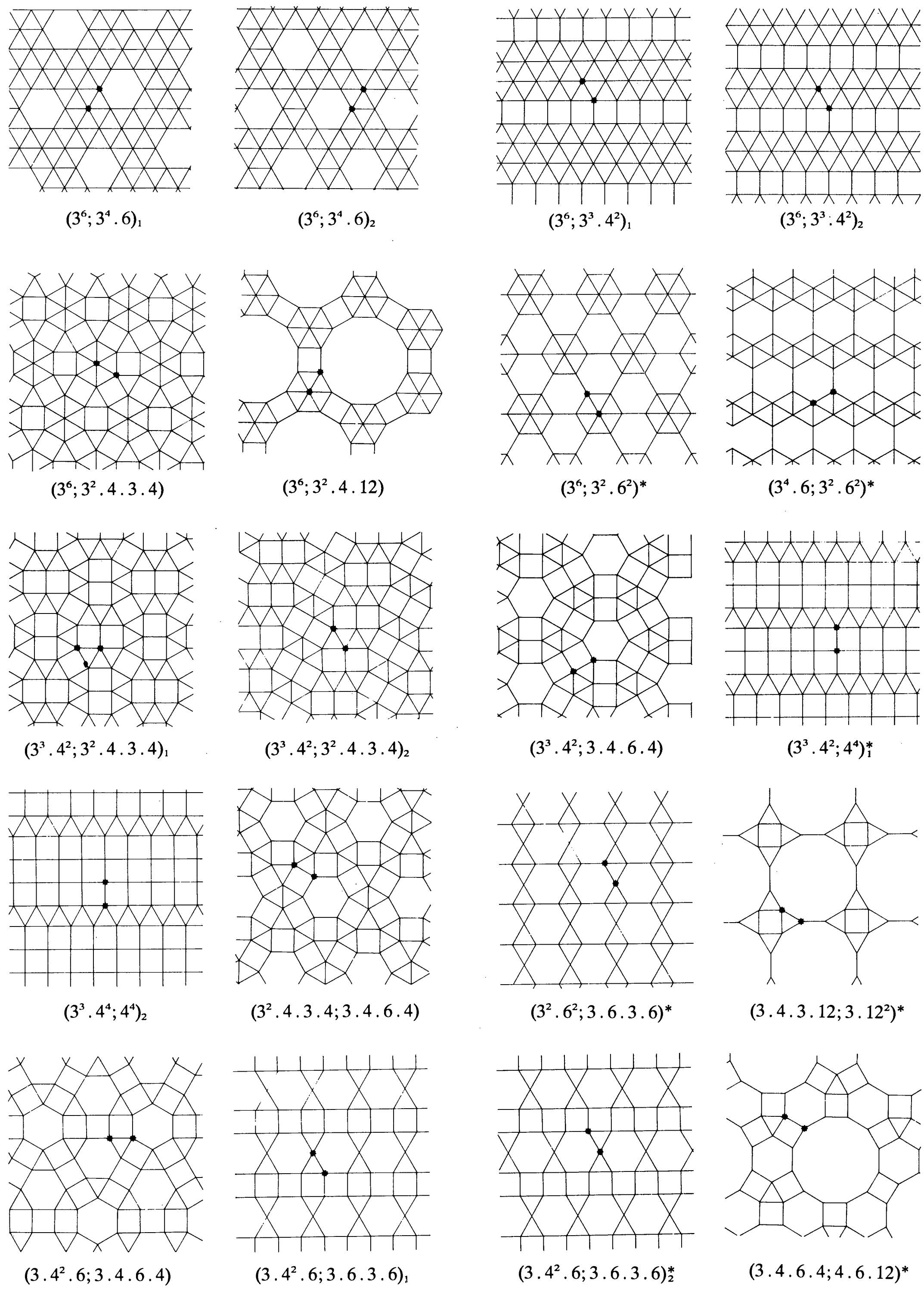
In the same vein, it may easily be verified that the three regular tilings have the following strong transitivity property. If a triplet consisting of a polygon, one of its edges, and a vertex of that edge is called a **flag,** then any two flags of a regular tiling are equivalent under the symmetries of the tiling. From now on, the "regularity" of "regular tilings" will always be understood in this sense, which is becoming more widespread in many related areas; see, for example, Coxeter [1975], Grünbaum [1976]. We should stress that flag-transitivity is more restrictive than requiring that a tiling be vertex-, edge- and tile-transitive: there is exactly one tiling by polygons (FIGURE 16a) which has the latter three kinds of transitivity, but which fails to be regular.

2. *k*-uniform tilings

The observation that the Archimedean tilings are uniform suggests the following possibility of generalization. A tiling is called ***k*-uniform** if its vertices form precisely k transitivity classes with respect to the group of all symmetries of the tiling. Clearly, uniform tilings coincide with 1-uniform tilings. If the types of vertices in the k classes are $a_1.b_1.c_1.\dots; a_2.b_2.c_2.\dots;\dots; a_k.b_k.c_k.\dots$, we will designate the tiling by the symbol $(a_1.b_1.c_1.\dots; a_2.b_2.c_2.\dots;\dots; a_k.b_k.c_k.\dots)$, with the obvious shortening through the use of superscripts, and with subscripts to distinguish tilings in which the same types of vertices appear. *There exist* 20 *distinct types of* 2*-uniform edge-to-edge tilings by regular polygons.* They are shown in FIGURE 9. The proof of this fact may be carried out along lines analogous to those explained in connection with the 11 uniform tilings. However, the details are here much more intricate; it appears that the only published version of the proof is found in the paper of Krötenheerdt [1969].

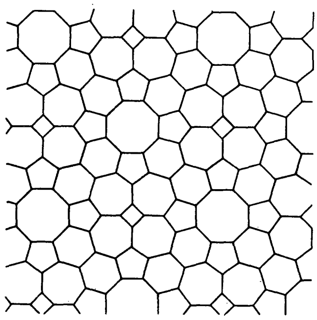
It is not hard to see that there exist k-uniform tilings for each $k \geqq 1$. Examples are given in Krötenheerdt [1969] for $k = 3, 4, 5, 6$ and 7, and in FIGURE 10 for $k = 3, 4$. However, even for $k = 3$ it is not known how many distinct 3-uniform tilings exist, nor is any kind of asymptotic estimate available for the number of k-uniform tilings with large k.

A closely related notion was also examined by Krötenheerdt [1969], [1970a], [1970b]. He considered those k-uniform tilings in which the k transitivity classes of vertices consist of k *distinct*



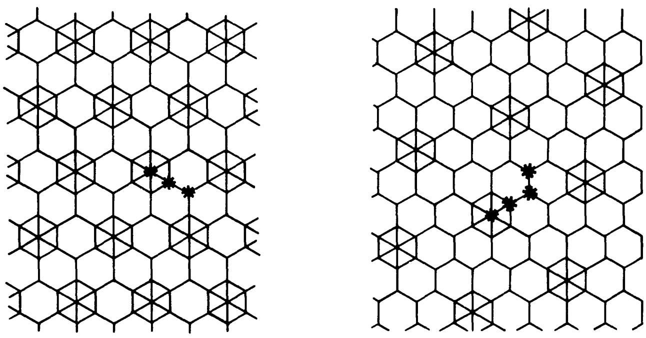
The 20 different types of 2-uniform tilings. The tiling $(3^6; 3^4.6)_2$ exists in mirror-symmetric forms, only one of which is shown. Tilings marked by an asterisk are homogeneous in the sense defined in Section 3. One vertex of each transitivity class is marked.

FIGURE 9



A fake tiling with regular polygons, adapted from a children's coloring book *Altair Design* (Holiday [1970]).

FIGURE 8



Two examples of homogeneous tilings; one is also 3-uniform, the other is 4-uniform. One vertex of each transitivity class is marked.

FIGURE 10

types of vertices. While it is easily seen that for $k = 1$ and for $k = 2$ these coincide with the k-uniform ones, Krötenheerdt's condition is actually restrictive for $k \geqq 3$. Denoting by $K(k)$ the number of distinct Krötenheerdt tilings, he established that $K(1) = 11$, $K(2) = 20$, $K(3) = 39$, $K(4) = 33$, $K(5) = 15$, $K(6) = 10$, $K(7) = 7$ and $K(k) = 0$ for each $k \geqq 8$. Krötenheerdt's method of proof is a natural extension of the one used in the determination of the uniform tilings.

3. Homogeneous and edge-transitive tilings

Departing from the terminology used by some authors, we shall say that an edge-to-edge tiling of the plane by regular polygons is ***k*-homogeneous** if the tiles form precisely k transitivity classes under the symmetries of the tiling. We shall also say that a tiling is **homogeneous** if all tiles that are mutually congruent form one transitivity class. It is easily verified that all the uniform tilings are homogeneous, except $(3^4.6)$, which is 3-homogeneous. Other homogeneous tilings are the seven 2-uniform tilings marked by an asterisk in FIGURE 9.

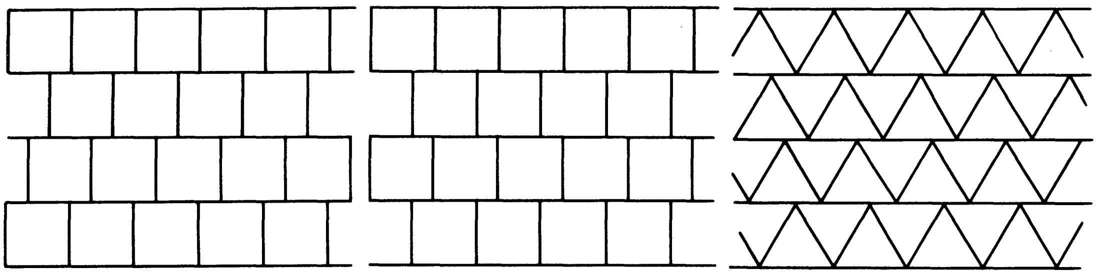
It is rather surprising that there seems to be no consideration in the literature of the homogeneous or k-homogeneous tilings. It appears reasonable to expect that for each k there exists a least number $h(k)$ such that every k-uniform tiling is also h-homogeneous for some $h \leqq h(k)$. Likewise, there probably exists a least number $k(h)$ such that every h-homogeneous tiling is also k-uniform for some $k \leqq k(h)$. From the above remarks and from FIGURE 9 it is easy to see that $h(1) = 3$ and $h(2) = 5$. On the other hand clearly $k(1) = 1$, while the examples of FIGURE 10 show that $k(2) \geqq 4$.

The determination of all 2-homogeneous tilings (all of which are, obviously, homogeneous) should not be very hard, and even the determination of all homogeneous tilings is probably possible with a little patience. We conjecture that the 19 homogeneous tilings shown in FIGURES 7, 9 and 10 are the only ones possible, and that, in consequence, there are just fourteen 2-homogeneous tilings, and that $k(2) = 4$.

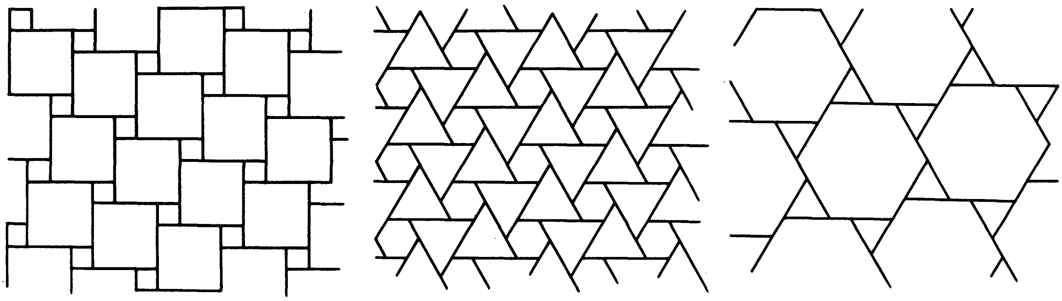
Similar problems arise if we consider transitivity classes of edges. If there are j such classes in a tiling we shall call it a ***j*-edge-transitive tiling.** We mention this idea because we believe that it also is not considered in the literature. There appear to be just four 1-edge-transitive tilings by regular polygons (namely (3^6), (4^4), (6^3) and $(3.6.3.6)$) and four 2-edge-transitive tilings (namely $(3^2.4.3.4)$, $(3.4.6.4)$, (3.12^2) and (4.8^2)).

4. Tilings that are not edge-to-edge

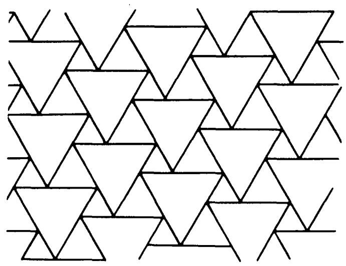
We now consider tilings by regular polygons without the requirement that the tiles meet edge-to-edge. Kepler briefly considered this possibility (see drawings *Bb* and *Kk* in FIGURE 1), but no further consideration seems to have been given to the mathematical possibilities for several centuries.



$0<\alpha\leqq\frac{1}{2}$ $0<\alpha\leqq\frac{1}{2}$ $0<\alpha\leqq\frac{1}{2}$



$0<\alpha<1$ $0<\alpha<1$ $0<\alpha<1$



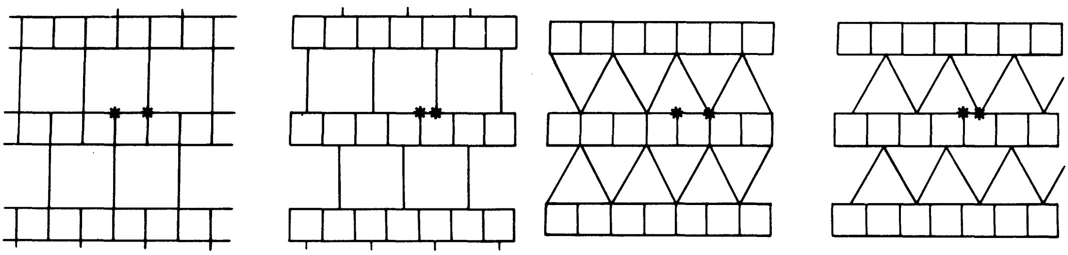
$0<\alpha\leqq\frac{1}{2}$

Representative examples of the seven families of uniform tilings that are not edge-to-edge. Explanation of the parameters is given in Section 4.

FIGURE 11

In a tiling that possibly is not edge-to-edge we shall call each point that is a vertex of some tile a **node** of the tiling, and we shall call a tiling **uniform** provided the symmetries of the tiling act transitively on its nodes. It is not hard to prove that all uniform tilings by regular polygons that are not edge-to-edge may be arranged in seven families, each family depending on a real-valued parameter α. These seven families are illustrated in FIGURE 11. In the first three families each tiling uses only mutually congruent tiles, and the parameter indicates the fraction of overlap between edges of adjacent tiles. The tilings of the next three families use two non-congruent kinds of tiles, and the parameter indicates the ratio of their edge-lengths. In the last family three sizes of triangles appear and the parameter α denotes the ratio of the side of the smallest triangle to that of the largest. (If $\alpha = 1/2$ only two sizes of triangles occur).

It is obviously possible to apply the definitions of k-uniformity, homogeneity, etc., to tilings that are not edge-to-edge. In FIGURE 12 we show several 2-uniform and homogeneous tilings of this kind and it is easy to construct additional examples of a similar character. A complete enumeration of homogeneous, 2-uniform tilings by regular polygons is probably obtainable with moderate effort. Many ornamental designs contain uniform tilings that are not edge-to-edge; see, for example, Dye [1937, FIGURES C15b, K5a, Y2b, Y3a, &a la].

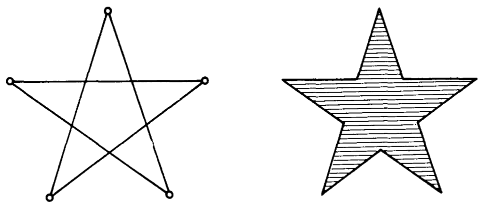


Examples of homogeneous and 2-uniform tilings that are not edge-to-edge. One vertex of each transitivity class is marked.

FIGURE 12

5. Tilings that use star-polygons

The drawings reproduced in FIGURE 1 show that Kepler had a rather pragmatic and experimental approach to tilings. He was looking for various more or less regular tilings, and although his main concern was with tilings that have vertices of a single species, several other possibilities are evident. One such variant, considered by Kepler but apparently not discussed in this form since, deals with edge-to-edge tilings that include star-polygons. In the first book of Kepler [1619] star-polygons are obtained by extending the sides of regular convex polygons. In a rather modern spirit, Kepler treats as vertices of star-polygons only the endpoints of these extended edges, not the vertices of the original convex polygon. Thus the pentagram (FIGURE 13a) has only 5 vertices and five edges. However, when dealing with tilings in Book 2 (and to some extent also later, in connection with the regular non-convex "Kepler–Poinsot" polyhedra), Kepler treats the star-pentagon (FIGURE 13b) as a non-convex decagon which may be called a pentacle and uses other star-polygons in the same way. It



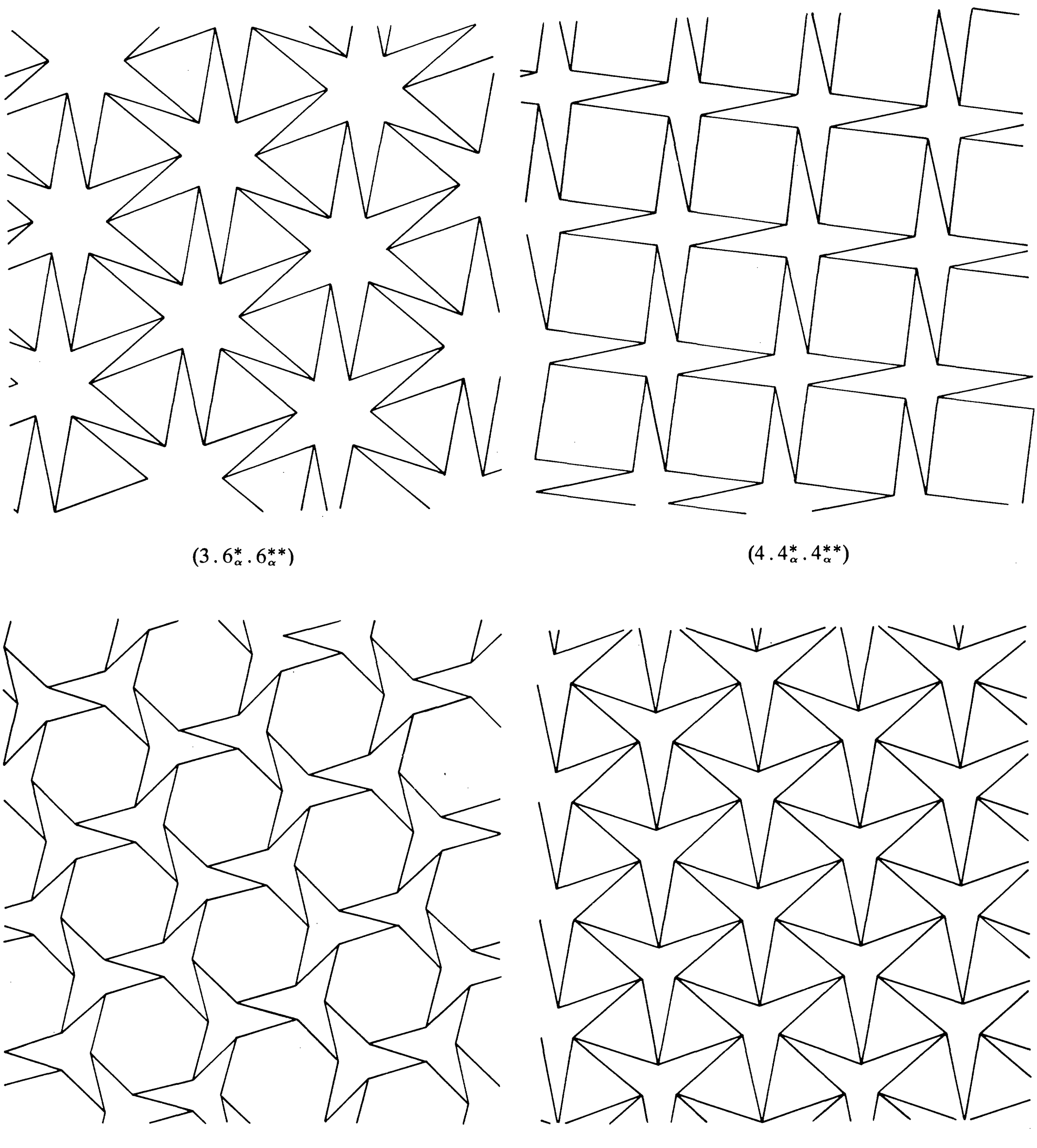
(a) (b)

The two interpretations of regular star-pentagons: (a) the pentagram, consisting of just 5 vertices and 5 edges; (b) the pentacle, a five-pointed "patch" of the plane.

FIGURE 13

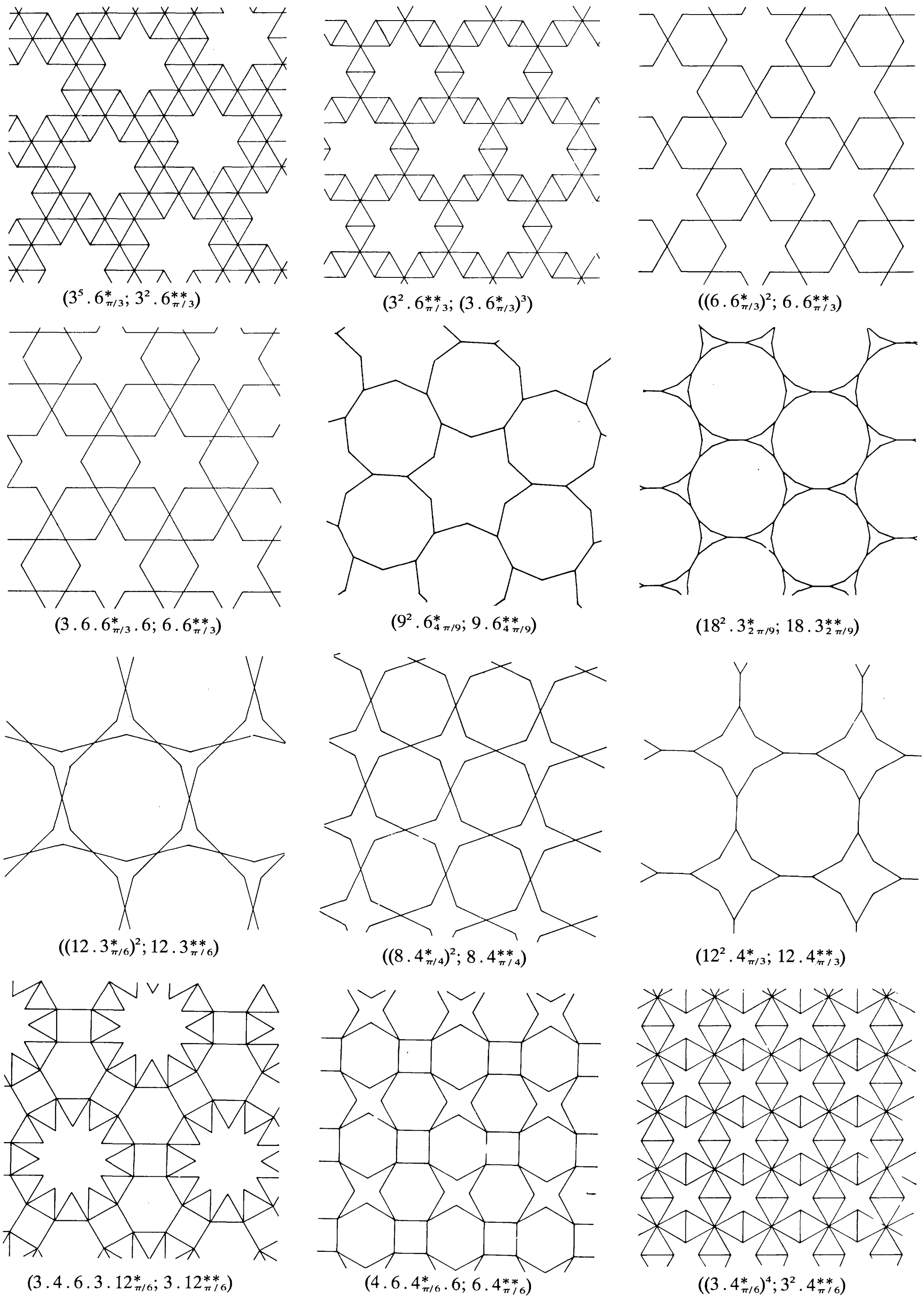
is never made quite clear exactly what rules must be followed or what polygons may be used. In Book 1 Kepler speaks only of what today would be called regular star polygons $\{n/d\}$, with n and d coprime. In other words, the edges of the $\{n/d\}$ form a single circuit. In the tiling K of FIGURE 1, however, Kepler not only allows six-pointed stars — which in the regular case would be just "hexagrams", each composed of two triangles — but even permits stars that have angles of $\pi/6$ at their points. At any rate, Kepler missed several possibilities and it is amusing to try to complete his list of tilings under some definite sets of rules.

One possibility is to allow as "regular" all n-pointed star-polygons that have the same symmetries as the regular convex n-gon. Such an n-pointed star, denoted $\{n_\alpha\}$ in the sequel, has n vertices of angle α (where $0<\alpha<(n-2)\pi/n$), n vertices of angle $2(n-1)\pi/n-\alpha$, and $2n$ mutually equivalent edges. Extending the definitions of k-uniform tilings to the case in which regular star-polygons are allowed, it is easy to see that there are precisely four families of 1-uniform tilings, each depending on a real parameter α; these families are illustrated in FIGURE 14. By using single and double asterisks to distinguish between the two kinds of angles in the star-polygons, we can denote these four families by $(3.6^*_\alpha.6^{**}_\alpha)$, $(4.4^*_\alpha.4^{**}_\alpha)$, $(6.3^*_\alpha.3^{**}_\alpha)$ and $(3.3^*_\alpha.3.3^{**}_\alpha)$. (Each of the first three of these families comes in two enantiomorphic forms.) There are many possibilities for 2-uniform tilings that include star-polygons, such as Kepler's *K*, *T*, *Nn*, and those shown in FIGURE 15. Most of these are also homogeneous, if the definition of this term is extended to cover star-polygons. With some patience it should be possible to determine all 2-uniform (and also all homogeneous) tilings that include star-polygons.



Representative examples for the four 1-uniform families of tilings that include star-polygons. Explanation of the notation is given in section 5.

FIGURE 14

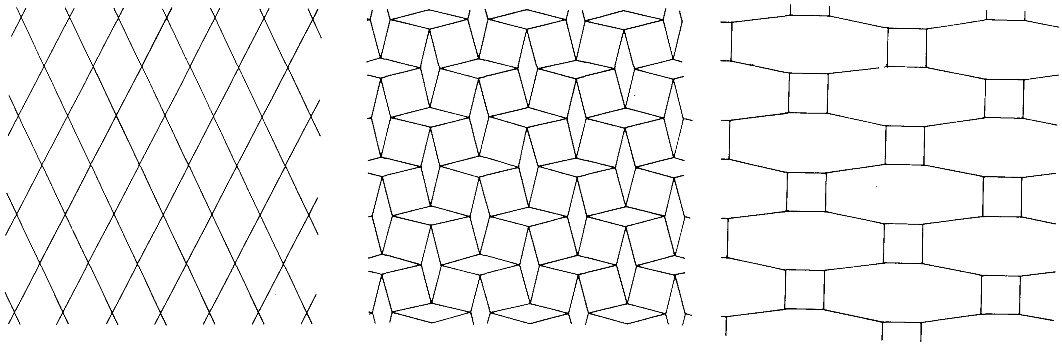


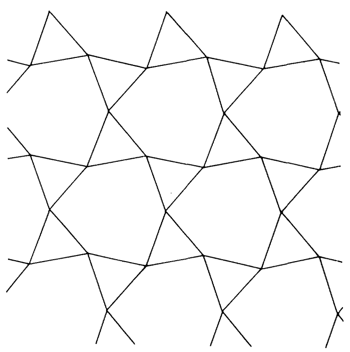
Examples of 2-uniform tilings that include star-polygons. Note the occurrence of 9-gons and 18-gons. All tilings except $(3^5 . 6^{*}_{\pi/3}); 3^2 . 6^{**}_{\pi/3})$ are homogeneous.

FIGURE 15

Tilings with star-polygons occur frequently in Islamic art. For example, slightly distorted versions of $(4\,.\,4_{\alpha}^{*}\,.\,4_{\alpha}^{**})$ may be seen on Plates 104, 117 and 118 in Ipşiroğlu [1971], Plate 45 of which shows the 2-uniform tiling $(3\,.\,6\,.\,6_{\pi/3}^{*}\,.\,6;\ 6\,.\,6_{\pi/3}^{**})$. Similarly, the tiling $(6\,.\,6_{\pi/3}^{**};\ 6\,.\,6_{\pi/3}^{*}\,.\,6\,.\,6_{\pi/3}^{*})$ of FIGURE 15 is shown in Plate 1 of Bourgoin [1879]. The tiling $(3^2\,.\,4_{\pi/6}^{**};\ 3\,.\,4_{\pi/6}^{*}\,.\,3\,.\,4_{\pi/6}^{*}\,.\,3\,.\,4_{\pi/6}^{*}\,.\,3/4_{\pi/6}^{*})$ also occurs as the design of an early American patchwork quilt known as "windmill blades" (Safford & Bishop [1972, FIGURE 173]).

It could be argued that the star-polygons $\{n_\alpha\}$ should actually be called (non-regular) $(2n)$-gons, and that in any case similar treatment should be given to the analogously defined convex (but not regular) polygons $\{n_\alpha\}$ with $(n-2)\pi/n < \alpha < (n-1)\pi/n$ and $n \geqq 2$, in which larger and smaller angles alternate. There is nothing illogical in this suggestion, and it may even not go far enough. It is probably possible (with a reasonable amount of effort) to determine all uniform and homogeneous tilings by arbitrary polygons, in other words, to find all tilings in which several kinds of (not necessarily regular) polygons may be present, but in which all congruent polygons form one equivalence class with respect to the symmetries of the tiling and all vertices are mutually equivalent. Examples of such tilings are shown in FIGURE 16. Even more general problems of a related nature have been considered in the literature, mostly with a crystallographic motivation. For example, attempts were made to determine all tile-transitive tilings (see, for example, Haag [1911], Hilbert & Cohn-Vossen [1932, p. 72], Delone [1959], Heesch [1968]) and all vertex-transitive tilings (Šubnikov [1916], Sauer [1937], Subnikov & Koptsik [1972]), but the claims of success are not justified. Detailed treatments of these questions are given in Grünbaum & Shephard [1977a], [1977b].





Examples of uniform and homogeneous tilings with non-regular polygons. Observe that the first tiling is transitive on vertices, edges, and tiles. All but the last tiling are also transitive on the edges.

FIGURE 16

6. History

The three regular tilings and several uniform tilings were used as decorations in antiquity and during the Middle Ages; the first mathematical treatment appears to be that of Kepler [1619]. He found all 11 uniform tilings as well as many other kinds of tilings and considered them — in a very modern way — as analogues of the Platonic (regular) and Archimedean polyhedra. (The drawing M in FIGURE 1 does not represent $(3^3\,.\,4^2)$ but Kepler's text describes it.) It is therefore strange, almost unbelievable, to find that this part of Kepler's work was completely forgotten for almost 300 years! Although Kepler was frequently quoted by authors interested in regular polyhedra, the first reference to the fact that Kepler determined the 11 uniform tilings appears to be in a note appended by Sommerville to his paper of 1905. Meanwhile, other authors dealt with the topic, usually in connection with investigations of Archimedean or related kinds of polyhedra, but the going was unaccountably slow. Gergonne [1818] obtained several of the uniform tilings; his work was extended, and completeness claimed for the result obtained by Badoureau [1878], [1881]. But although the latter paper is very interesting from several points of view (see Section 7(ii) below) his list of uniform tilings does not contain $(3^4.6)$. Badoureau's defective treatment was uncritically accepted by Lévy [1891] and by Brückner [1900]. The first correct determinations of the 11 uniform tilings in modern times were carried out — independently of each other and blissfully unaware of any of the previous work — by Sommerville [1905] and Andreini [1907]. The proof given by Sommerville that no other uniform tilings are possible is essentially the one we hinted at in Section 1. (The arguments are mentioned also in Ahrens [1901, pp. 66–71], but without a final list of uniform tilings.) Andreini [1907] uses the same method, but in a very cavalier way. He "finds" that there are just 10 (!) possible species of vertices, and the impression is inevitable that he let his "knowledge" of the 11 uniform tilings influence his judgment concerning the possibility of existence of various species. Similarly inadequate is the treatment in Šubnikov [1916]. The proofs or hints given in Kraitchik [1942, p. 203], Bilinski [1948], Critchlow [1970, p. 60], and Williams [1972, p. 42] are similar to the hint given in Section 1. A very nice treatment of this topic and many related questions is given in the refreshingly different text O'Daffer & Clemens [1976]. Several other works present the 11 uniform tilings without proofs (Fejes Tóth [1953, Section 7], [1965, pp. 45–49], Steinhaus [1950, Chapter 4], Cundy & Rollett [1951, Section 2.9]).

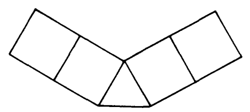
While many erred — as we have seen — in missing some of the uniform tilings, a modern text on architectural design (Borrego [1968, pp. 132, 134]) has too many. It is claimed that the tiling $(3^2.4.3.4)$ exists — like $(3^4.6)$ — in two enantiomorphic forms, not equivalent by motions without reflections!

Many authors have considered ways of generalizing Archimedean tilings by relaxing the requirement that all vertices be of the same type. Actually, even Kepler was interested in such tilings. For example, regarding vertices of species 3 Kepler remarks that they lead to two uniform tilings as well as to the tiling he denotes by O (see FIGURE 1) that "may be continued non-uniformly". His tilings R, Dd, Ee are in our notation the 2-uniform $(3^2.6^2; 3.6.3.6)$, $(3^6; 3^2.4.12)$ and $(3.4.3.12; 3.12^2)$, while his Cc may be extended to a 4-uniform $(3^2.4.3.4; 3.4.3.12; 3.4.6.4; 3.4.6.4)$. It is curious that Kepler states that his figure Kk cannot be extended without "mixing in" vertices of different species, while actually it appears to be part of the 2-uniform $(3.4^2.6; 3.4.6.4)$, all vertices of which are of species 6.

Kepler did not make precise what kinds of tilings he was interested in, other than the uniform ones. Several later authors were similarly vague, indicating only the desire to limit the species (or the types) of permitted vertices, or trying to obtain more or less symmetric tilings. Such discussions may be found in Lévy [1891], [1894] and especially in Sommerville [1905], while Kraitchik [1942, pp. 205–207] and Steinhaus [1950, Chapter 4] present several examples. Critchlow [1970, p. 60] presents 14 nonuniform tilings and asserts that these are the only possible ones. This assertion is repeated by Williams [1972, p. 43].

As we mentioned in Section 1, not much can be said in way of enumerating all tilings with vertices of just one species. Hence also the papers of Lévy and Sommerville reach no reasonable conclusions. However, there are several lines of investigation that appear to be challenging and promising. They deal with the extension to a tiling of the plane of a given "patch", that is, a finite part of the plane covered by regular polygons without overlaps and without enclosed gaps.

Given a patch such that all the vertices in it are of a species that allows a uniform tiling of the plane, is it always possible to extend the patch to a tiling using only vertices of the same species? Lévy [1891] mentions this question for vertices of species 6; Sommerville [1905] discusses in some detail the possibilities for species 3 and some others.



A "patch" (involving only vertices of species 3) that may be extended at each vertex separately but may not be extended at all vertices simultaneously.

FIGURE 17

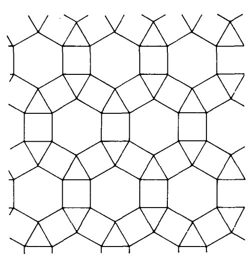
The answer is negative at least in some cases. To see this, we consider the very simple patch in FIGURE 17. Each of its vertices is of species 3, or may be completed to be of species 3. However, such completions may not be carried out simultaneously, so the patch is not part of a tiling with vertices of species 3. Similar examples exist for species 4. The answer is not known for a patch which involves only vertices of species 6. If the answer to this question is affirmative, is it always possible to choose the extension so that the resulting tiling is k-uniform for some k, or to have at least the symmetries of the original patch? Finally, for any variant of these questions, is there an algorithmic decision procedure that would allow the separation of the patches that have extension from the others?

7. Generalizations

We have discussed a number of variants of the theme "more-or-less regular tilings by regular polygons"; nevertheless we have barely scratched the surface of the topic. This final section is devoted to very brief hints at other variants, each of which would deserve a full article (or book) to describe its ramifications.

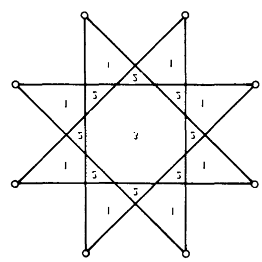
(1) *Multiple tilings.* Petersen [1888] considered the possibility of multiple coverings of the plane by congruent regular polygons of the same kind, subject to the condition that each edge of one tile is an edge of precisely one other. He found that the only way in which this could happen is by superimposing several copies of the regular tilings (3^6), or (4^4), or (6^3). However, it seems that the analogous problem for uniform or Archimedean multiple tilings is still undecided. Moreover, even consideration of multiple tilings with just triangles, or squares, or hexagons — which by Petersen's result consist of superimposed copies of (3^6), (4^4), or (6^3) — leads to the following open question. What multiplicities m are possible in regular tilings? Here "regular" means "flag-transitive", as explained at the end of Section 1. In case of (4^4) probably only $m = k^2$ and $m = 2k^2$ are possible, with integral k; the problem appears to be related to regular maps on the torus (see Coxeter-Moser [1957, Chapter 8]). Multiple tilings by non-regular convex polygons are discussed by Marley [1974].

(ii) *Tiles with densities.* Consider the tiling shown in FIGURE 18. It may be interpreted as the uniform tiling $(3.4.6.4)$ in which every point of the plane which does not lie on an edge is an interior point of exactly one tile. But the same drawing may be interpreted as another uniform tiling, by triangles, hexagons and 12-gons. We must assign a "density" of $+1$ to each hexagon and 12-gon, and a density of -1 to each triangle. With this interpretation the tiling is edge-to-edge, but two tiles sharing an edge may lie on the same side of it (if their densities have different signs). Now each point of the plane (not on an edge) can be assigned a density equal to the sum of the densities of the tiles of which



The uniform tiling $(-3.12.6.12)$ of density 2.

FIGURE 18

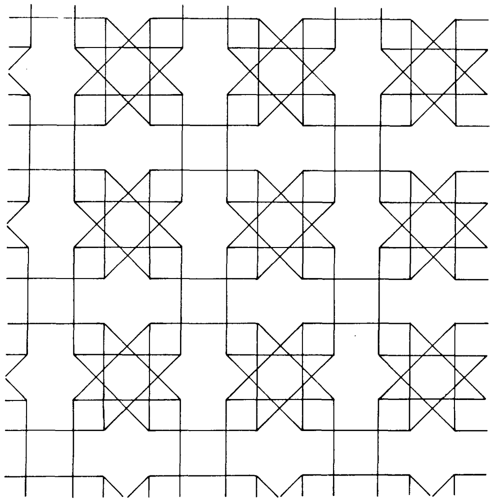


The regular star-polygon $\{8/3\}$, with the "density" of the various regions indicated.

FIGURE 19

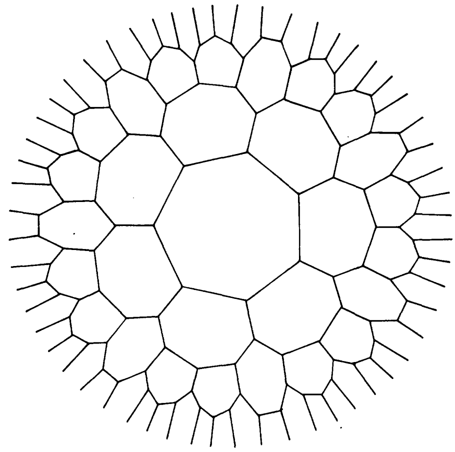
it is an interior point. In this case, it is easy to see that at every such point the density is 2, and so we say that the tiling has density 2. A reasonable symbol for this uniform tiling is $(-3.12.6.12)$.

Analogously, let us consider the regular star-polygon {8/3} (in the modern interpretation, see FIGURE 19), whose 8 vertices are marked by small circles and whose 8 edges connect them in pairs. It is reasonable to assign to the central region a density 3, to the regions near the vertices a density 1, and to the intermediate triangular regions a density 2. Then FIGURE 20 may be interpreted as a uniform tiling $(-4.8.-8/3)$ of density 1, in which each {4} has "density" -1, each {8} density 1, and each {8/3} has densities -1, -2, -3 in its various regions. An approach to tilings related to this (but distinct from it) was followed by Badoureau [1878], [1881] and is also touched upon in Coxeter & Longuet-Higgins & Miller [1954], but there are several complications and a full treatment of such tilings is still not available.



The uniform tiling $(-4.8.-8/3)$ of density 1, first found by Badoureau in 1881.

FIGURE 20



A topologically regular map of type (7^3). Similar topologically regular maps (p^q) exist for all p, q such that $1/p + 1/q < 1/2$.

FIGURE 21

(iii) *Topological uniformity.* In the determination of Archimedean or of uniform convex polyhedra, Euler's equation may be used to provide necessary conditions for the existence of various types. Since many authors treat uniform tilings of the plane together with the Archimedean or uniform polyhedra, the temptation to apply some "limiting form" or "modification" of the Euler relation to such tilings is great. If it were valid such an approach would also have the advantage that it would apply to "topologically uniform" tilings — that is, "maps" in which "symmetries" are not restricted to isometric transformations but may be affected by arbitrary homeomorphisms. However, the execution of such an approach — although feasible — is rather tricky and has to be done with great care (see, for example, Laves [1931], Delone [1959]). The "limiting form" of Euler's theorem usually quoted is, not surprisingly, $V - E + T = 0$, where $V : E : T$ are proportional to the "numbers" of vertices, edges and tiles in the tiling. But this equation only applies to tilings of very restricted kinds. For example, for the "topologically regular map" (7^3) shown in FIGURE 21, in which three heptagonal countries meet at each vertex, it is easy to verify that $V:E:T = 7/3:7/2:1$, which do not satisfy Euler's theorem. Although the heptagons in this map are not congruent, any two are equivalent under a homeomorphism of the plane that carries the map onto itself. In fact, it deserves the designation "topologically regular" since its self-homeomorphisms act transitively on its flags. While it is easy to verify that there exist "topologically regular maps" (p^q) in which q p-gons meet at each vertex whenever $1/p + 1/q \leqq 1/2$, the question of what "uniform maps" exist is open, as is the question of "Archimedean maps". These two notions are probably distinct; at any rate, while it is easy to verify that no topologically uniform map (3.5^3) exists, the existence of an Archimedean map with all vertices of type 3.5^3 is undecided. These possibilities have escaped many writers, such as Andreini [1907], Šubnikov [1916], Walsh [1972], Loeb [1976, p. 92], who used Euler's relation without due care and reached the conclusion that the only possible "topologically Archimedean maps" of the plane are of the same types as the Archimedean tilings by regular polygons. As the examples of the topologically regular maps show, this is false.

(iv) *Non-Euclidean tilings*. Another variant deserving attention deals with regular convex polygons tiling the sphere or the hyperbolic (non-Euclidean Lobačevski) plane. On the sphere the situation is well known — the uniform or Archimedean tilings may be obtained as central projections of Platonic, uniform, or Archimedean polyhedra. But in this case the distinction between "Archimedean" and "uniform" tilings (or polyhedra) is more than semantic. Besides the uniform (3.4^3) there exists an Archimedean but non-uniform (3.4^3). This appears to have been observed first by Sommerville [1905], and has been rediscovered (often with vehement priority claims) many times since then (Ball [1938, Chapter V], Aškinuze [1957], [1963, p. 430], Lyusternik [1956]). Multiple regular tilings correspond to the Kepler–Poinsot regular non-convex polyhedra, while multiple uniform tilings correspond to polyhedra studied by many authors. See, in particular, Coxeter & Longuet-Higgins & Miller [1954] and Skilling [1975], where references to the earlier literature may be found. Some non-edge-to-edge tilings of the sphere by regular polygons have been considered by Brun [1972]. Digons $\{2\}$ (which are legitimate regular polygons on the sphere) may be used to construct uniform (non-edge-to-edge) tilings of the sphere consisting, for any $n \geqq 3$, of two n-gons and n congruent digons; besides depending on a real-valued parameter, these tilings come in enantiomorphic pairs.

In the hyperbolic plane there exist regular tilings (p^q) whenever $1/p + 1/q < 1/2$ (see, for example, Fejes Tóth [1965, p. 85], Coxeter & Moser [1957, Chapter 5]). There also exist many uniform and Archimedean tilings, but no complete classification is known. This is related to the problem discussed at the end of (iii) above, since the hyperbolic plane is homeomorphic to the Euclidean plane. Detailed consideration of these questions, and partial results, may be found in Bilinski [1948].

(v) *Unbounded polyhedral surfaces.* Finally, as a natural extension of regular or uniform tilings of the plane by regular polygons we may consider the formation of unbounded polyhedral surfaces in 3-dimensional space that are composed of regular convex polygons. Various requirements on the polygons and on the vertices regarding transitivity under symmetries of the surface may be imposed. While it is known (Coxeter [1937]) that only three such surfaces deserve the adjective "regular", there are many that are "uniform", "homogeneous", "Archimedean", etc. Using (p^q) to denote a uniform polyhedral surface in which q regular p-gons $\{p\}$ meet at each vertex, the three regular (so-called Petrie–Coxeter) polyhedra are of types (4^6), (6^4), and (6^6). Uniform polyhedral surfaces are known for types (3^6), (3^7), (3^8), (3^9), (3^{10}), (3^{12}), (4^4), (4^5), (4^6), (5^5), (6^4), and (6^6) (see Gott [1967], Wells [1969], Wachman, Burt & Kleinmann [1974]). Although it is probable that no other types (p^q) are possible, no proof of this conjecture is known. For a discussion of some other related questions see Grünbaum [1977].

Research for this paper was supported by the National Science Foundation Grant MPS74–07547 A01, and by the National Research Council of Canada Grant A 7071.

References

W. Ahrens

1901 Mathematische Unterhaltungen und Spiele. Teubner, Leipzig 1901.

A. Andreini

1907 Sulle reti di poliedri regolari e semiregolari e sulle corrispondenti reti correlative. Soc. Ital. delle Scienze, Memorie. Ser. 3, Vol. 14 (1907), pp. 75–129.

V. G. Aškinuze

1957 On the number of semiregular polyhedra. [In Russian] Matematičeskoe Prosvešč. 1957, No. 1, pp. 107–118.

1963 Polygons and polyhedra. [In Russian] "Enciklopediya elementarnoĭ matematiki" Vol. 4, Geometry, edited by V. G. Boltyanskiĭ and I. M. Yaglom, Moscow 1963, pp. 382–447. German translation: "Enzyklopädie der Elementarmathematik" Vol. 4, Geometrie, pp. 393–456. VEB, Berlin 1969.

A. Badoureau

1878 Sur les figures isoscèles. C. R. Acad. Sci. Paris 87 (1878), pp. 823–825.

1881 Mémoire sur les figures isoscèles. J. de l'École Polytechnique 30 (1881), pp. 47–172.

W. W. R. Ball

1939 Mathematical Recreations and Essays. Eleventh ed., revised by H. S. M. Coxeter. Macmillan, London and New York 1939. Twelfth edition, by W. W. R. Ball and H. S. M. Coxeter, University of Toronto Press, 1974.

S. Bilinski

1948 Homogene mreže ravnine. [In Serbo-Croat] (= Homogeneous nets in the plane.) Rad Jugoslav. Akad. Znan. Umjet. Zagreb, 271 (1948) pp. 1–119.

J. Borrego

1968 Space Grid Structures. MIT Press, Cambridge, Mass. 1968.

J. Bourgoin

1879 Les Éléments de l'Art Arabe: le Trait des Entrelacs. Firmin-Didot et Cie., Paris 1879. New edition of plates: Arabic Geometrical Pattern and Design, Dover, New York 1973.

M. Brückner

1900 Vielecke und Vielflache. Teubner, Leipzig 1900.

V. Brun

1972 Noen Setninger om Kuleflatens Inndeling, Inspirert av Virusforskningen. [In Norwegian] (Some theorems on the partitioning of the sphere, inspired by virus research.) Nordisk Mat. Tidskr. 20 (1972), pp. 87–91, English summary, p. 120.

H. S. M. Coxeter

1937 Regular skew polyhedra in three and four dimensions, and their topological analogues. Proc. London Math. Soc. (2) 43 (1937), pp. 33–62.

1975 Regular Complex Polytopes. Cambridge Univ. Press, London and New York 1975.

H. S. M. Coxeter, M. S. Longuet-Higgins and J. C. P. Miller

1954 Uniform polyhedra. Philos. Trans. Roy. Soc. London, A 246 (1954), pp. 401–450.

H. S. M. Coxeter and W. O. J. Moser

1957 Generators and Relation for Discrete Groups. Springer, Berlin 1957. Third edition, 1972.

K. Critchlow

1970 Order in Space. Viking Press, New York, 1970.

H. M. Cundy and A. P. Rollett

1952 Mathematical Models. Clarendon Press, Oxford 1951. Second edition 1961.

B. N. Delone

1959 Theory of planigons. [In Russian] Izv. Akad. Nauk SSSR, Ser. Matem., 23 (1959), pp. 365–386.

D. S. Dye

1937 Chinese Lattice Designs. Harvard University Press, 1937. Dover reprint, New York 1974.

L. Fejes Toth

1953 Lagerungen in der Ebene, auf der Kugel und in Raume. Springer, Berlin 1953. Second edition, 1972.

1965 Reguläre Figuren. Akadémiai Kiądo, Budapest 1965. English translation: Regular Figures. Pergamon, New York 1965.

J. D. Gergonne

1819 Recherches sur les polyèdres, renfermant en particulier un commencement de solution du problème proposé à la page 256 du septième volume des Annales. Annales de Math. pures et appl. 9 (1819), pp. 321–

J. R. Gott, III

1967 Pseudopolyhedrons. Amer. Math. Monthly 74 (1967), pp. 497–504.

B. Grünbaum

1977 Regular polyhedra — old and new. Aequationes Math. (to appear).

B. Grünbaum and G. C. Shephard

1977a The eighty-one types of isohedral tilings in the plane. Math. Proc. Cambridge Phil. Soc. (to appear).

1977b The ninety-one types of isogonal tilings in the plane. Trans. Amer. Math. Soc. (to appear).

F. Haag

1911 Die regelmässigen Planteilungen. Zeitschr. für Kristallographie 49 (1911), pp. 360–369.

H. Heesch

1968 Reguläres Parkettierungsproblem. Westdeutscher Verlag, Köln 1968.

D. Hilbert and S. Cohn-Vossen

1932 Anschauliche Geometrie. Springer, Berlin 1932. English translation: "Geometry and the Imagination." Chelsea, New York 1952.

E. Holiday
1970 Altair design. Pantheon 1970.
M. S. Ipşiroğlu
1971 Das Bild im Islam. Anton Schroll & Co., Wien und München, 1971.
J. Kepler
1619 Harmonice Mundi. Gesammelte Werke, Band VI. Beck, München 1940. Many other editions, among them the easily available Harmonices Mundi Libri V. Facsimile edition, Culture et Civilisation, Bruxelles 1968. (In some copies of this edition the drawings referred to in our text have been omitted by mistake.)
M. Kraitchik
1942 Mathematical Recreations. W. W. Norton & Co., New York 1942. Sec. ed., Dover, New York 1953.
O. Krötenheerdt
1969 Die homogenen Mosaike n-ter Ordnung in der euklidischen Ebene. I, II, III. Wiss. Z. Martin-Luther-
1970ab Univ. Halle-Wittenberg Math.-Natur. Reihe 18 (1969), pp. 273–290, 19 (1970), pp. 19–38 and 97–122.
F. Laves
1931 Ebenenteilung und Koordinationszahl. Zeitschrift für Kristallographie 78 (1931), pp. 208–241.
L. Lévy
1891 Sur les pavages à l'aide de polygones réguliers. Bull. de la Société Philomatique de Paris, (8) 3 (1891), pp. 46–50.
1894 Question 262. Interméd. Math. 1 (1894), p. 147 and 7 (1900), p. 153.
A. L. Loeb
1976 Space Structures. Addison-Wesley, Reading, Massachusetts 1976.
L. A. Lyusternik
1956 Vypuklie figury i mnogogranniki. [In Russian] (= Convex figures and polyhedra.) Moscow 1956. English translations: By T. J. Smith, Dover, New York 1963, and by D. L. Barnett, Heath, Boston 1966.
G. C. Marley
1974 Multiple subdivisions of the plane. This MAGAZINE 47 (1974), pp. 202–206.
P. G. O'Daffer and S. R. Clemens
1976 Geometry: An Investigative Approach. Addison-Wesley, Menlo Park 1976.
A. W. Petersen
1888 On Planers Bedaekning med Hjaelp af et System af regulaere n-Kanter. [In Danish] (= On covering the plane with systems of regular polygons.) Tidsskrift for Mathematik (Kopenhagen), (5) 6 (1888), pp. 182–184.
C. L. Safford and R. Bishop
1972 America's Quilts and Coverlets. Dutton, New York 1972.
R. Sauer
1937 Ebene gleicheckige Polygongitter. Jber. Deutsch. Math.-Verein. 47 (1937), pp. 115–124.
J. Skilling
1975 The complete set of uniform polyhedra. Philos. Trans. Roy. Soc. London A 278 (1975), pp. 111–135.
D. M. Y. Sommerville
1905 Semi-regular networks of the plane in absolute geometry. Trans. Roy. Soc. Edinburgh, 41 (1905), pp. 725–747 and 12 plates.
H. Steinhaus
1950 Mathematical Snapshots. Oxford Univ. Press, New York 1950. Second Edition, 1960.
A. Šubnikov
1916 K voprosu o stroenii kristallov. [In Russian] (= On the problem of structure of crystals.) Bull. Acad. Imper. Sci., Ser. 6., 10 (1916), pp. 755–779.
A. V. Šubnikov and V. A. Kopcik
1972 Simmetriya v nauke i iskustve. [In Russian] Nauka Press, Moscow 1972. English translation: A. V. Shubnikov and V. A. Koptsik, Symmetry in Science and Art. Plenum Press, New York 1974.
A. Wachman, M. Burt and M. Kleinmann
1974 Infinite polyhedra. Technion — Israel Institute of Technology, Haifa 1974.
R. T. S. Walsh
1972 Characterizing the vertex neighborhoods of semi-regular polyhedra. Geometriae Dedicata 1 (1972), pp. 117–123.
A. F. Wells
1969 The geometrical basis of crystal chemistry. X. Further study of three-dimensional polyhedra. Acta Cryst., B25 (1969), pp. 1711–1719.
R. Williams
1972 Natural Structure. Eudaemon Press, Moorpark, California, 1972.