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# The eighty-one types of isohedral tilings in the plane

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1. A *tiling* is a collection $\mathscr{T} = \{T_i | i = 1, 2, \ldots\}$ of closed topological discs which covers the Euclidean plane $E^2$, and of which the individual *tiles* $T_i$ have disjoint interiors. We shall assume throughout that the intersection of any two tiles is a connected set. If each tile is *congruent* (directly or reflectively isometric) to a given set $T$, then the tiling $\mathscr{T}$ is called *monohedral* and $T$ is called the *prototile* of $\mathscr{T}$. Clearly every monohedral tiling is locally finite.

The tiling $\mathscr{T}$ is *isohedral* if the symmetry group $S(\mathscr{T})$ (that is, the group of isometries which leave $\mathscr{T}$ invariant) is transitive on the tiles. An isohedral tiling is necessarily monohedral.

The purpose of this paper is to enumerate and describe all possible types of monohedral tilings of the plane which are isohedral. It is surprising that no previous account of the enumeration appears in the literature. In fact several authors ((15), (2), (7)) have attempted to do this but failed for various reasons. On the other hand, the hardest part of the proof dealing with 'marked tilings', was essentially carried out as long ago as 1938 by Sinogowitz (15). The details of his enumeration have not been published and his method appears not to have had any influence on later research. Here we shall present a new derivation which is based on an extension of Delone's 'adjacency symbols' (2), and then show how the 81 types of isohedral tilings can be derived from the 'marked tilings'.

In the next section we shall introduce some additional terminology and notation, and, in particular, will elucidate what we mean by a 'type' of tiling. Previous attempts at enumeration failed to clarify this point. Intuitively, two tilings will be said to be of the same type if each tile of one bears the same relation to each of its neighbours as each tile of the other. The third section will be devoted to the statement and proof of the main theorem. We shall present the characteristics of the different tilings in

Table 1, and we shall show representatives of each type in Fig. 4. In the last section we discuss some related results and problems.

2. Denote by $S(\mathscr{T}; T_i)$ the subgroup of the symmetry group $S(\mathscr{T})$ that leaves a particular tile $T_i$ invariant. Since, by transitivity, the groups $S(\mathscr{T}; T_i)$ are isomorphic for all $i$, we may refer to the corresponding abstract group as the *induced group* of $\mathscr{T}$ and denote it by $I(\mathscr{T})$. Clearly $S(\mathscr{T}; T_i)$ is a subgroup of $S(T_i)$, the symmetry group of the tile $T_i$. The number of possible groups $S(\mathscr{T})$ and $I(\mathscr{T})$ is very limited: $S(\mathscr{T})$ is one of the 17 two-dimensional crystallographic groups (see, for example, (9), (1), (4) or (7)) and $I(\mathscr{T})$ is one of the following ten groups: $E$ (the trivial group), $C_n$ (the cyclic group of order $n$) for $n = 2, 3, 4$ or 6, and $D_m$ (the dihedral group of order $2m$) for $m = 1, 2, 3, 4$ or 6. This restriction on $I(\mathscr{T})$ arises because these groups are the only ones that occur as subgroups that leave a point fixed of the 17 crystallographic groups.

Also associated with $\mathscr{T}$ is its 1-skeleton or *net* $N(\mathscr{T})$. This is the graph consisting of *nodes* or *vertices* (where 3 or more tiles meet) and *edges* (where two tiles intersect). The part of the boundary of a tile that lies between two adjacent vertices will be called a *side* of the tile, so that each edge of the tiling coincides with sides of two tiles. If each tile in a monohedral tiling has $r$ sides and $r$ vertices, then we shall call it an *r-gon* – this word not implying convexity of the tile or even that its sides are line segments. Although many of the published accounts are fallacious, it is an easy consequence of Euler's theorem ((14), (12), (2)) that $3 \leqslant r \leqslant 6$. Moreover, since $\mathscr{T}$ is isohedral, only eleven topological types of distinct nets are possible. These are the *Laves nets* (sometimes called *regular* or *Šubnikov* nets) (12), illustrated in Fig. 1 along with symbols denoting the valences of the vertices. We remark in passing that the Laves nets are familiar as the duals of the nets of the eleven types of uniform tilings ((3), (6), (16)).

Clearly we must consider two plane tilings as of different types if they differ in $S(\mathscr{T})$, $I(\mathscr{T})$ or $N(\mathscr{T})$, but we shall see that this classification is not sufficiently fine. For example, tilings IH 43 and IH 44 in Fig. 4 cannot be distinguished by these three parameters, yet, for the intuitive reasons stated above, they should be considered as different types. To explain the finer classification we need to introduce the concept of an 'adjacency symbol'. This is implicit in the work of Sinogowitz (15) but first used explicitly by Delone (2) – though only in the case $I(\mathscr{T}) = E$. We show here how the method extends in a simple and elegant manner to the general case.

Let $T_i$ be any given tile in $\mathscr{T}$ and let us assign a symbol, say $a$, to any *directed* (oriented) side of $T_i$. Since $\mathscr{T}$ is isohedral, applying the operations of $S(\mathscr{T})$ will then yield a corresponding assignation of the same symbol to at least one side of every other tile in $\mathscr{T}$. Not only may two or more directed sides of a tile be assigned the same symbol, but it may also happen (if the side under consideration is reversed by an operation of $I(\mathscr{T})$) that the same symbol is assigned a second time to the same side of $T_i$, but with a reversed direction. In this case we consider the symbol $a$ to be a label for an *undirected side* of $T_i$.

If there are further sides of tiles to which no symbol is attached, then we proceed to assign a new symbol, say $b$, to one of the free sides. We proceed in the same way until a symbol has been assigned to every side of every tile. Of course, since each tile is an $r$-gon with $r \leqslant 6$, it is never necessary to use more than 6 distinct symbols.

Consider, for example, Fig. 2. Here we have allocated symbols in the manner described above to the sides of the tiles of two tilings. In the first case all the sides are directed, the symbols are distinct, and $I(\mathscr{T}) = E$. In the second case, $I(\mathscr{T}) = D_1$ and we have two symbols $a$, $d$ corresponding to undirected sides and two symbols $b$, $c$ corresponding to directed sides. The *tile symbol* is obtained by reading off the symbols in cyclic order round the tile. A superscript $^+$ or $^-$ denotes whether the side is coherently or oppositely oriented, and no superscript implies that the corresponding side is not directed. Thus in Fig. 2 the tile symbols are $a^+b^+c^+d^+e^+f^+$ and $ab^+c^+dc^-b^-$ respectively. In general we shall direct our sides and define the tile symbol in such a way that as many superscripts $^+$ as possible occur at the beginning of the symbol. Also we shall not consider two tile symbols as distinct if one can be obtained from the other by permutation or reversal.

After assigning to each side of each tile a symbol in the manner described above, we can define the adjacency symbol. To do this we consider, in sequence, the sides $a, b, c, \ldots$ of $T_i$ in cyclic order. Since each edge is common to two tiles, it follows that the side $a$ of $T_i$ is also a side, say $x$, of some adjacent tile. If $a$ is undirected, then necessarily so is $x$, and then $x$ is the first component of the adjacency symbol. On the other hand, if $a$ is directed, then there are two possibilities: *either* $x$ may be directed in the *opposite* direction from that of $a$, and in that case the first component is $x^+$; *or* $x$ may be directed in the *same* direction as $a$, and in this case the first component is $x^-$. For the second, third, ... components of the adjacency symbol we assign letters in the same way using $b, c, \ldots$ until all distinct letters in the tile symbol have been exhausted. The final sequence of letters so constructed is called the *adjacency symbol* of the tiling.

For example, the adjacency symbols of the tilings of Fig. 2, with the letters allocated as shown, are $a^+e^+d^-c^-b^+f^+$ and $dc^-b^-a$ respectively.

*Definition.* Two isohedral tilings are said to be *of the same type* if they have the same adjacency symbols.

We do not distinguish, of course, two adjacency symbols that differ trivially – by cyclic permutation or by change of notation. It will be seen that this adjacency symbol is a mathematical formulation of the intuitive concept of 'type of tiling' mentioned in Section 1.

3. We can now state our main result.

THEOREM. *There exist eighty-one types of isohedral tilings of the plane.*

The proof is in two stages. The first is to enumerate all possible tile symbols and their corresponding adjacency symbols, and the second is to see which of these correspond to actual tilings.

At first sight, the first of these two steps appears formidable. For example, in the case of 6-gons with tile symbol $a^+b^+c^+d^+e^+f^+$, there appear to be $6!\,3^6 = 524\,880$ possible adjacency symbols, produced by taking all permutations of $a, b, \ldots, f$ and then adding to each letter a superscript $^+$, $^-$, or neither. But the vast majority of these can be eliminated immediately.

To begin with, we need only consider those permutations which consist of a number of disjoint transpositions; if edge $u$ abuts on edge $v$, then edge $v$ must abut on edge $u$, and so the permutation contains the transposition $vu$ of $uv$. If we also eliminate the variants produced by cyclic changes and by reversals of order, we are finally left with just 15 permutations to be considered. And even then, many possible ways of allocating superscripts are clearly inadmissible. For example, if the transposition $vu$ of the symbols $uv$ occurs, then the only possible superscripts are $v^+u^+$ and $v^-u^-$. Similar restrictions apply to letters without superscripts.

Next, we must see if each adjacency symbol is *combinatorially* possible. For this we take the Laves net and assign letters according to the adjacency symbol we have chosen. It may prove to be a consistent labelling scheme – in which case we have a possible tiling – or some inconsistency may arise, in which case it is rejected. In fact, to make sure that the tiling is possible it is only necessary to show that letters may be assigned consistently to all the tiles incident (at edge or vertex) with one given tile.

Eventually our 524 880 symbols for the hexagonal net will be reduced to 7, and these appear in the first seven rows of Table 1. We proceed similarly with all other Laves nets and with all other possible groups $I(\mathscr{T})$ to arrive finally at the 93 adjacency symbols listed in Table 1.

A convenient way to present the 93 tilings corresponding to the 93 adjacency symbols is by means of *marked tiles*. Examples are given in Fig. 3 of marked tilings corresponding to numbers IH 5, IH 12, IH 10 and IH 11 in our list (Table 1) and further examples are given in Fig. 5. In effect all we have to do is to assign a mark to any one tile (the mark may be chosen arbitrarily so long as its symmetry group is $E$ – we have used an $\llcorner$) and then apply the operations of $S(\mathscr{T})$ to mark all the other tiles. If $I(\mathscr{T}) \neq E$ then each tile may carry more than one mark. Thus in Fig. 3 ($b$), we have two $\llcorner$'s superimposed to form $\top$, and more complicated cases arise in Figs. 3 ($c$), ($d$) and 5.

It will be appreciated that the existence of these 93 marked tilings does not imply the existence of the same number of 'unmarked' tilings, that is tilings as originally defined. It is strange that this point has been overlooked by all authors previously attempting to enumerate the isohedral types of tilings.

So our next and final task is to find whether, for each of the marked tilings, the tile *shape* can be chosen in such a way that the corresponding tiling is of the type under consideration. If we proceed through the list we find that in fact 12 tilings cannot be so represented. For example consider tiling number IH 19. Here the symmetry group of the tile is $D_3$, and the symmetries of the tiling that leave any vertex invariant form also the group $D_3$. Now it is clear that these two conditions cannot be met unless the tile is a regular hexagon – and then it is not of type IH 19 but is of type IH 20. Hence tiling number IH 19 can only be realized by marked tiles. In Table 1 the 12 tilings of this kind are marked by an $M$ in column (9).

The general technique for finding a tile shape corresponding to a given type can be briefly described as follows. First we see whether we can distort the net so that the group is unaltered but, so far as possible, different transitivity classes of edges are of different lengths. We then replace each edge by an arc according to the following rules:

($a$) If an oriented edge $x^+$ is against an edge also marked $x$ but in the opposite direction, then we replace it by any centrally symmetric arc, the centre of symmetry being the midpoint of the edge.

(*b*) If an unoriented edge $x$ is against an unoriented edge with a different symbol, then we replace it by any arc which possesses the perpendicular bisector of the edge as a line of reflective symmetry.

(*c*) If an oriented edge $x^+$ is against an edge bearing a different symbol then we replace it by any asymmetric arc.

In the other cases we must leave the edge as a straight-line segment. The only restriction on the choice of arcs in cases (*a*), (*b*) and (*c*) is that they shall be disjoint, except possibly at their endpoints.

In this way we finally arrive at our list of 81 tilings. Representatives of each type are given in Fig. 4, completing the proof of the theorem. In each case, the tiles have been chosen to be of as general a shape as possible and an $a^+$ side is marked by $\nearrow$ or an $a$ side is marked by /.

4. There have been many attempts to enumerate and classify plane tilings, starting with the classic work of Kepler in 1619 (**10**). In addition to the works (**2**), (**7**), (**15**) quoted above, we must mention the papers of Fedorov (**3**), Haag (**6**), Wollny (**17**) and the books of MacMahon (**13**), Hilbert–Cohn-Vossen (**9**) and Heesch–Kienzle (**8**). These authors, however, imposed various and differing conditions and so obtained results which are not directly comparable with the enumeration given in this paper. In fact, the correct interpretation of some of these works seems rather obscure. It is strange that the more obvious restrictions seem to have been overlooked, or have led to incomplete enumerations. For example, it is still not known how many types of monohedral tilings by *convex polygons* exist ((**11**), (**5**)). The case of *isohedral tilings by convex polygons* will be discussed in a forthcoming paper by the authors.

The methods of this paper also lead to a classification of all plane patterns. By a *pattern* we mean any connected *motif* together with all its images under the operations of one of the 17 crystallographic groups, so long as all these images are disjoint. To do so we consider the Dirichlet region of each motif and so obtain a tile-transitive marked tiling of the plane, the motifs providing the necessary marks. Hence there can be at most 93 types of patterns. It remains to be shown that each of these can actually arise. For 81 types, those which correspond to 'unmarked' (shaped) tilings, the existence is established trivially by observing that in each case we need only take the interior of the tile itself as the motif. For the remaining 12 types, those marked by $M$ in column (9) of Table 1, we may choose motifs as in Fig. 5, where the Dirichlet regions are also indicated.

It seems to be an unsolved problem whether every type of isohedral tiling with convex prototile can be realized as the set of Dirichlet regions of a point set in the plane.

We remark that the corresponding enumeration problem for tile-transitive tilings in three dimensions is essentially different, since it is trivial to show that there exist infinitely many types.

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Table 1. *The* 93 *adjacency symbols*

Columns (2)–(4) are explained in the text. When a dihedral group $D_n$ occurs in column (3) several cases may arise, and these were distinguished as follows: $D_n^l$ when each reflexion leaves a pair of opposite vertices invariant, $D_n^s$ when a pair of opposite edges is invariant; $D_1^{l*}$ and $D_1^{s*}$ distinguish between the two possible types of reflexions in the net [3.6.3.6]. Column (5) indicates the symmetry group of the tiling using the international crystallographic notation (see (1), or (7)). Column (6) indicates the transitivity classes of the vertices induced by the symmetries of the tiling. The various classes are denoted by $\alpha$, $\beta$, ..., and the symbol relates to the vertices taken in cyclic order round the tile. Analogously, column (7) indicates the transitivity classes of the edges. Two tiles are said to be of the same *aspect* if one is a translation of the other; we have borrowed the term from MacMahon (13). Column (8) indicates the number of aspects that occur in the tiling under consideration, where $D$ and $R$ mean direct and reflected aspects. Column (9) indicates the way in which each of the tilings can be realised. $C$ means that there exists a tiling of the given type with a convex prototile. $N$ indicates that in any realization the prototile is non-convex. $M$ means that the tiling is realizable by marked tiles only, and not by 'shaped tiles'. Column (10) refers to the number of the corresponding tiling in the lists of Heesch (H) (7), where we have numbered the tilings consecutively; Heesch–Kienzle (HK) (8); Wollny (W) (17); Sinogowitz (S) (15); and Delone (D), where again we have numbered them consecutively in the order in which they appear in (2). An asterisk means that the data given in the reference contains errors.

| List no. (1) | Laves net (2) | Induced group and tile symbol (3) | Adjacency symbol (4) | Crystal-lographic group (5) | Vertex transitivity (6) | Edge transitivity (7) | Aspects (8) | Realiza-tions (9) | References (10) |
|---|---|---|---|---|---|---|---|---|---|
| IH 1 | $[3^6]$ | $E\ a^+b^+c^+d^+e^+f^+$ | $d^+e^+f^+a^+b^+c^+$ | $p1$ | $\alpha\beta\alpha\beta\alpha\beta$ | $\alpha\beta\gamma\alpha\beta\gamma$ | $1D$ | $N$ | H12, HK2, S20, D7 |
| IH 2 | | | $b^-a^-f^+e^-d^-c^+$ | $pg$ | $\alpha\alpha\alpha\beta\beta\beta$ | $\alpha\alpha\beta\gamma\gamma\beta$ | $1D$, $1R$ | $C$ | H15, HK18, S19, D6 |
| IH 3 | | | $c^-e^+a^-f^-b^+d^-$ | $pg$ | $\alpha\beta\alpha\beta\alpha\beta$ | $\alpha\beta\alpha\gamma\beta\gamma$ | $1D$, $1R$ | $C$ | H13, HK20, S18, D3 |
| IH 4 | | | $a^+e^+c^+d^+b^+f^+$ | $p2$ | $\alpha\alpha\beta\beta\beta\alpha$ | $\alpha\beta\gamma\delta\beta\epsilon$ | $2D$ | $C$ | H17, HK7, S15, D4 |
| IH 5 | | | $a^+e^+d^-c^-b^+f^+$ | $pgg$ | $\alpha\alpha\beta\beta\beta\alpha$ | $\alpha\beta\gamma\gamma\beta\delta$ | $2D$, $2R$ | $C$ | H16, HK24, S17, D5 |
| IH 6 | | | $a^+e^-c^+f^-b^-d^-$ | $pgg$ | $\alpha\alpha\beta\beta\alpha\beta$ | $\alpha\beta\gamma\delta\beta\delta$ | $2D$, $2R$ | $C$ | H18, HK 28, S16*, D2 |
| IH 7 | | | $b^+a^+d^+c^+f^+e^+$ | $p3$ | $\alpha\beta\alpha\gamma\alpha\delta$ | $\alpha\alpha\beta\beta\gamma\gamma$ | $3D$ | $C$ | H20, HK9, S14, D1 |
| IH 8 | | $C_2\ a^+b^+c^+a^+b^+c^+$ | $a^+b^+c^+$ | $p2$ | $\alpha\alpha\alpha\alpha\alpha\alpha$ | $\alpha\beta\gamma\alpha\beta\gamma$ | $1D$ | $C$ | H7, S7 |
| IH 9 | | | $a^+c^-b^-$ | $pgg$ | $\alpha\alpha\alpha\alpha\alpha\alpha$ | $\alpha\beta\beta\alpha\beta\beta$ | $1D$, $1R$ | $C$ | H5, S8 |
| IH 10 | | $C_3\ a^+b^+a^+b^+a^+b^+$ | $b^+a^+$ | $p3$ | $\alpha\beta\alpha\beta\alpha\beta$ | $\alpha\alpha\alpha\alpha\alpha\alpha$ | $1D$ | $N$ | H10, S5 |
| IH 11 | | $C_6\ a^+a^+a^+a^+a^+a^+$ | $a^+$ | $p6$ | $\alpha\alpha\alpha\alpha\alpha\alpha$ | $\alpha\alpha\alpha\alpha\alpha\alpha$ | $1D$ | $N$ | H2, S2 |
| IH 12 | | $D_1^s\ ab^+c^+dc^-b^-$ | $dc^-b^-a$ | $cm$ | $\alpha\alpha\alpha\alpha\alpha\alpha$ | $\alpha\beta\beta\alpha\beta\beta$ | 1 | $N$ | H6, S10 |
| IH 13 | | | $db^+c^+a$ | $pmg$ | $\alpha\alpha\alpha\alpha\alpha\alpha$ | $\alpha\beta\gamma\alpha\gamma\beta$ | 2 | $C$ | H8, S9 |

Table 1 (*cont.*)

| List no. (1) | Laves net (2) | Induced group and tile symbol (3) | Adjacency symbol (4) | Crystal-lographic group (5) | Vertex transitivity (6) | Edge transitivity (7) | Aspects (8) | Realiza-tions (9) | References (10) |
|---|---|---|---|---|---|---|---|---|---|
| IH 14 | $[3^6]$ (*cont.*) | $D_1^l\ a^+b^+c^+c^-b^-a^-$ | $c^-b^-a^-$ | *cm* | $\alpha\beta\alpha\beta\alpha\beta$ | $\alpha\beta\alpha\alpha\beta\alpha$ | 1 | *N* | H11, S13 |
| IH 15 | | | $a^+b^-c^+$ | *pmg* | $\alpha\alpha\beta\beta\beta\alpha$ | $\alpha\beta\gamma\gamma\beta\alpha$ | 2 | *C* | H14, S12 |
| IH 16 | | | $a^-c^+b^+$ | *p*31*m* | $\alpha\beta\gamma\beta\gamma\beta$ | $\alpha\beta\beta\beta\beta\alpha$ | 3 | *C* | H19, S11 |
| IH 17 | | $D_2\ ab^+b^-ab^+b^-$ | $ab^+$ | *cmm* | $\alpha\alpha\alpha\alpha\alpha\alpha$ | $\alpha\beta\beta\alpha\beta\beta$ | 1 | *C* | H4, S6 |
| IH 18 | | $D_3^s\ ababab$ | $ba$ | *p*31*m* | $\alpha\alpha\alpha\alpha\alpha\alpha$ | $\alpha\alpha\alpha\alpha\alpha\alpha$ | 1 | *N* | H3, S3 |
| IH 19 | | $D_3^l\ a^+a^-a^+a^-a^+a^-$ | $a^-$ | *p*3*m*1 | $\alpha\beta\alpha\beta\alpha\beta$ | $\alpha\alpha\alpha\alpha\alpha\alpha$ | 1 | *M* | H9, S4 |
| IH 20 | | $D^6\ aaaaaa$ | $a$ | *p*6*m* | $\alpha\alpha\alpha\alpha\alpha\alpha$ | $\alpha\alpha\alpha\alpha\alpha\alpha$ | 1 | *C* | H1, S1 |
| IH 21 | $[3^4.6]$ | $E\ a^+b^+c^+d^+e^+$ | $e^+c^+b^+d^+a^+$ | *p*6 | $\alpha\beta\gamma\beta\beta$ | $\alpha\beta\beta\gamma\alpha$ | 6*D* | *C* | H21, HK13, S21, D8 |
| IH 22 | $[3^3.4^2]$ | $E\ a^+b^+c^+d^+e^+$ | $a^-e^+d^-c^-b^+$ | *cm* | $\alpha\alpha\beta\beta\beta$ | $\alpha\beta\gamma\gamma\beta$ | 1*D*, 1*R* | *C* | H23, S26, D12 |
| IH 23 | | | $a^+e^+c^+d^+b^+$ | *p*2 | $\alpha\alpha\beta\beta\beta$ | $\alpha\beta\gamma\delta\beta$ | 2*D* | *C* | H26, HK6, S23, D9 |
| IH 24 | | | $a^-e^+c^+d^+b^+$ | *pmg* | $\alpha\alpha\beta\beta\beta$ | $\alpha\beta\gamma\delta\beta$ | 2*D*, 2*R* | *C* | H25, S24, D11 |
| IH 25 | | | $a^+e^+d^-c^-b^+$ | *pgg* | $\alpha\alpha\beta\beta\beta$ | $\alpha\beta\gamma\gamma\beta$ | 2*D*, 2*R* | *C* | H24*, HK23, S25, D10 |
| IH 26 | | $D_1\ ab^+c^+c^-b^-$ | $ab^-c^+$ | *cmm* | $\alpha\alpha\beta\beta\beta$ | $\alpha\beta\gamma\gamma\beta$ | 2 | *C* | H22, S22 |
| IH 27 | $[3^2.4.3.4]$ | $E\ a^+b^+c^+d^+e^+$ | $a^+d^-e^-b^-c^-$ | *pgg* | $\alpha\alpha\beta\alpha\beta$ | $\alpha\beta\gamma\beta\gamma$ | 2*D*, 2*R* | *C* | H28*, S29, D14 |
| IH 28 | | | $a^+c^+d^+e^+d^+$ | *p*4 | $\alpha\alpha\beta\alpha\gamma$ | $\alpha\beta\beta\gamma\gamma$ | 4*D* | *C* | H29, HK16, S28, D13 |
| IH 29 | | $D_1\ ab^+c^+c^-b^-$ | $ac^+b^+$ | *p*4*g* | $\alpha\alpha\beta\alpha\beta$ | $\alpha\beta\beta\beta\beta$ | 4 | *C* | H27, S27 |
| IH 30 | $[3.4.6.4]$ | $E\ a^+b^+c^+d^+$ | $a^-b^-d^+c^+$ | *p*31*m* | $\alpha\beta\alpha\gamma$ | $\alpha\beta\gamma\gamma$ | 3*D*, 3*R* | *C* | H73, S32, D32 |
| IH 31 | | | $b^+a^+d^+c^+$ | *p*6 | $\alpha\beta\alpha\gamma$ | $\alpha\alpha\beta\beta$ | 6*D* | *N* | H72, HK12, S31, D31 |
| IH 32 | | $D_1\ a^+a^-b^+b^-$ | $a^-b^-$ | *p*6*m* | $\alpha\beta\alpha\gamma$ | $\alpha\alpha\beta\beta$ | 6 | *C* | H71, S30 |
| IH 33 | $[3.6.3.6]$ | $E\ a^+b^+c^+d^+$ | $d^+c^+b^+a^+$ | *p*3 | $\alpha\beta\gamma\beta$ | $\alpha\beta\beta\alpha$ | 3*D* | *N* | H70*, HK8, S37, D33 |
| IH 34 | | $C_2\ a^+b^+a^+b^+$ | $b^+a^+$ | *p*6 | $\alpha\beta\alpha\beta$ | $\alpha\alpha\alpha\alpha$ | 3*D* | *N* | H68, S34 |
| IH 35 | | $D_1^{s*}\ a^+b^+b^-a^-$ | $a^-b^-$ | *p*3*m*1 | $\alpha\beta\gamma\beta$ | $\alpha\beta\beta\alpha$ | 3 | *M* | H67*, S35 |
| IH 36 | | $D_1^{l*}\ a^+a^-b^+b^-$ | $b^-a^-$ | *p*31*m* | $\alpha\beta\alpha\beta$ | $\alpha\alpha\alpha\alpha$ | 3 | *N* | H69, S36 |
| IH 37 | | $D_2\ a^+a^-a^+a^-$ | $a^-$ | *p*6*m* | $\alpha\beta\alpha\beta$ | $\alpha\alpha\alpha\alpha$ | 3 | *C* | H66, S33 |
| IH 38 | $[3.12.12]$ | $E\ a^+b^+c^+$ | $a^-c^+b^+$ | *p*31*m* | $\alpha\alpha\beta$ | $\alpha\beta\beta$ | 3*D*, 3*R* | *N* | H91, S76, D46 |
| IH 39 | | | $a^+c^+b^+$ | *p*6 | $\alpha\alpha\beta$ | $\alpha\beta\beta$ | 6*D* | *N* | H92, S75, D45 |
| IH 40 | | $D_1\ ab^+b^-$ | $ab^-$ | *p*6*m* | $\alpha\alpha\beta$ | $\alpha\beta\beta$ | 6 | *C* | H90, S74 |

Table 1 (*cont.*)

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| IH 41 | $[4^4]$ | $E\ a^+b^+c^+d^+$ | $c^+d^+a^+b^+$ | $p1$ | $\alpha\alpha\alpha\alpha$ | $\alpha\beta\alpha\beta$ | $1D$ | $N$ | H44, HK1, W1, S73, D23 |
| IH 42 | | | $c^+b^-a^+d^-$ | $pm$ | $\alpha\beta\beta\alpha$ | $\alpha\beta\alpha\gamma$ | $1D$, $1R$ | $N$ | H58, W8, S69, D28 |
| IH 43 | | | $c^-d^+a^-b^+$ | $pg$ | $\alpha\alpha\alpha\alpha$ | $\alpha\beta\alpha\beta$ | $1D$, $1R$ | $N$ | H45, HK19, W10, S71, D25 |
| IH 44 | | | $b^-a^-d^-c^-$ | $pg$ | $\alpha\alpha\alpha\alpha$ | $\alpha\alpha\beta\beta$ | $1D$, $1R$ | $N$ | H38, HK17, W11, S72, D17 |
| IH 45 | | | $c^-b^-a^-d^-$ | $cm$ | $\alpha\beta\alpha\beta$ | $\alpha\beta\alpha\gamma$ | $1D$, $1R$ | $N$ | H55, W12, S70, D29 |
| IH 46 | | | $a^+b^+c^+d^+$ | $p2$ | $\alpha\alpha\alpha\alpha$ | $\alpha\beta\gamma\delta$ | $2D$ | $C$ | H49, HK4, W3, S65, D15 |
| IH 47 | | | $c^+b^+a^+d^+$ | $p2$ | $\alpha\beta\beta\alpha$ | $\alpha\beta\alpha\gamma$ | $2D$ | $N$ | H57, HK5, W2, S64, D25 |
| IH 48 | | | $a^-b^-c^-d^-$ | $pmm$ | $\alpha\beta\gamma\delta$ | $\alpha\beta\gamma\delta$ | $2D$, $2R$ | $M$ | H65, W18, S60, D30 |
| IH 49 | | | $a^-b^+c^-d^+$ | $pmg$ | $\alpha\beta\beta\alpha$ | $\alpha\beta\gamma\delta$ | $2D$, $2R$ | $C$ | H60, W22, S62, D20 |
| IH 50 | | | $c^+b^-a^+d^+$ | $pmg$ | $\alpha\beta\beta\alpha$ | $\alpha\beta\alpha\gamma$ | $2D$, $2R$ | $C$ | H59, W21, S63, D27 |
| IH 51 | | | $c^-b^+a^-d^+$ | $pgg$ | $\alpha\alpha\alpha\alpha$ | $\alpha\beta\alpha\gamma$ | $2D$, $2R$ | $C$ | H48, HK25, W15, S68, D18 |
| IH 52 | | | $c^-d^-a^-b^-$ | $pgg$ | $\alpha\beta\alpha\beta$ | $\alpha\beta\alpha\beta$ | $2D$, $2R$ | $N$ | H54*, HK26, W14, S66, D26 |
| IH 53 | | | $b^-a^-c^+d^+$ | $pgg$ | $\alpha\alpha\alpha\alpha$ | $\alpha\alpha\beta\gamma$ | $2D$, $2R$ | $C$ | H46, HK22, W16, S67, D16 |
| IH 54 | | | $a^-b^-c^-d^+$ | $cmm$ | $\alpha\beta\gamma\alpha$ | $\alpha\beta\gamma\delta$ | $2D$, $2R$ | $C$ | H64, W26, S61, D19 |
| IH 55 | | | $b^+a^+d^+c^+$ | $p4$ | $\alpha\beta\alpha\gamma$ | $\alpha\alpha\beta\beta$ | $4D$ | $N$ | H62*, HK15, W5, S59, D21 |
| IH 56 | | | $b^+a^+c^-d^-$ | $p4g$ | $\alpha\beta\alpha\gamma$ | $\alpha\alpha\beta\gamma$ | $4D$, $4R$ | $C$ | H63, W30, S58, D22 |
| IH 57 | | $C_2\ a^+b^+a^+b^+$ | $a^+b^+$ | $p2$ | $\alpha\alpha\alpha\alpha$ | $\alpha\beta\alpha\beta$ | $1D$ | $C$ | H40, W4, S46 |
| IH 58 | | | $a^-b^+$ | $pmg$ | $\alpha\alpha\alpha\alpha$ | $\alpha\beta\alpha\beta$ | $1D$, $1R$ | $C$ | H41, W25, S48 |
| IH 59 | | | $b^-a^-$ | $pgg$ | $\alpha\alpha\alpha\alpha$ | $\alpha\alpha\alpha\alpha$ | $1D$, $1R$ | $N$ | H35, W17, S49 |
| IH 60 | | | $a^-b^-$ | $cmm$ | $\alpha\beta\alpha\beta$ | $\alpha\beta\alpha\beta$ | $D$, $1R$ | $M$ | H53, W27, S47 |
| IH 61 | | | $b^+a^+$ | $p4$ | $\alpha\beta\alpha\beta$ | $\alpha\alpha\alpha\alpha$ | $2D$ | $N$ | H51, W7, S45 |
| IH 62 | | $C_4\ a^+a^+a^+a^+$ | $a^+$ | $p4$ | $\alpha\alpha\alpha\alpha$ | $\alpha\alpha\alpha\alpha$ | $1D$ | $N$ | H31, W6, S40 |
| IH 63 | | | $a^-$ | $p4g$ | $\alpha\alpha\alpha\alpha$ | $\alpha\alpha\alpha\alpha$ | $1D$, $1R$ | $M$ | H32, W32, S39 |
| IH 64 | | $D_1^s\ ab^+cb^-$ | $cb^-a$ | $pm$ | $\alpha\alpha\alpha\alpha$ | $\alpha\beta\alpha\beta$ | 1 | $N$ | H42, W9, S53 |
| IH 65 | | | $ab^-c$ | $pmm$ | $\alpha\alpha\beta\beta$ | $\alpha\beta\gamma\beta$ | 2 | $M$ | H56, W19, S50 |
| IH 66 | | | $cb^+a$ | $pmg$ | $\alpha\alpha\alpha\alpha$ | $\alpha\beta\alpha\beta$ | 2 | $N$ | H43*, W24, S52 |
| IH 67 | | | $ab^+c$ | $cmm$ | $\alpha\alpha\alpha\alpha$ | $\alpha\beta\gamma\beta$ | 2 | $C$ | H47, W28, S51 |
| IH 68 | | $D_1^l\ a^+b^+b^-a^-$ | $b^-a^-$ | $cm$ | $\alpha\alpha\alpha\alpha$ | $\alpha\alpha\alpha\alpha$ | 1 | $N$ | H36, W13, S57 |
| IH 69 | | | $a^+b^+$ | $pmg$ | $\alpha\alpha\alpha\alpha$ | $\alpha\beta\beta\alpha$ | 2 | $C$ | H37, W23, S56 |
| IH 70 | | | $a^-b^-$ | $p4m$ | $\alpha\beta\gamma\beta$ | $\alpha\beta\beta\alpha$ | 4 | $M$ | H61, W34, S54 |
| IH 71 | | | $b^+a^+$ | $p4g$ | $\alpha\beta\alpha\beta$ | $\alpha\alpha\alpha\alpha$ | 4 | $N$ | H52, W31, S55 |
| IH 72 | | $D_2^s\ abab$ | $ab$ | $pmm$ | $\alpha\alpha\alpha\alpha$ | $\alpha\beta\alpha\beta$ | 1 | $C$ | H39, W20, S42 |
| IH 73 | | | $ba$ | $p4g$ | $\alpha\alpha\alpha\alpha$ | $\alpha\alpha\alpha\alpha$ | 2 | $N$ | H34, W33, S41 |

Table 1 (*cont.*)

| List no. (1) | Laves net (2) | Induced group and tile symbol (3) | Adjacency symbol (4) | Crystal-lographic group (5) | Vertex transitivity (6) | Edge transitivity (7) | Aspects (8) | Realiza-tions (9) | References (10) |
|---|---|---|---|---|---|---|---|---|---|
| IH 74 | $[4^4]$ (*cont.*) | $D_2^l\ a^+a^-a^+a^-$ | $a^+$ | *cmm* | $\alpha\alpha\alpha\alpha$ | $\alpha\alpha\alpha\alpha$ | 1 | *C* | H33, W29, S44 |
| IH 75 | | | $a^-$ | *p4m* | $\alpha\beta\alpha\beta$ | $\alpha\alpha\alpha\alpha$ | 2 | *M* | H50, W35, S43 |
| IH 76 | | $D_4\,aaaa$ | $a$ | *p4m* | $\alpha\alpha\alpha\alpha$ | $\alpha\alpha\alpha\alpha$ | 1 | *C* | H30, W36, S38 |
| IH 77 | [4.6.12] | $E\ a^+b^+c^+$ | $a^-b^-c^-$ | *p6m* | $\alpha\beta\gamma$ | $\alpha\beta\gamma$ | 6*D*, 6*R* | *C* | H93, S77, D40 |
| IH 78 | $[4.8^2]$ | $E\ a^+b^+c^+$ | $a^+b^-c^-$ | *cmm* | $\alpha\alpha\beta$ | $\alpha\beta\gamma$ | 2*D*, 2*R* | *C* | H88*, S82, D41 |
| IH 79 | | | $a^+c^+b^+$ | *p4* | $\alpha\alpha\beta$ | $\alpha\beta\beta$ | 4*D* | *N* | H87, HK14, S81, D42 |
| IH 80 | | | $a^-b^-c^-$ | *p4m* | $\alpha\beta\gamma$ | $\alpha\beta\gamma$ | 4*D*, 4*R* | *M* | H89, S79, D44 |
| IH 81 | | | $a^-c^+b^+$ | *p4g* | $\alpha\alpha\beta$ | $\alpha\beta\beta$ | 4*D*, 4*R* | *N* | H86, S80, D43 |
| IH 82 | | $D_1\ ab^+b^-$ | $ab^-$ | *p4m* | $\alpha\alpha\beta$ | $\alpha\beta\beta$ | 4 | *C* | H85, S78 |
| IH 83 | $[6^3]$ | $E\ a^+b^+c^+$ | $b^-a^-c^-$ | *cm* | $\alpha\alpha\alpha$ | $\alpha\alpha\beta$ | 1*D*, 1*R* | *N* | H78, S93, D36 |
| IH 84 | | | $a^+b^+c^+$ | *p2* | $\alpha\alpha\alpha$ | $\alpha\beta\gamma$ | 2*D* | *C* | H81, HK3, S91, D34 |
| IH 85 | | | $a^-b^+c^+$ | *pmg* | $\alpha\alpha\alpha$ | $\alpha\beta\gamma$ | 2*D*, 2*R* | *C* | H80, S90, D37 |
| IH 86 | | | $b^-a^-c^+$ | *pgg* | $\alpha\alpha\alpha$ | $\alpha\alpha\beta$ | 2*D*, 2*R* | *N* | H79, HK21, S92, D35 |
| IH 87 | | | $a^-b^-c^-$ | *p3m1* | $\alpha\beta\gamma$ | $\alpha\beta\gamma$ | 3*D*, 3*R* | *M* | H84, S89, D39 |
| IH 88 | | | $b^+a^+c^+$ | *p6* | $\alpha\beta\alpha$ | $\alpha\alpha\beta$ | 6*D* | *N* | H83, HK11, S88, D38 |
| IH 89 | | $C_3\ a^+a^+a^+$ | $a^-$ | *p31m* | $\alpha\alpha\alpha$ | $\alpha\alpha\alpha$ | 1*D*, 1*R* | *M* | H75, S84 |
| IH 90 | | | $a^+$ | *p6* | $\alpha\alpha\alpha$ | $\alpha\alpha\alpha$ | 2*D* | *N* | H76, S85 |
| IH 91 | | $D_1\ ab^+b^-$ | $ab^+$ | *cmm* | $\alpha\alpha\alpha$ | $\alpha\beta\beta$ | 2 | *C* | H77, S87 |
| IH 92 | | | $ab^-$ | *p6m* | $\alpha\alpha\beta$ | $\alpha\beta\beta$ | 6 | *M* | H82, S86 |
| IH 93 | | $D_3\ aaa$ | $a$ | *p6m* | $\alpha\alpha\alpha$ | $\alpha\alpha\alpha$ | 2 | *C* | H74, S83 |

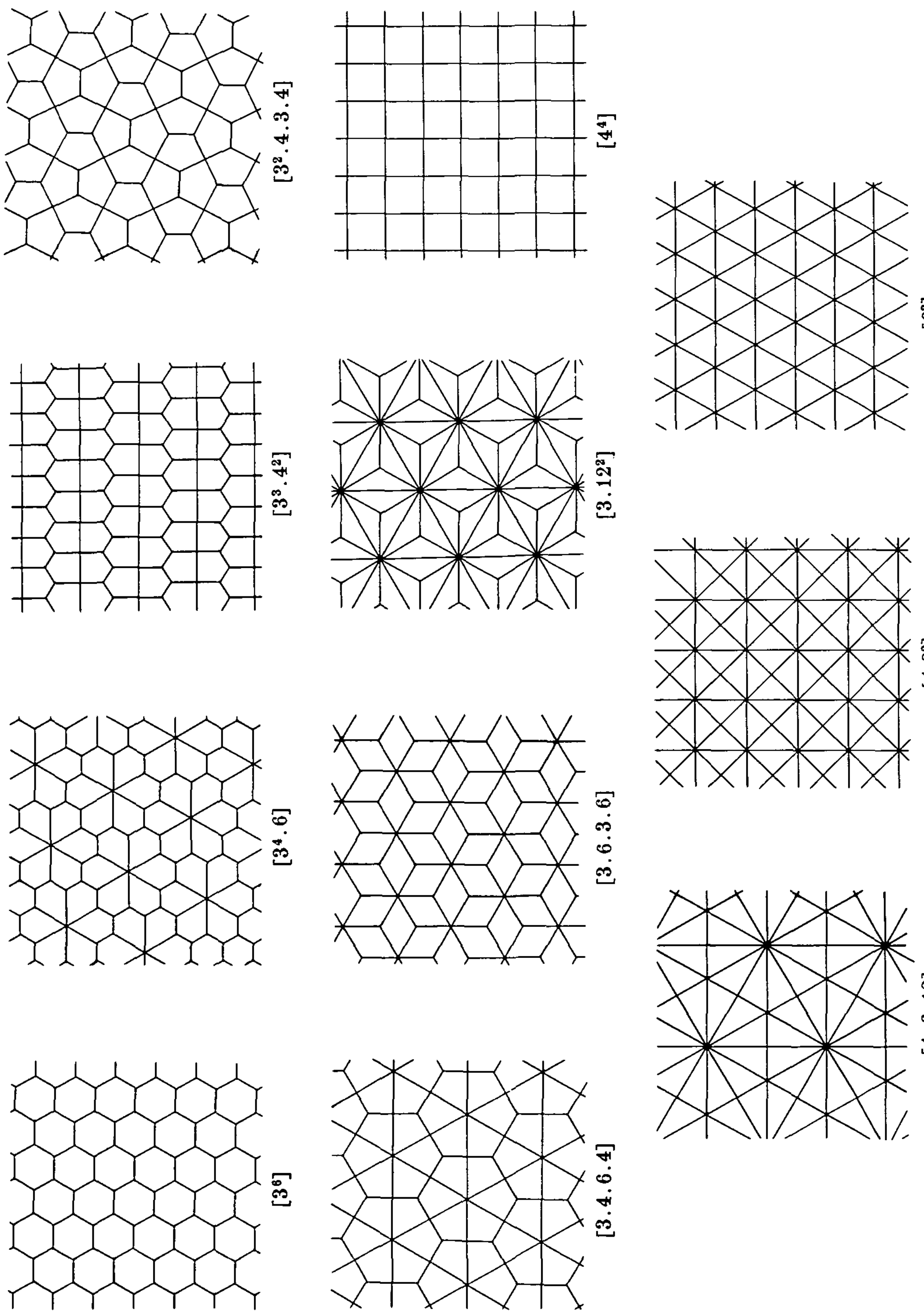
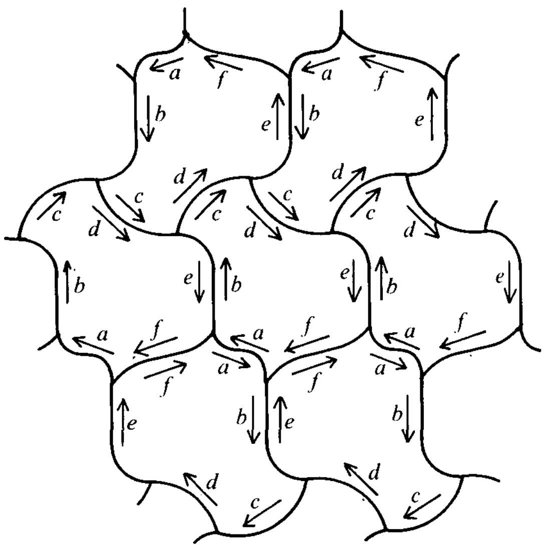
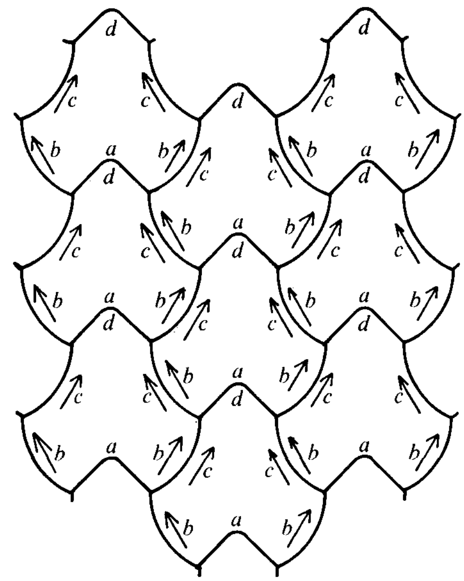



*Fig.* 1. *The eleven Laves nets.*



Tile symbol: $a^+b^+c^+d^+e^+f^+$
Adjacency symbol: $a^+e^+d^-c^-b^+f^+$

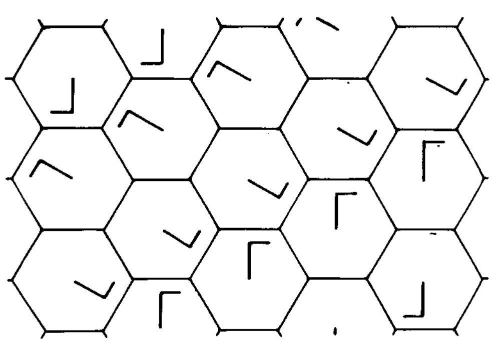
(*a*)



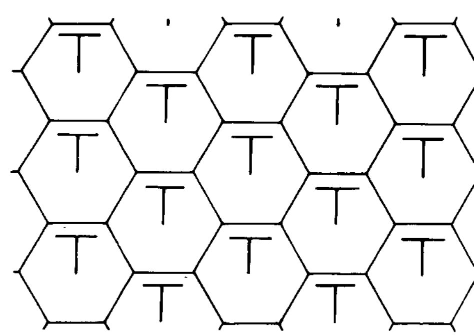
Tile symbol: $ab^+c^+dc^-b^-$
Adjacency symbol: $dc^-b^-a$

(*b*)

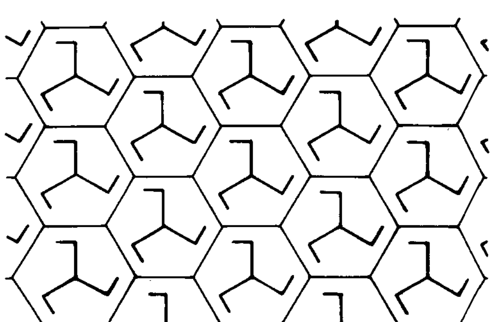
Fig. 2



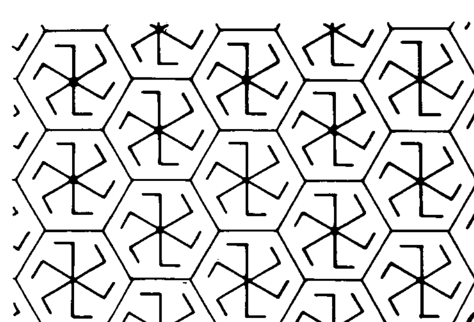
(*a*) IH 5



(*b*) IH 12



(*c*) IH 10



(*d*) IH 11

Fig. 3. Examples of marked isohedral tilings.

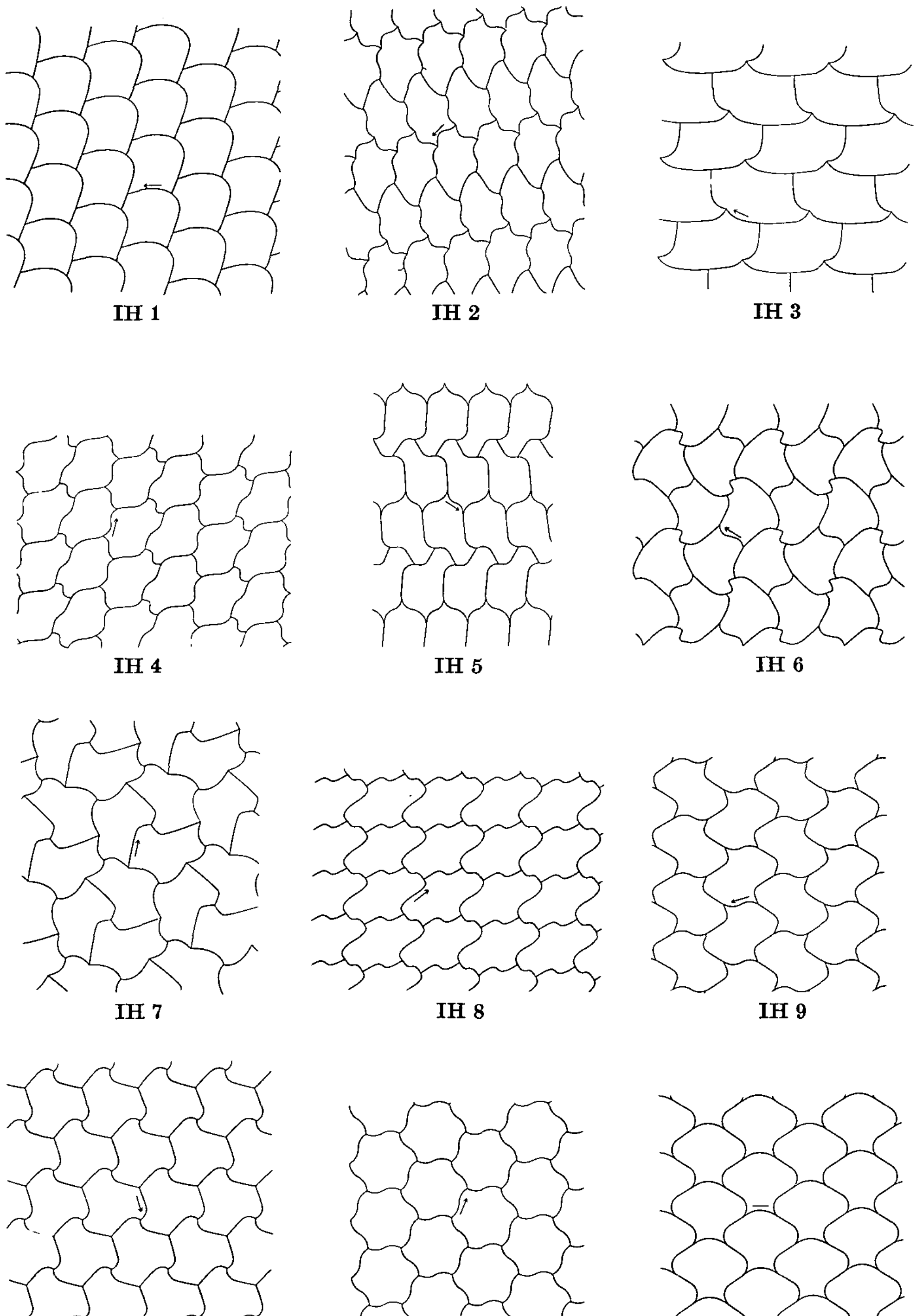
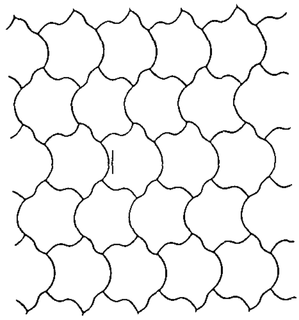
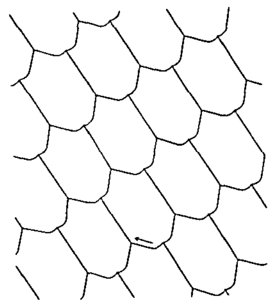


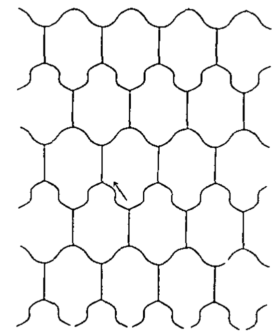

Fig. 4. Examples of the 81 types of isohedral tilings. In each tiling either an $a^+$ edge is marked $\nearrow$, or an $a$ edge is marked $/$.



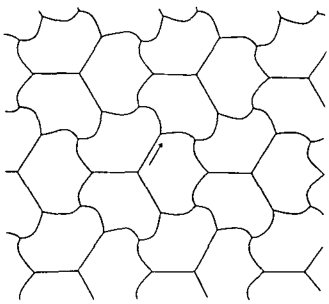
IH 13



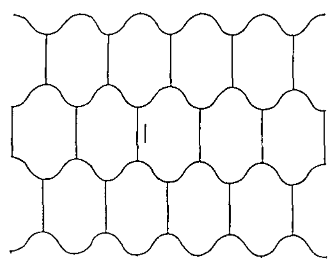
IH 14



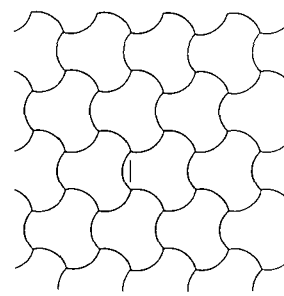
IH 15



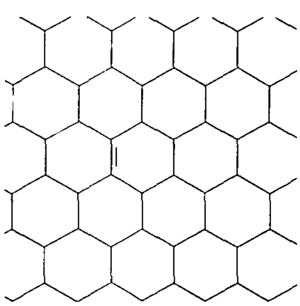
IH 16



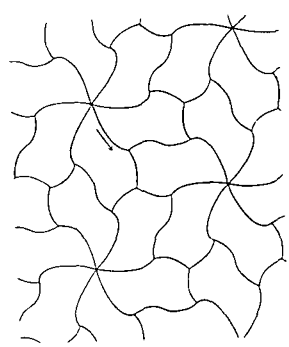
IH 17



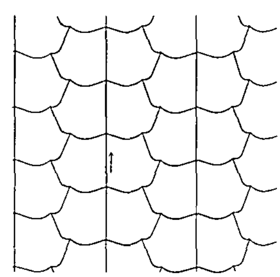
IH 18



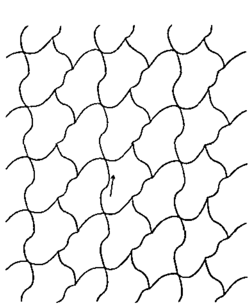
IH 20



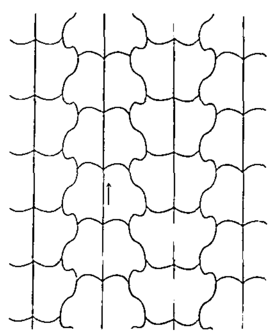
IH 21



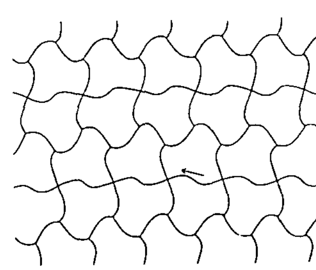
IH 22



IH 23

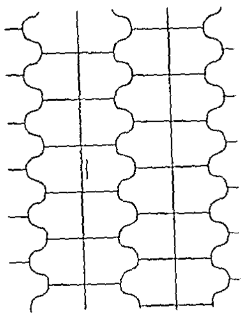


IH 24

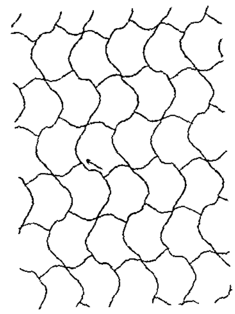


IH 25

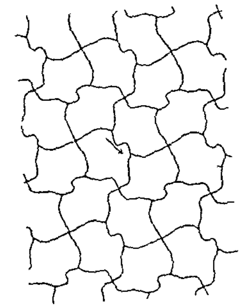
Fig. 4 (*cont.*)



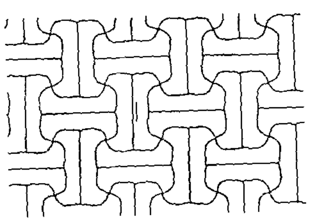
IH 26



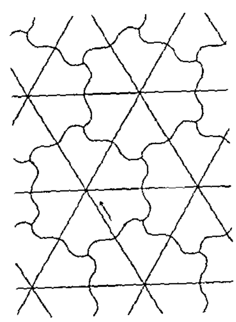
IH 27



IH 28



IH 29



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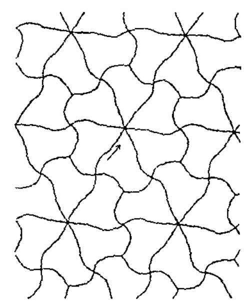

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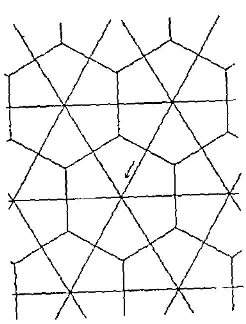
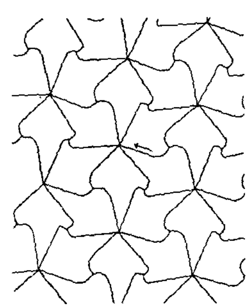


IH 32



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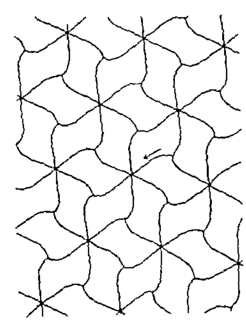

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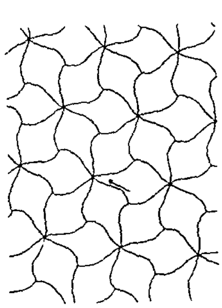

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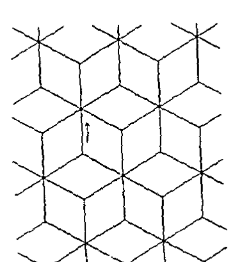
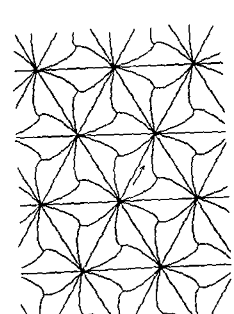
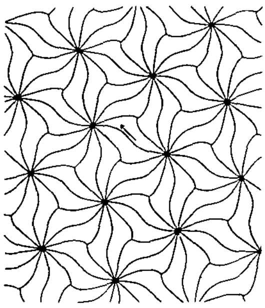


IH 37



IH 38

Fig. 4 (*cont.*)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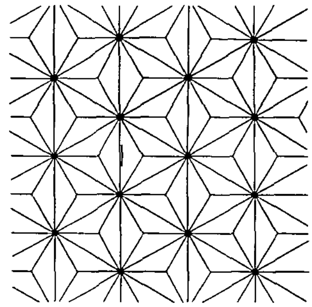
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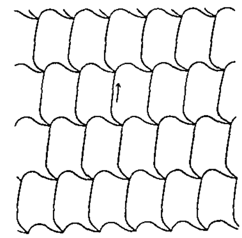
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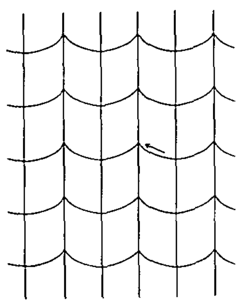
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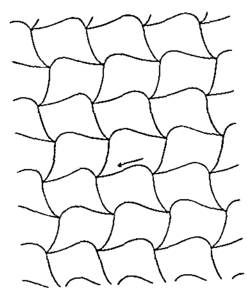
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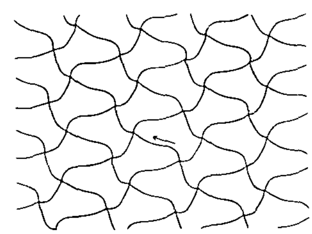
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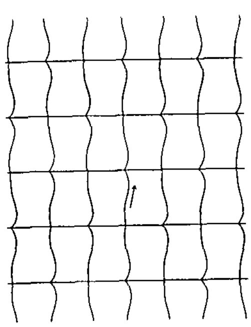
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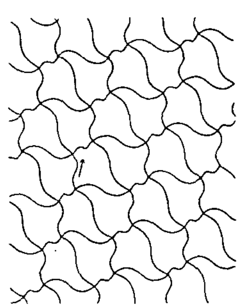
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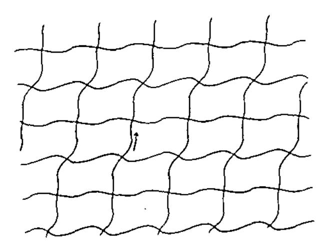
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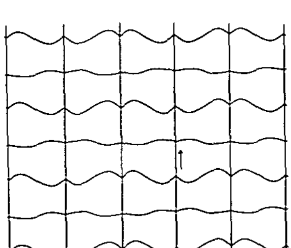
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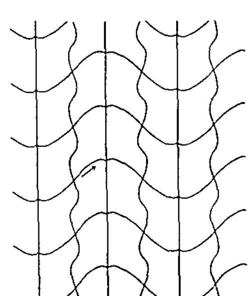
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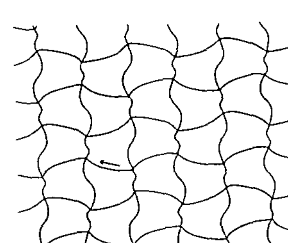
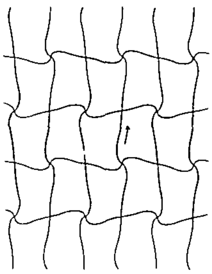

IH 51

Fig. 4 (*cont.*)



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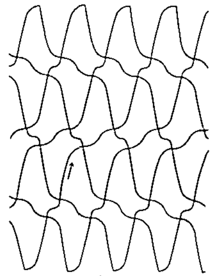

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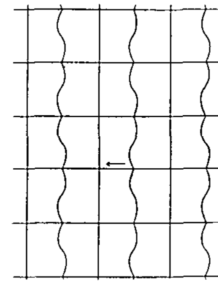

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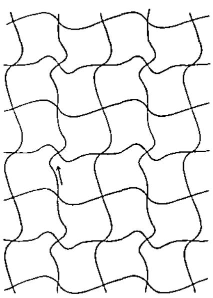

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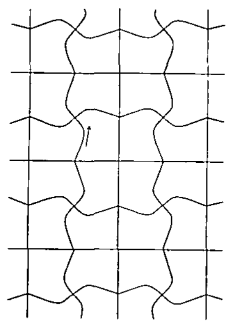

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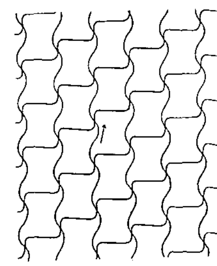

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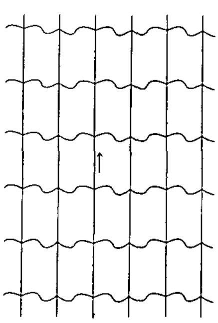
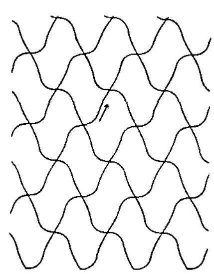


IH 58



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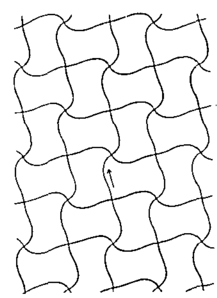
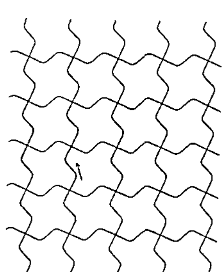


IH 61



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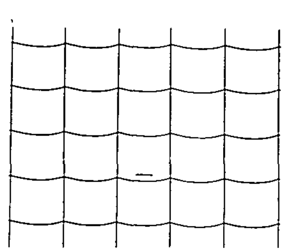

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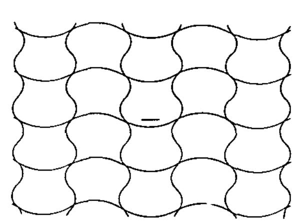
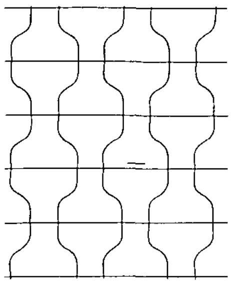


IH 66

Fig. 4 (*cont.*)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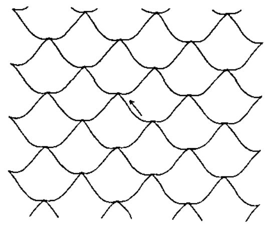
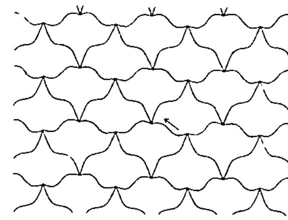

IH 68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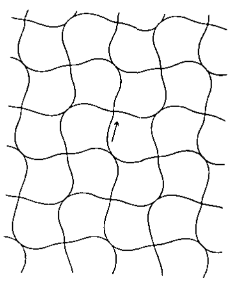
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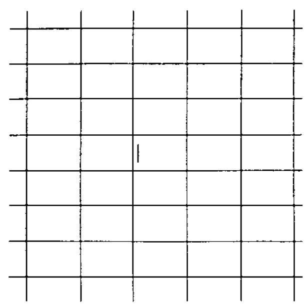
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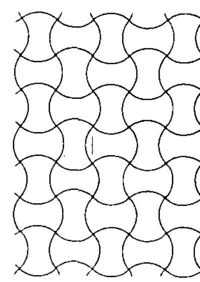
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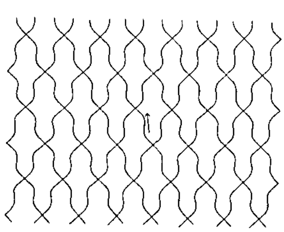
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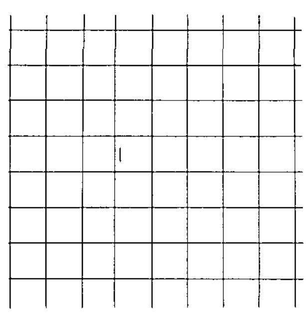
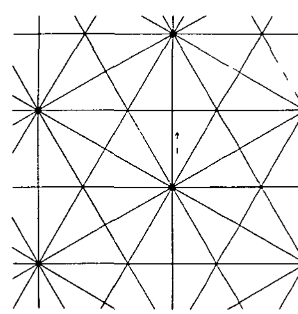

IH 76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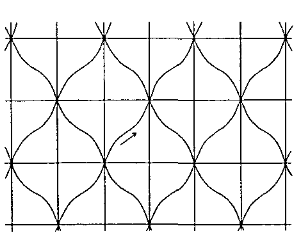
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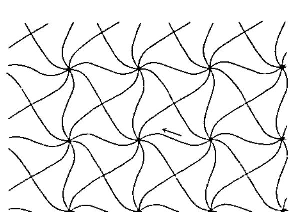
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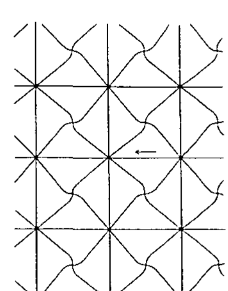
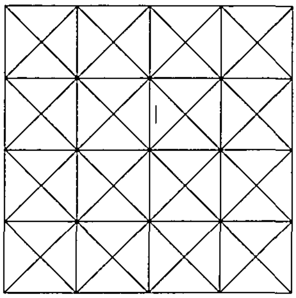
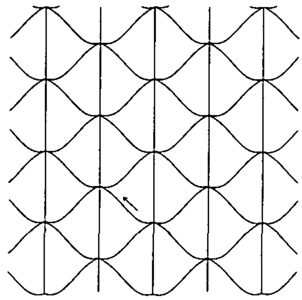

IH 81

Fig. 4 (*cont.*)



**IH 82**



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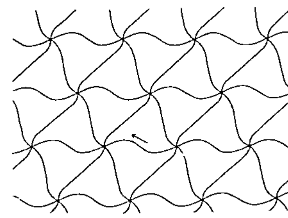

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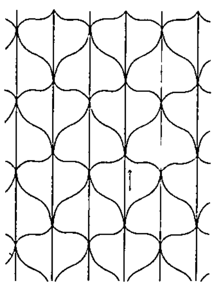

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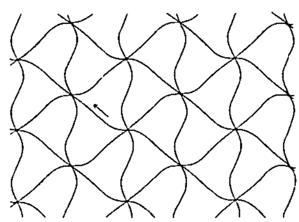

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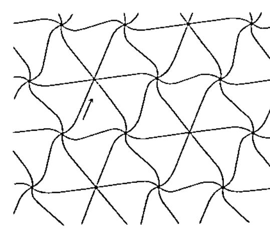

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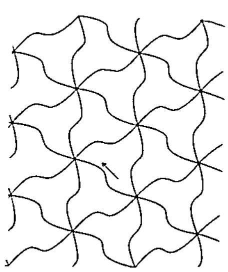

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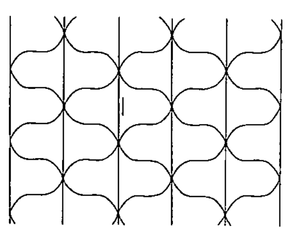

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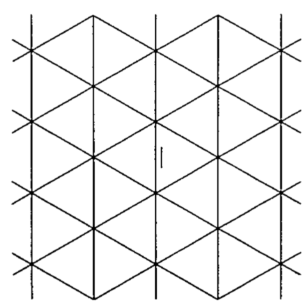

**IH 93**

Fig. 4 (*cont.*)

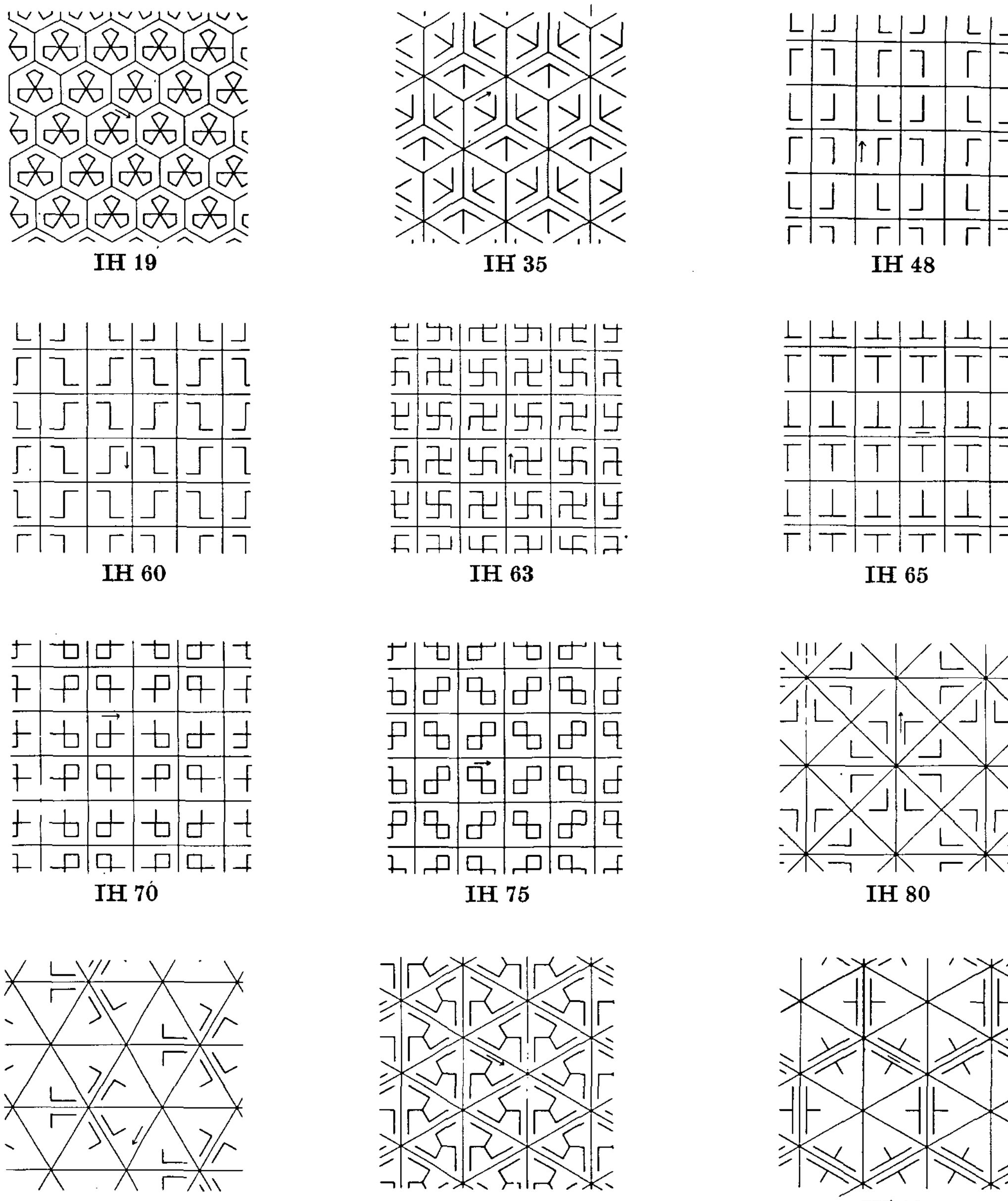


Fig. 5. Examples of the twelve types of marked isohedral tilings that have no unmarked representatives.