Joe said, "there must be two people, at least, with exactly the same number of hairs on their heads." How could Joe be so sure?

B. If an equilateral triangle of side 1 is decomposed into 3 subsets, show that at least one subset must have a diameter greater than or equal to $1/\sqrt{3}$. (The diameter of a plane set is the maximum distance between any two of its points.)

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Patch-determined tilings

B. GRÜNBAUM AND G. C. SHEPHARD

A *tiling* of the plane is a family of sets, called *tiles*, that cover the plane without gaps or overlaps. Usually we are concerned with tilings whose tiles are of a small number of different shapes; familiar examples are the regular and uniform tilings (see, for example, [1]).

Although the mathematical theory of tiling is very old, it still contains a rich supply of interesting and challenging problems. The purpose of this article is to describe some of these. In many cases the mathematical aspect is enhanced by the aesthetic appeal of the resulting tilings.

Let S be any set in the plane, and T a given tiling. Then we denote by $T(S)$ the set of tiles which have non-empty intersection with S. The set $T(S)$ will be called a *patch* of tiles, the name being suggested by the fact that the patches with which we shall be concerned here consist of just a few contiguous tiles. Now if we are given $T(S)$ it will clearly be possible in at least one way (namely T itself) to 'adjoin' tiles, of the same shapes as those used in T, to $T(S)$ in such a way as to complete a tiling of the whole plane. But, in general, there will be many ways of doing this. If the tiling may be completed in a *unique* way, then we shall say that it is *completely determined* by the patch $T(S)$. The problem with which we shall be concerned is to find examples of very small patches which completely determine a tiling.

At first it may seem surprising that such patches can exist, apart from trivial cases such as the following. If there is a unique way in which the plane can be tiled with a given shape (for example there is a unique tiling by regular hexagons of a given size) then a patch consisting of just *one* tile will completely determine the tiling. Other examples of tiles with the same uniqueness property appear in Fig. 1. On the other hand, in the case of

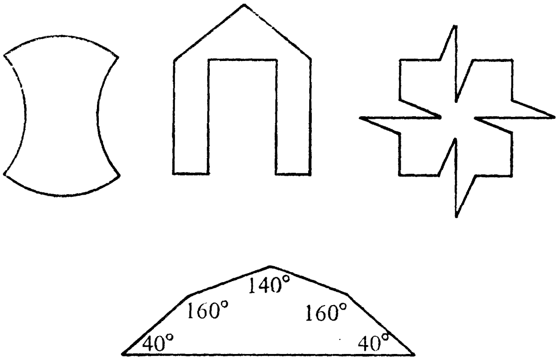


FIGURE 1.

squares, there is an uncountable infinity of distinct tilings which can be produced by displacement of rows (or columns) as in Fig. 2. In this case, the tiling cannot be completely determined by any finite patch of tiles.

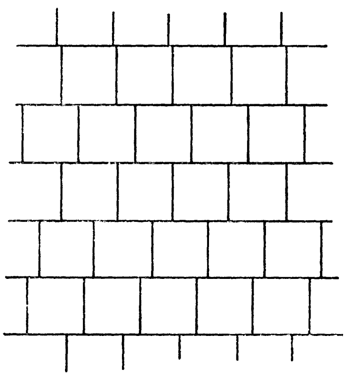
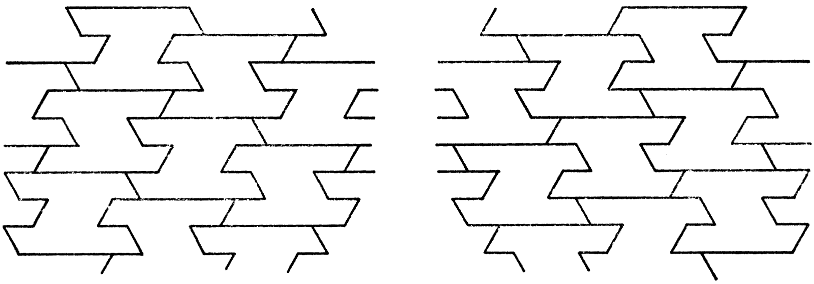


FIGURE 2.

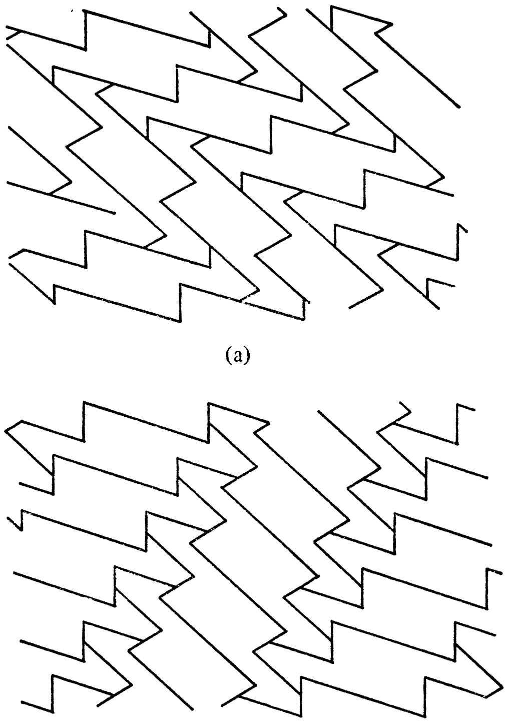
Now consider the tilings of Fig. 3. Here we use only one shape of tile, but it is necessary to 'turn the tile over', that is, to use reflective congruence. The tile has ten equal sides and two which are four times as long; it is easily constructed as the union of twenty equilateral triangles. One of the tilings is a mirror-image of the other, and either is completely determined by a patch consisting of any two contiguous tiles. A similar situation occurs with the



(a) (b)

FIGURE 3.

tilings of Fig. 4. Again only two tilings exist, but here they are quite distinct and each coincides with its mirror-image. It is simple to show that no other tilings are possible with this shape, and again, each is determined by a patch



(a)

(b)

FIGURE 4.

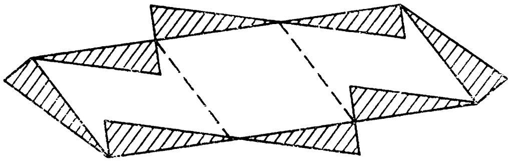
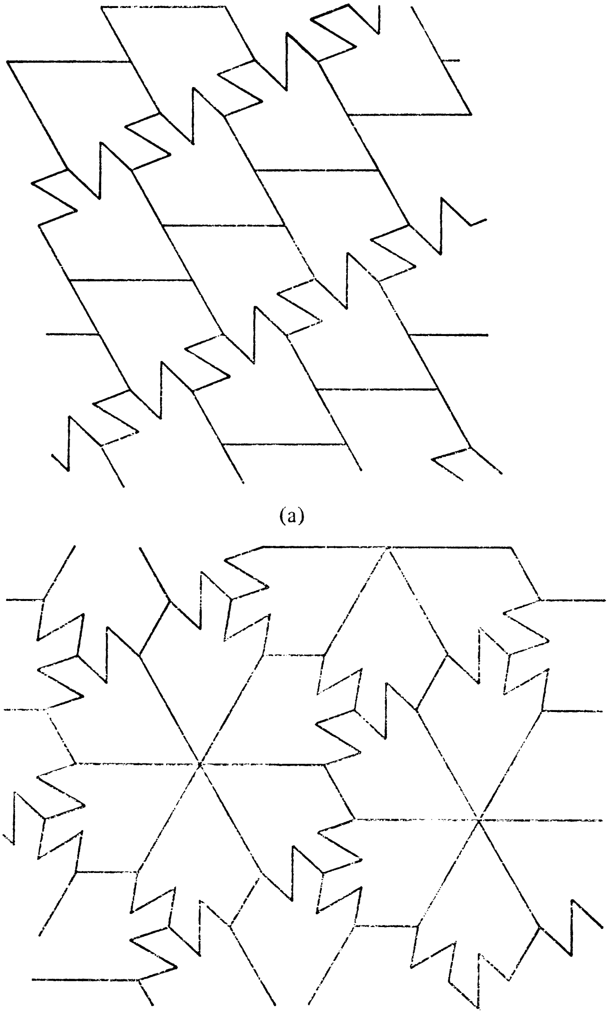


FIGURE 5.



(a)

(b)

FIGURE 6.

consisting of two contiguous tiles. In Fig. 5 we show how this tile is constructed. It is based on a parallelogram of which one pair of edges is three times as long as the other. To this we add, or cut away, triangles, as shown. The exact shape of the triangle is not important—we may replace it by a curve or polygonal arc if we wish—so long as it does not possess any symmetries. The triangles force one or other of the two patterns and prevent

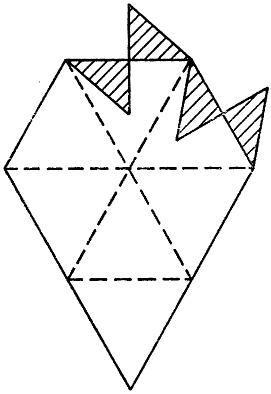


FIGURE 7.

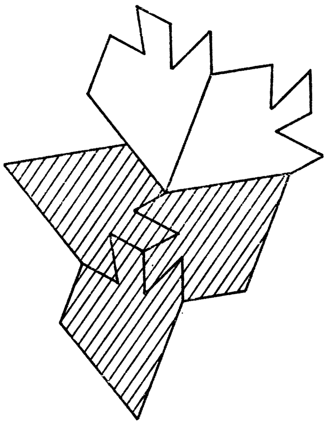
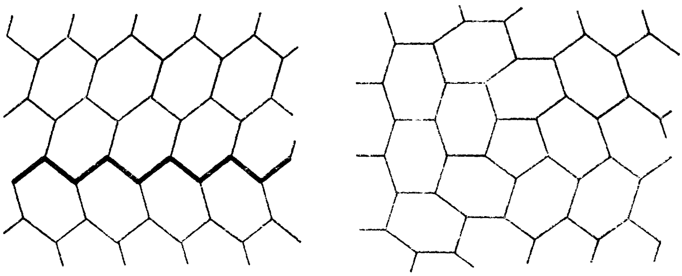


FIGURE 8.

any other arrangement of the parallelograms, such as laying them out in parallel rows.

The tilings of Fig. 6 are, in a way, more interesting. The tile is based on a set of seven equilateral triangles, and two edges of the resulting pentagon are replaced by Z-shaped arcs as shown in Fig. 7. Here we use only one shape of tile and reflections are *not* allowed. It can be shown that only two distinct tilings are possible, those of Fig. 6, and that the tiling is completely determined by a patch consisting of two contiguous tiles. A curious feature of this tiling is that as one builds it up, tile by tile, starting from the original patch, it often happens that there appears to be a choice how the next tiles are to be laid. Thus in Fig. 8, let us assume that the shaded tiles are already in place. Then we can insert two further tiles into the 'gap' as shown. On the other hand, this same gap can also be filled with a single tile. This choice is, in fact, spurious. If one lays the tiles the wrong way, then several steps later it will be found impossible to complete the tiling. So there is no real choice open to us, and that is why the tiling is completely determined.



(a) (b)

FIGURE 9.

Now consider tilings in which two shapes of tile are used. In Fig. 9 we show two tilings using (i) regular pentagons, and (ii) hexagons with angles 108°, 108°, 144°, 108°, 108°, 144°. All the edges are of the same length. If we use hexagons alone, then we can arrange them as in Fig. 9a. It will be noticed that the zigzag line (thickened in the figure) separates rows of hexagons tilted one way from rows tilted the other way. These rows of hexagons can be varied in width and number, leading to an uncountable infinity of different tilings. On the other hand, if one of the pentagonal tiles is used, then there are only two possible tilings, namely Fig. 9b and its mirror-image. No tilings exist with more than one pentagon. Hence a patch consisting of the pentagon and one of the contiguous hexagons completely determines the tiling. This example is of interest partly because of the simple shapes of tiles and partly because a single tile (the pentagon) *nearly* determines the tiling completely—it determines it within a reflection.

Our final example also uses two shapes of tile. These are based on a regular pentagon and a rhomb with angles 18°, 162°, 18°, 162°. In each case the edges are replaced by a 'saw-tooth' leading to the shapes of tiles shown in Fig. 10. As in the previous example, there is an uncountable infinity of tilings using the 'rhombs' alone, but only two if a 'pentagonal' tile is used, and none with two or more 'pentagons'. Hence, as before, a patch consisting of the 'pentagon' and an adjacent tile completely determines the tiling. The reason why we introduce this example is because of its striking appearance as a five-armed spiral. It is similar in construction, in some respects, to Goldberg's "vortex tesselations" [2]. We remark that if we remove the saw-tooth and use pentagons and rhombs with straight-line edges, then a much larger patch is required to determine a tiling completely.

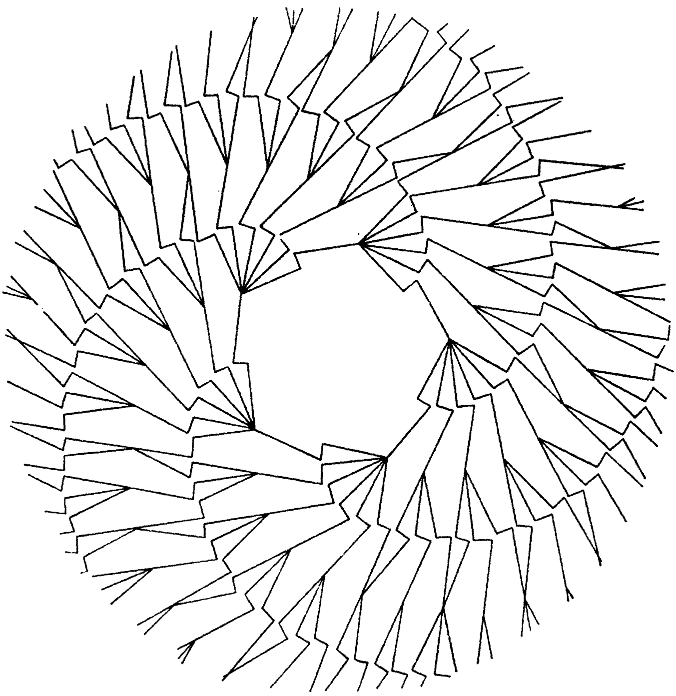


FIGURE 10.

There are many open questions concerning the problems we have considered here. Examples have been given of a shape of tile which admits precisely two different tilings. Do examples exist for which precisely n tilings are possible, where $n = 3\dagger, 4, 5, \ldots$? What restrictions are there on the values of n both in the case where the tiles are of one shape, and in the cases where 2, 3, ..., different shapes are used? Can examples be found with the properties mentioned in this note if we restrict attention to tiles which are convex polygons? The extent of our mathematical ignorance in this area is shown by the fact that the number of different types of tiling by a single

† Since this was written, an example has been found which will tile the plane in exactly *three* different ways.

shape of convex pentagon is still not known. Certainly the list given in [3] is not complete; see [4].

The only previous treatments in the literature of the problem of extending patches of tiles appear to be in early papers of Lévy [5] and Sommerville [6]. They, however, were concerned with rather different problems using only regular polygons.

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Classifying triangles and quadrilaterals

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Among the definitions at the beginning of Book I in Euclid's *Elements* [1] there are several that pick out special kinds of triangles and quadrilaterals. In his commentary [2] on Book I, Proclus observes that Euclid classifies triangles in two ways: firstly 'by sides' into equilateral, isosceles and scalene triangles; and secondly 'by angles' into right-angled, obtuse-angled and acute-angled triangles. With regard to quadrilaterals, Proclus ([2], p. 134) attributes to Posidonius the classification scheme on p. 39, which is to be found in Heath's edition of the *Elements* ([1], p. 189). Thus the ancient classification of triangles and quadrilaterals produces three (or six) species of triangles and seven species of quadrilaterals.

My plan here is to consider the problem of classifying triangles and quadrilaterals afresh, using simple ideas of topology and symmetry. The results