EDITED BY RICHARD GUY

In this department the MONTHLY *presents easily stated unsolved problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada T2N 1N4.*

CAN ALL TILES OF A TILING HAVE FIVE-FOLD SYMMETRY?*

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It is well known that the plane can be tiled by equilateral triangles, by squares, or by regular hexagons, but that there exists no such tiling by regular pentagons. More generally it is not hard to see that the plane cannot be tiled by congruent figures each with 5-fold symmetry, that is, admitting as symmetries rotations about a point through angles of 72°, 144°, 216° and 288°. However, if one does not insist that the tiles are congruent, and one allows a variety of different shapes and sizes, then the question of tiling by tiles with 5-fold symmetry becomes much more interesting and leads to several open problems.

We recall that one of the basic results in geometric crystallography is the *crystallographic restriction*, which asserts that in the Euclidean plane no figure with a discrete symmetry group can have more than one center of five-fold rotational symmetry. Similar assertions can be made about k-fold rotations with $k \geqslant 7$. These facts lead to the well-known result that if a planar figure admits translational symmetries but does not admit arbitrarily short translations as symmetries, then the only possible rotational symmetries are k-fold rotations with $k = 2$, 3, 4 or 6 (see, for example, Buerger [**3**, p. 33], Coxeter [**5**, Sect. 4.5], Fejes Tôth [**9**, Sect. I.1.4]).

However, the crystallographic restriction does not prove the impossibility of a tiling in which each individual tile has 5-fold symmetry. Fruitless efforts by mathematicians to construct such tilings go back at least to Kepler [**11**]. Fig. 1 shows some of Kepler's attempts; he notes that in each case a tiling can be obtained only at the price of introducing "monsters" (such as the "fused decagons" in Fig. 1a) which do not have 5-fold symmetry. Explanations of Kepler's tilings and variants of them have been discussed by several authors (Caspar [**4**, p. 374], Bindel [**1**], Eberhart [**7**], Grünbaum & Shephard [**10**, Sect. 2.5]). Even earlier attempts at constructing tilings with tiles that have 5-fold symmetry are discernible in Islamic art (see, for example, Bourgoin [**2**], Critchlow [**6**], Wade [**14**], El-Said & Parman [**8**]).

It is therefore somewhat surprising to find that such tilings can easily be constructed by a simple inductive procedure. In the example indicated in Fig. 2, the tiles used are congruent either to the pentagon A shown in Fig. 2a, or to the nonconvex 20-sided polygon B shown in Fig. 2b, or to a polygon 3^nB similar to B in ratio 3^n, where $n = 1,2,3,\ldots$. The construction of the tiling, indicated in Fig. 2c, is based on the fact that five copies of A yield with B a pentagon 3A similar to A in ratio 3; five of these and 3B yield a pentagon 3^2A, etc.

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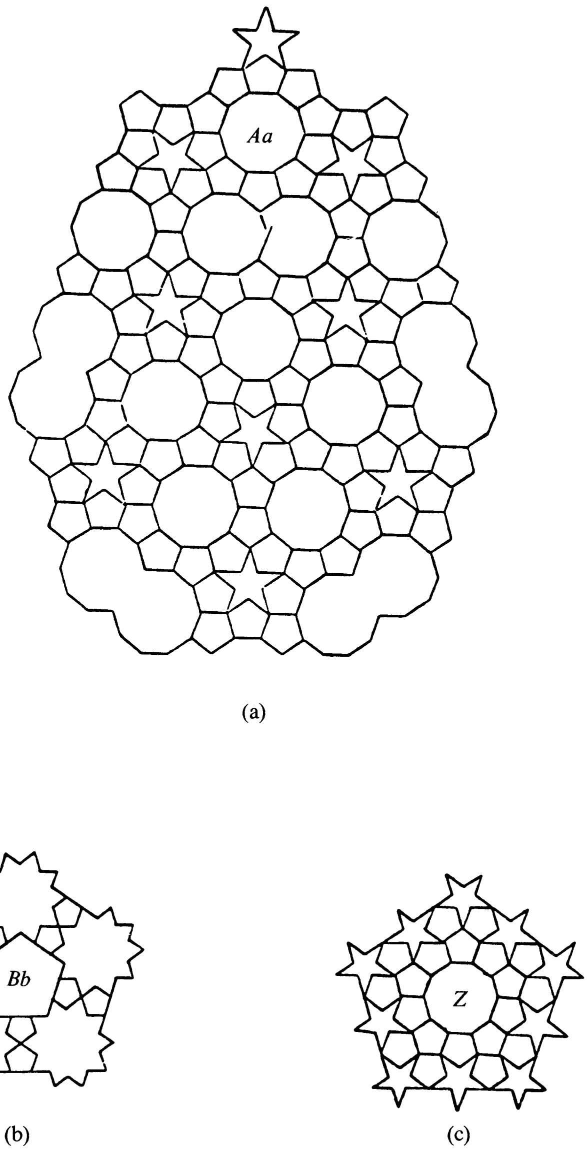

FIG. 1. Some of the patches devised by Kepler in 1619 in attempts to tile the plane by tiles with 5-fold symmetry.

The tiles in this example are not uniformly bounded in diameter. It seems that the answer to the question changes if the diameters of the tiles cannot grow without bounds. To make this problem precise from now on we restrict attention to tilings in which the tiles are closed topological disks of diameter at most 1, covering the plane without gaps or overlaps of their interiors. We conjecture that *the Euclidean plane admits no tiling in which each tile has five-fold symmetry*. If true, this conjecture is precariously balanced against various indications that seem to lend support to the opposite view. For example, tilings by congruent regular pentagons are possible on the sphere, in the elliptic plane and in the hyperbolic plane. The plane can be tiled by affinely regular pentagons, all mutually similar and of only two sizes (see Fig. 3). Many kinds of equilateral convex pentagons—some very close in shape to regular pentagons—can be used to tile

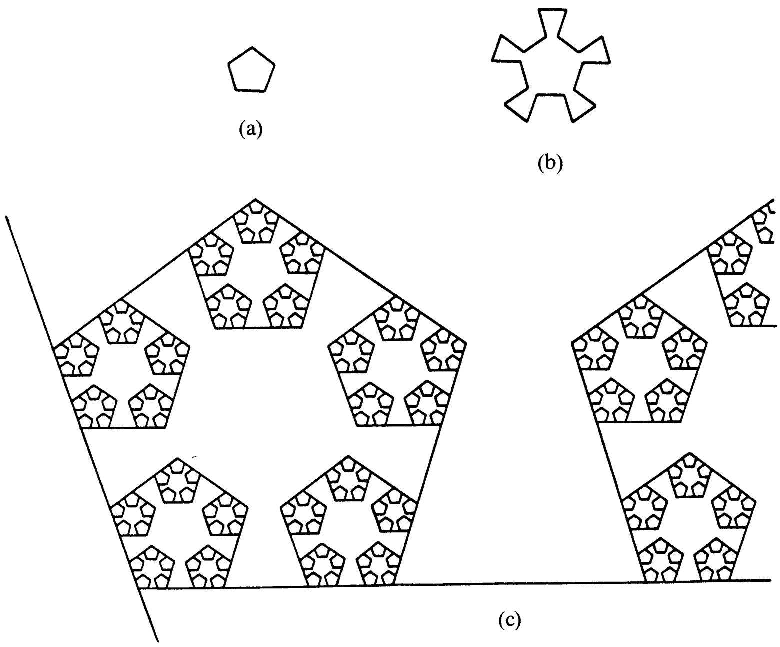

FIG. 2. The construction of a tiling (c) in which each tile is congruent either to the regular pentagon in (a), or to the 20-sided polygon in (b), or to an enlarged version of this polygon.

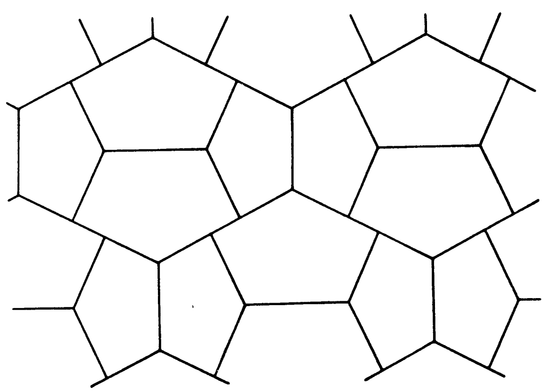

FIG. 3. A tiling of the plane by affinely regular pentagons, all mutually similar and of only two sizes.

the plane by congruent copies (see the surveys in Schattschneider [**12**], [**13**]). Probably even more telling are examples like the one in Fig. 4, which *appears* to show a tiling in which each tile is a regular pentagon. Actually, the pentagons in Fig. 4 form a Cantor-type set, and do not cover the plane; hence they do not form a tiling in the sense considered here.

Large regions of the plane can be covered, in many ways, by *patches* of tiles, in which each tile has 5-fold symmetry or even reflective 5-fold symmetry. In order to compare in a meaningful way the sizes of such patches, various measures can be used. Possibly the most appropriate one is the ratio ρ of the diameter of the largest circular disk covered by the patch to the diameter of the smallest circular disk that can cover each of the tiles in the patch. By adding around Kepler's patch in Fig. 1b ten of the larger pentagons and fifteen of the smaller ones, a patch with $\rho \approx 3.1$ is

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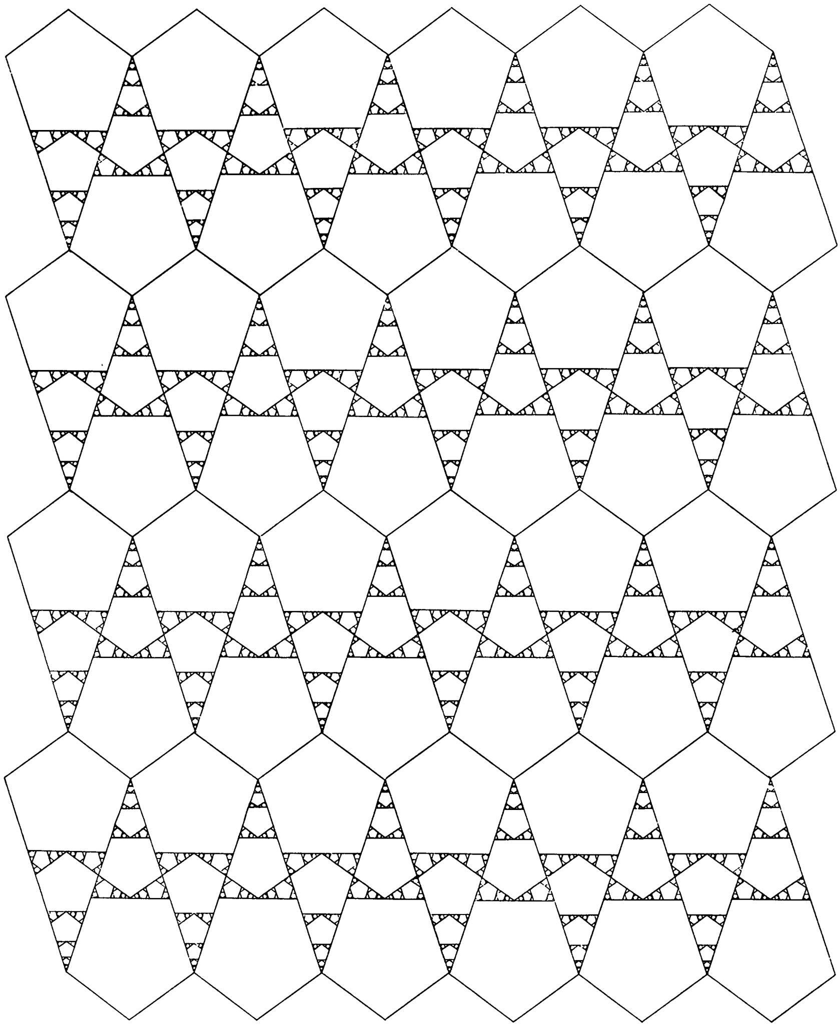

FIG. 4. A tiling which appears to consist of regular pentagons (the triangular spaces being "filled" by smaller and smaller pentagons). However, it can be shown that the union of the pentagons is a set of Cantor type, which does not cover the plane.

obtained. Similarly, the patch in Fig. 1c can be extended (by adding ten pentagons) to a patch with $\rho \approx 2.6$, while deleting from Fig. 1a the tiles which do not have 5-fold symmetry leads to a patch with $\rho \approx 3.9$.

The largest known patch made up of tiles which are regular pentagons, decagons, and pentagrams is shown in Fig. 5. Here $\rho = \sqrt{(83 + 37\sqrt{5})/8} \approx 4.6$. Patches with larger values of ρ can be constructed using the process of "decomposition"—a method which has proved useful in other tiling problems (see, for example, [**10**, Chapter 10]). This may be explained as follows. Let the edge-length of the small pentagons in Fig. 5 be denoted by s, and let each vertex of this patch be the center of a regular decagon with edge-length s/τ^3, where $\tau = \frac{1}{2}(1 + \sqrt{5})$ is the golden section ratio. Then the parts of the tiles outside these decagons can be partitioned into smaller tiles, each with 5-fold symmetry. In Fig. 6 we show this process applied to (one tenth of) the patch of Fig. 5. To obtain the whole patch we have to mirror the sector shown in the dotted lines and

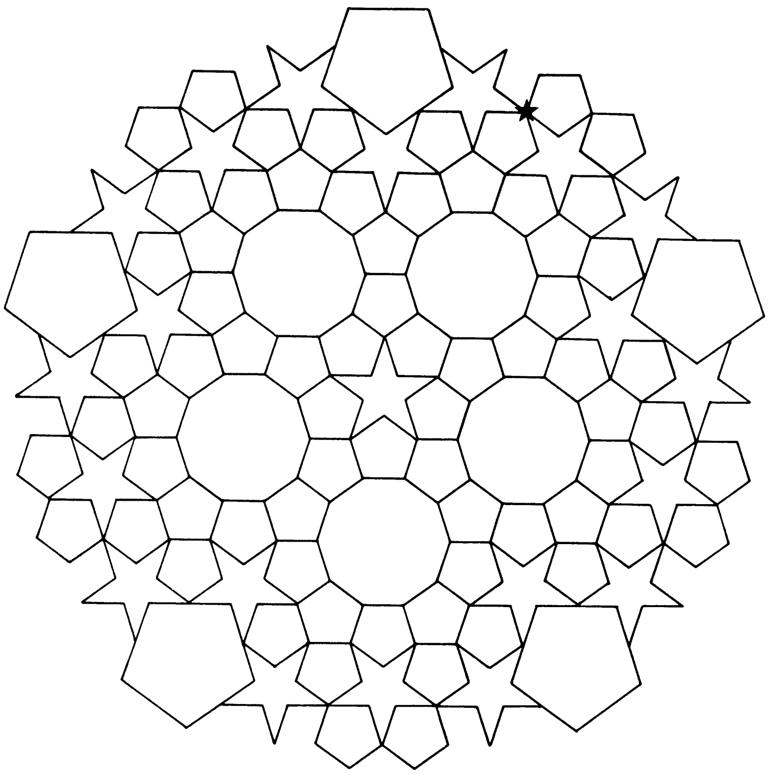

FIG. 5. A patch with the largest known $\rho \approx 4.55157\ldots$ among patches consisting of regular pentagons, decagons and pentagrams. The largest circle covered by the patch passes through the point marked by an asterisk and through its nine homologues.

their images. The original (larger) tiles are indicated by thin lines, and the tiles obtained by the decomposition process are shown by thicker lines.

Decomposition increases ρ for two reasons: the diameters of the tiles are decreased and also the patch can be extended a little using the smaller tiles. The patch in Fig. 6 has $\rho \approx 8.6$.

Repeating the process of decomposition leads to patches with larger values of ρ; in this way we can obtain patches with up to $\rho \approx 13$. Using other starting patches, we can get as high as $\rho \approx 38$. We do not know whether this is anywhere near the maximum; indeed, we cannot even prove the existence of a value $\omega < \infty$ such that $\rho \leqslant \omega$ for each patch of the kind under consideration. We conjecture that no such ω exists.

Similar problems can be raised concerning tilings and patches in which each tile has k-fold symmetry for some $k \geqslant 7$, or for any combination of such tiles and tiles with 5-fold symmetry.

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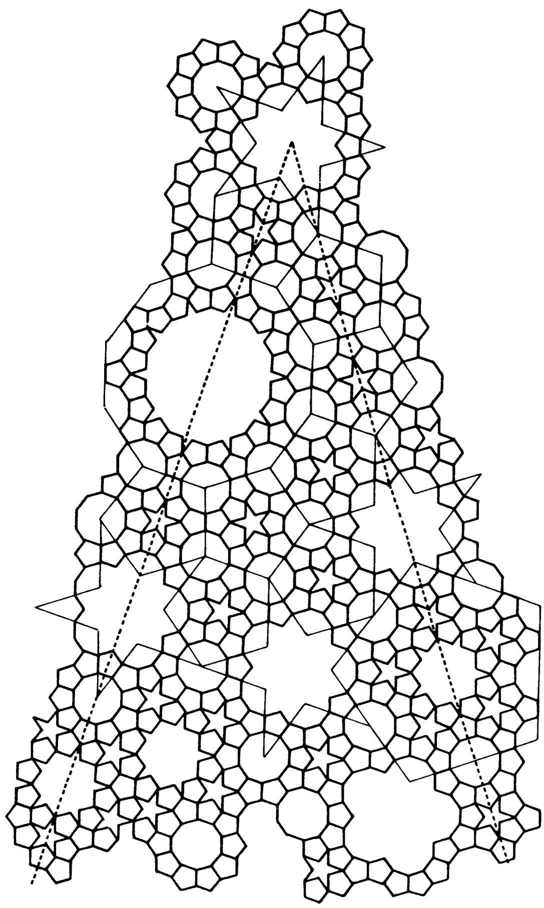

FIG. 6. A patch with $\rho \approx 8.6$ obtained by "decomposition" from the tiling in Fig. 5 and addition of some tiles around the boundary.

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MISCELLANEA

83. Mathematics, like Dialectics, is an organ of the inner, higher mind; in practice, it is an art like eloquence. In both, nothing counts but the form; the content is irrelevant.

—J. W. v. Goethe, *Maximen und Reflexionen*, no. 605.