

3-connected configurations (n_3) with no Hamiltonian circuit

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Abstract. For more than a century there have been examples of (n_3) configurations without a Hamiltonian circuit. However, all these examples were only 2-connected. It has been believed that all geometric 3-connected configurations (n_3) admit Hamiltonian circuits. We present a 3-connected configuration (25_3) of points and lines in the Euclidean plane which has no Hamiltonian circuit.

1. Introduction. A *combinatorial configuration* (n_3) is a collection of n distinct symbols, which we call "points", together with n (unordered) triplets of "points". Each triplet is called a "line", and we shall assume throughout that each "point" appears in precisely three "lines", and that each pair of "points" appears in at most one "line". We shall also say that a "line" and each of its three "points" are mutually incident. "Points" and "lines" are collectively known as *elements* of the configuration. A configuration is *connected* if any two of its elements are the end terms of a finite chain, any two adjacent elements in which are mutually incident. A configuration is *k-connected* if it contains at least $k+1$ elements and it remains connected even if the deletion of $k-1$ or fewer of its elements cannot disconnect it.

A combinatorial configuration C is *geometrically realizable* if there exists a mapping of the elements of C into the Euclidean plane such that each "point" is mapped to a point, each "line" is mapped to a (straight) line, and two incident elements of C are mapped to a point and a line that contains the point.

In the sequel we shall dispense with the quotation marks when discussing the elements of a combinatorial configuration. Whether the intention is to combinatorial or to geometric configurations will be clear from the context, and will be explicitly stated when required for clarity.

We shall say that a configuration C is *Hamiltonian*, or that it has a *Hamiltonian circuit*, if it is possible to arrange all elements of C in a cycle in which each element appears precisely once, and pairs of adjacent elements are mutually incident. Figure 1 shows the three non-isomorphic configurations (9_3) , each with a Hamiltonian circuit.

The study of configurations was very popular during the last quarter of the 19th century, but only few writers were concerned with them during most of the 20th century. Interest has picked up during the last 20 or so years, with several old problems solved and some of the old results shown to be invalid.

In his 1881 paper [7], S. Kantor determined that there are ten non-isomorphic configurations (10_3) . These he determined combinatorially, and then presented diagrams purporting to show geometric realizations of all ten. (Schröter [10] showed in 1889 that, in fact, only nine of these ten are geometrically realizable.) More relevant to the present topic is the fact that Kantor, on page 1305 of [7], states as a theorem that every (n_3) configuration has a Hamiltonian circuit. Kantor gives no proof, simply stating that this is a consequence

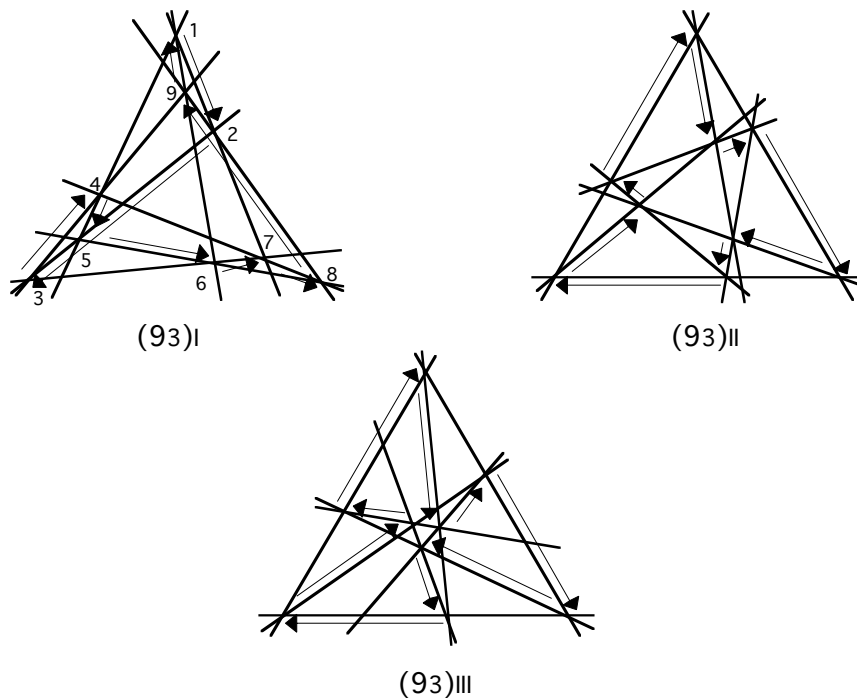


Figure 1. Hamiltonian circuits in the three configurations (9_3) .

of a theorem of Listing — but without giving any details or references. (I tried hard, but was not successful in tracking this down.) As illustration of the concepts, Kantor lists Hamiltonian circuits for each of the ten configurations (10_3) . Schroeter [10] repeats Kantor's claim, mentions Listing again without reference, and provides Hamiltonian circuits for the nine geometrically realizable configurations (10_3) .

Neither Kantor nor Schroeter observe that a necessary condition for the existence of a Hamiltonian circuit in a configuration is that the configuration is connected. This was stressed by Steinitz [11] in 1897. Steinitz also mentions that he verified that all 31 types of configurations (11_3) are Hamiltonian. This was extended by Gropp [5] to the statement that all connected combinatorial configurations (n_3) with $n \leq 14$ are Hamiltonian.

As the main result of [11], Steinitz produces a 2-connected combinatorial configuration (28_3) which is non-Hamiltonian. A geometric 2-connected non-Hamiltonian configuration (22_3) was found in [2]; this is the smallest known geometric counterexample to Kantor's claim.

As recently as 2002, in a course I gave at the University of Washington, and in various talks, I believed that the following is valid:

Conjecture. All 3-connected geometric configurations (n_3) are Hamiltonian.

However, more recently I found a counterexample, and the present note presents this result.

Theorem. For every $n \geq 35$ there exist geometric configurations (n_3) that are 3-connected but have no Hamiltonian circuits.

This will be established by exhibiting a particular non-Hamiltonian 3-connected geometric configuration of 25 points and 25 (straight) lines, and then showing how it can be used to construct infinitely many such configurations. The construction of the 3-connected non-Hamiltonian (25_3) configuration proceeds in several steps. First, a graph devised by Georges [4] is interpreted as the Levi graph of a non-Hamiltonian 3-connected *combinatorial* configuration. Using a theorem of Steinitz [12], this configuration is shown to be geometrically realizable in the Euclidean plane by points and lines, all of which except one are straight. This is graphically carried out by Geometer's Sketchpad software, and a continuity argument is used to show the existence of a realization in which *all* lines are straight.

2. Georges' graph and configuration. Our construction starts with the non-Hamiltonian, 3-connected, bipartite graph shown in Figure 2. This

graph was constructed by Georges in [4], using a smaller graph devised by M. N. Ellingham and J. D. Horton in [3]. The main result of the construction that is relevant to the present aim is the fact that the Georges graph has girth 6. Since it is bipartite, it is the *Levi graph* of a combinatorial configuration. (The concept of Levi graph was introduced by W. F. Levi in the little known and hard-to-find book [9]. It gained wider attention, and the name, through Coxeter's paper [1].)

We recall that the Levi graph of an incidence structure associates a "black" point to every point of the structure, and a "white" point to every line. A black point and a white point determine an edge of the graph if and only if the point and line of the incidence structure are incident. A bipartite graph of girth at least six is the Levi graph of a combinatorial configuration.

I had known about Georges' paper from the review in *Mathematical Reviews*. However, nothing there indicated that the non-Hamiltonian 3-connected bipartite graph with 50 nodes constructed in the paper has girth 6, and hence can be interpreted as the Levi graph of a combinatorial configuration. Only after getting hold of the paper itself, and looking at a diagram of the graph in question (see Figure 2), did I realize that it is of girth 6, hence relevant to the present topic. As this is 3-connected and has no Hamiltonian circuit, the corresponding *combinatorial* configuration provides a counterexample to the *combinatorial* version of my conjecture. We shall call this configuration the **Georges configuration**. If the Georges configuration can be geometrically realized, it will provide a counterexample to conjecture as stated.

Georges' construction starts with the graph in Figure 3, which was shown in [3] to admit no Hamiltonian circuit that uses **both** heavily drawn

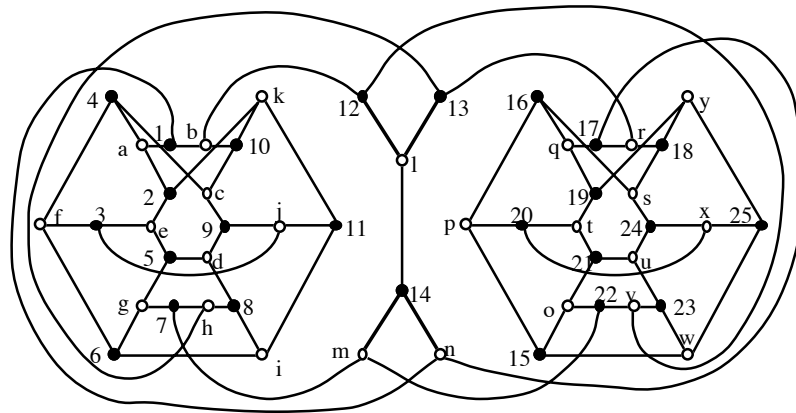


Figure 2. The 3-connected bipartite graph constructed by J. P. Georges has 50 vertices but admits no Hamiltonian circuit. The labels are explained below.

Georges' construction starts with the graph in Figure 3, which was shown in [3] to admit no Hamiltonian circuit that uses **both** heavily drawn edges. The proof of this is simple a follow-up of a few alternatives — the non-trivial, clever part is the *discovery* of the graph. The same comment (or compliment) applies to Georges' graph. In it, the two special edges of the graph in Figure 3 are subdivided by the addition of two nodes each, and two copies of this are connected to each other and to six additional nodes in the particular way shown in Figure 2. Again, the proof of non-Hamiltonicity is by examining several possibilities, and showing that neither leads to a Hamiltonian circuit. After the graph is constructed, this is a matter of routine checking, as are the other relevant properties.

3. Steinitz' geometric realization theorem. Since the Georges graph is the Levi graph of a 3-connected combinatorial configuration that is non-Hamiltonian, I wondered whether this combinatorial (25_3) configuration, the Georges configuration, has a geometric realization. To investigate this possibility, I followed the method of Steinitz [11]. The method is not well known, hence I will explain it briefly.

In order to start the Steinitz construction, we need to label the nodes of the Georges graph, so as to be able to write the configuration table of the corresponding configuration. Since I had no particular reason to do otherwise, I assigned the labels rather haphazardly, as shown in Figure 2. This yielded the configuration table shown in Table 1. (Please note that for ease of following the steps of the modifications, all tables are collected at the end of the paper.)

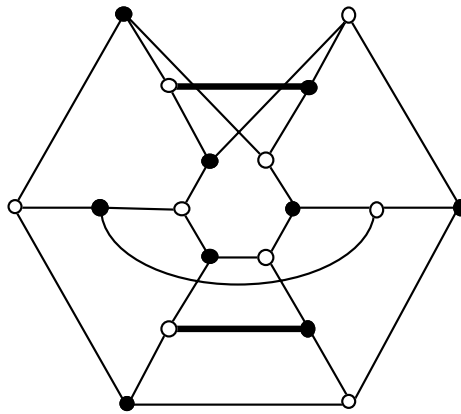


Figure 3. A bipartite graph found by Ellingham and Horton [3]; it admits no Hamiltonian circuit that uses both heavily drawn edges.

The first step in Steinitz's construction is to find an **orderly** configuration table for the configuration. A configuration table is called **orderly** if each vertex appears in every row, and therefore appears only once in every row. We begin by obtaining an orderly first row; this can be considered as a pairing of vertices of the configuration to lines, such that each point is incident in the configuration with the assigned line. In general, this is a complicated procedure which is dealt with at length in Steinitz [11]. But in the present case it turns out that a sort of "greedy algorithm" produces the desired pairing with no effort. Namely, in each column (that is, line) of the table, starting with the first column, we assign to it the first of its points that had not been assigned to a previous line. Clearly, such a procedure may get stuck in general, and Steinitz's algorithm deals with the general case. However, in the present instance there was no need to anything more complicated, and the "greedy" assignment worked for all lines and vertices. This is shown in Table 2, in which the entries in the columns have been rearranged so that the assigned vertex is in the first row, and the entries in the other two rows have been made orderly in the obvious way.

Table 3 shows the Georges configuration with columns (that is, lines) permuted so that a decomposition of the configuration in polygons (or more precisely, into "multilaterals" – that is circuits of lines) is obvious. This is accomplished by making the second entry in a column equal to the third entry in the preceding column, the first column being chosen arbitrarily. The exception is the last column of a multilateral, in which the last entry is the same as the second entry of the first column. In cases like the present one, where the first multilateral does not exhaust the columns, the first remaining column is used to start a new polygon. In the case of the George configuration, the first is a 22-gon, the second a triangle. Figure 4 provides an illustration.

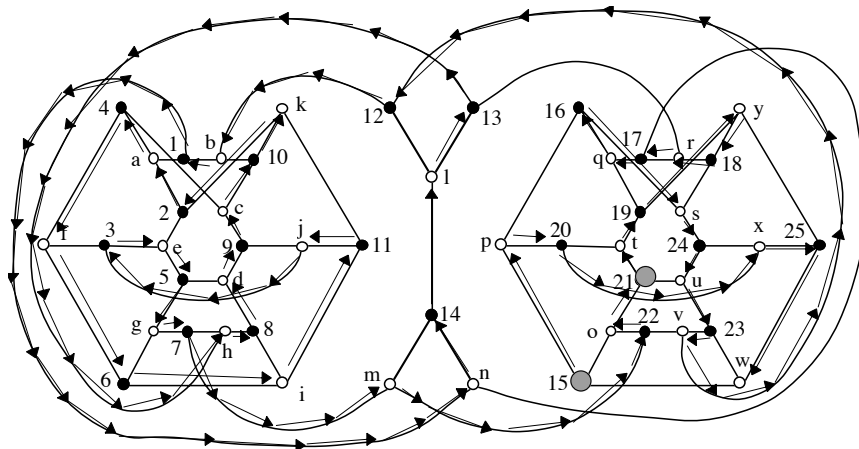


Figure 4. The decomposition of Georges graph that results from Steinitz's construction.

As a last step in the Steinitz algorithm before the geometric construction, we permute the columns (lines) once more. Some vertex of a column of the last (second in this case) polygon must be a vertex that appeared in a previous polygon, since otherwise the configuration would not be connected. We place that column as the first of the last polygon, and place as the last column of the previous polygon one of its columns that contains the same vertex. The other columns in both multilaterals are permuted accordingly, so as to preserve the multilaterals present. The result (one of the possible results) is shown in Table 4.

The geometric realization now proceeds very simply. It can be followed in Figure 5 which was obtained using “Geometer’s Sketchpad”™ and modified by ClarisDraw™. The idea is: Choose the vertices as arbitrary points, except when constrained to lie on one or two previously constructed lines. In this example we start with arbitrary points 21 and 19. Points 18 and 17 are also chosen freely, but the choice of 16 has to be on the line q through the previously determined points 19 and 17, and then 24 must be on the line s through 16 and 18; similarly, 23 must be on the line $u = 21, 24$. The points 12 and 1 can be chosen freely, but 14 must be on the line $n = 1, 17$, and 13 on the intersection point of the lines $r = 17, 18$ and $l = 12, 14$. Next, points 8, 9, 10, 2 are free, but 4 must be on the line $a = 1, 2$. The point 6 is free, while 11 is the intersection point of lines $i = 6, 8$ and $k = 10, 2$, and point 3 is the intersection point of the lines $f = 4, 6$ and $j = 9, 11$. Similarly, 5 is the intersection point of $d = 8, 9$ and $e = 2, 3$, and 7 is the intersection point of $h = 8, 13$ and $g = 5, 6$, while 22 is the intersection point of $v = 23, 12$ and $m = 7, 14$. This ends the construction of the first polygon. To start with the next (the triangle), we select 15 on the line $o = 22, 21$, then 20 as the intersection point of $t = 21, 19$ and $p = 16, 15$. The only remaining problem is the selection of point 25, which should be at the intersection of **three lines**, namely $y = 18, 19$, $x = 24, 20$, and $w = 23, 15$. It is to be expected that three lines do not have a common point. This is quite general, and this is the final solution given by Steinitz: the last line may need to be taken as a circle (or a parabola).

In the case of the Georges configuration, Figure 6 shows the outcome of selections in which the intersection point 25 of the lines y and x is very close to the third line w . In fact in this case — just as for many other configurations — by judicious choices of the free parameters one may find selections in which 25 is on a certain side of w , as well as selections where it is on the other side. By continuity, this implies that there is a position of coincidence. The final conclusion, therefore, is that the Georges configuration can be realized geometrically, by points and straight lines. Hence it is a 3-connected non-Hamiltonian configuration.

To complete the proof of the Theorem we shall now show how to obtain a 3-connected non-Hamiltonian (k_3) configuration for any $k \geq 35$. The

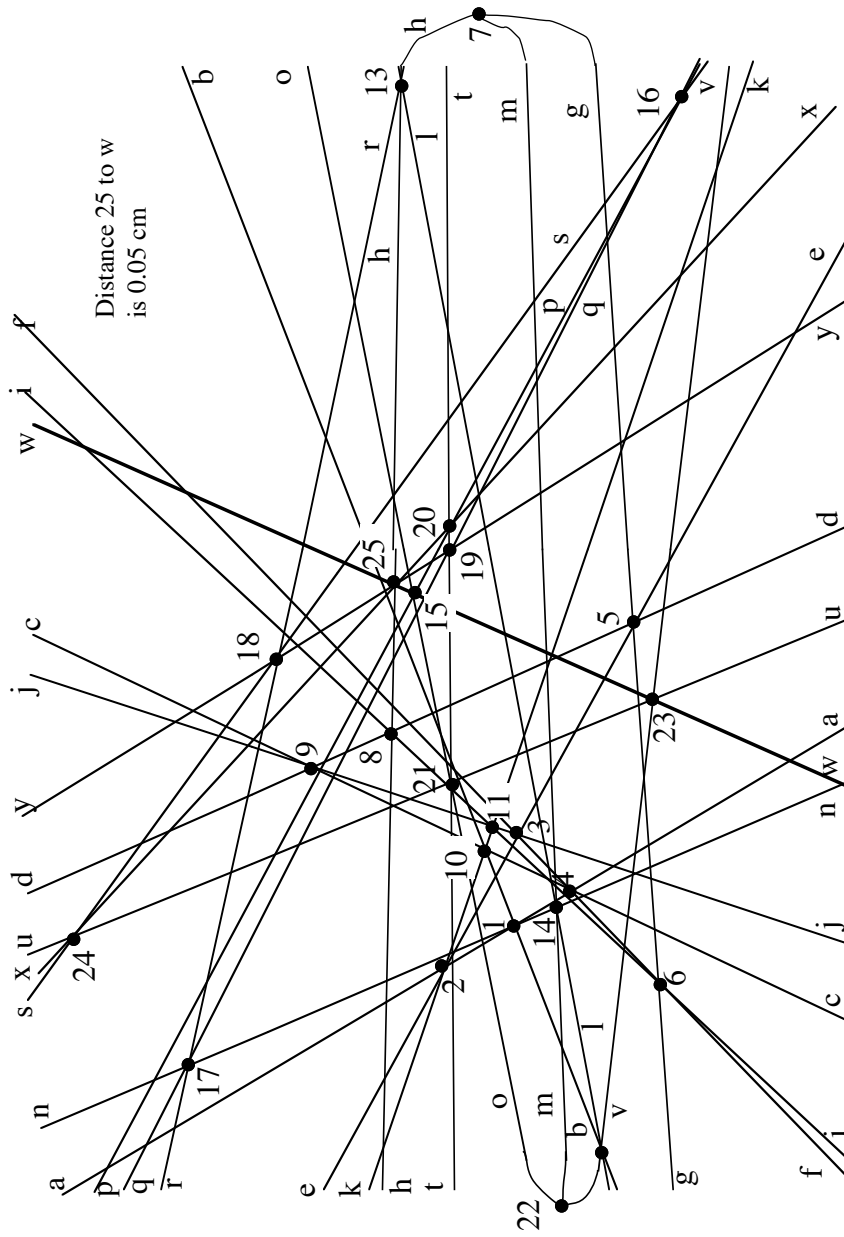


Figure 5. An example of a geometric “almost configuration” constructed using Steinitz’s method. See explanations in the text.

method is illustrated in Figure 6. Starting with any point of the (25_3) Georges configuration and the three lines incident with it (A in Figure 6(a)) and apply a *truncation*. That is, we delete that point and introduce three new points situated on the intersections of a new line L with the three starting lines (A' in Figure 6(b)). This is not a configuration, but can be turned into one by attaching to the same three points an analogous truncation of any geometric (q_3) configuration B (B' in Figure 6(c)). This yields a 3-connected $((26+q)_3)$ configuration. As is well known (see the survey [6]), such (q_3) configurations exist for all $q \geq 9$. If that $((26+q)_3)$ configuration were Hamiltonian, the part of the circuit that contains L would have to come to L from A' and exit to B' , with another part of the Hamiltonian circuit connecting A' to B' via the third point. But then the identification of the three points would yield a Hamiltonian circuit in the starting (25_3) configuration A . ■

4. Remarks. To conclude, a few notes.

4.1 Gropp [5] mentions Georges' paper [4], but makes no connection to configurations, and in particular, does not mention of the fact that the Georges graph is the Levi graph of a configuration. (It should be noted that in [4], the rendition

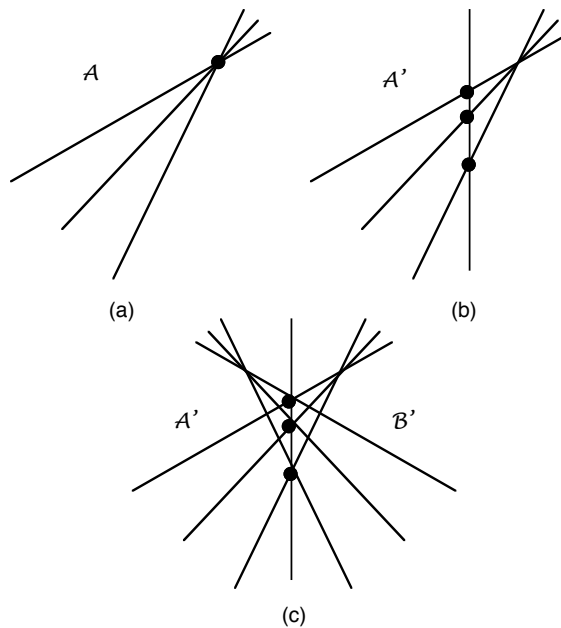


Figure 6. The construction of a 3-connected non-Hamiltonian $((n+q+1)_3)$ configuration from (n_3) and (q_3) configurations A and B of the same kind.

of the Ellingham-Horton graph, shown above in Figure 3, is missing one of the special edges.) Gropp also mentions that a result similar to Georges' has been found earlier by Kel'mans [8]. This may well be the case; however, I find the presentation in [8] too confusing to be able to decide whether the graph he constructs has girth 6. Like Georges, Kel'mans does not mention girth, or configurations. The claim in [5] that Kel'mans' 50-vertex graph is the same as Georges seems unjustified.

4.2 Although all the steps in Steinitz's proof [12] are correct (even though they are presented in a manner that is very hard to follow), he does not prove what he thought he proved, and what has been ascribed to him. More precisely, what Steinitz proved is that if one arbitrarily chosen line is omitted, the other lines and all points can be geometrically represented by straight lines and points in the Euclidean plane. "Geometrically represented" should be taken in the sense of the definition given at the start. However, what Steinitz claimed and believed to have proved is that the geometric representation can be taken in the more restricted sense – that a point is on a line if and *only if* their combinatorial counterparts are incident. As has been observed by T. Pisanski some years ago, this strengthening is not proved by Steinitz — in fact, there are counterexamples. A simple one is based on the "fake" configuration (16_3) shown in Figure 7. It was used in [2] for a different purpose, but yields a counterexample to the strong version of Steinitz's theorem if the heavily drawn line is chosen as the exceptional line. This happens because by the Pappus theorem of projective geometry the line L must be incident with the point P.

4.3. Although combinatorial configurations (7_3) and (8_3) exist, they cannot be realized by points and straight lines in the Euclidean (or the real projective)

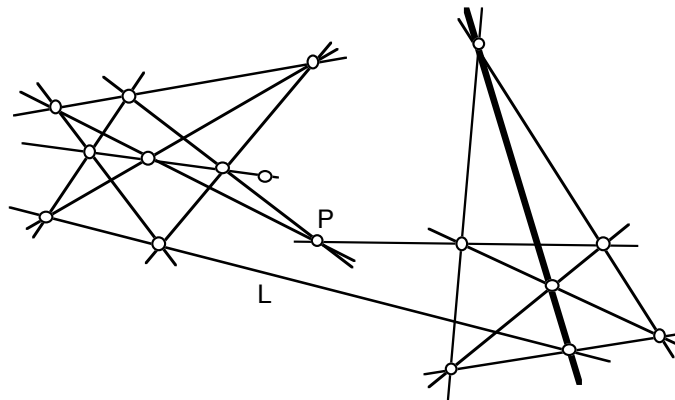


Figure 7. A "fake" configuration (16_3) from [2]. The nine points and lines in the left part of the figure form a Pappus configuration, hence the line L must contain the point P; this is a counterexample to the strong version of Steinitz's theorem on geometric realizations of combinatorial configurations (n_3) .

plane. However, the corresponding truncations are geometrically realizable, and can be used in the construction shown in Figure 6. Hence the Theorem is in fact valid for all $n \geq 33$.

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a	b	c	d	e	f	g	h	i	j	k	l	m
1	1	4	5	2	3	5	7	6	3	2	12	7
2	10	9	8	3	4	6	8	8	9	10	13	14
4	12	10	9	5	6	7	13	11	11	11	14	22

n	o	p	q	r	s	t	u	v	w	x	y
1	15	15	16	13	16	19	21	12	15	20	18
14	21	16	17	17	18	20	23	22	23	24	19
17	22	20	19	18	24	21	24	23	25	25	25

Table 1. A configuration table of the Georges configuration.

a	b	c	d	e	f	g	h	i	j	k	l	m
1	10	4	5	2	3	6	7	8	9	11	12	14
2	12	9	8	3	4	5	13	6	11	10	14	7
4	1	10	9	5	6	7	8	11	3	2	13	22

n	o	p	q	r	s	t	u	v	w	x	y
17	15	16	19	13	18	20	21	22	23	24	25
1	22	15	17	18	16	21	24	23	25	20	19
14	21	20	16	17	24	19	23	12	15	25	18

Table 2. An orderly configuration table for the Georges configuration.

a	f	i	j	e	g	m	o	t	y	r	q	s
1	3	8	9	2	6	14	15	20	25	13	19	18
2	4	6	11	3	5	7	22	21	19	18	17	16
4	6	11	3	5	7	22	21	19	18	17	16	24

u	v	b	n	l	h	d	c	k	p	x	w
21	22	10	17	12	7	5	4	11	16	24	23
24	23	12	1	14	13	8	9	10	15	20	25
23	12	1	14	13	8	9	10	2	20	25	15

Table 3. A rearrangement of the columns of the Georges configuration used to show a decomposition into multilaterals.

t	y	r	q	s	u	v	b	n	l	h	d	c
20	25	13	19	18	21	22	10	17	12	7	5	4
21	19	18	17	16	24	23	12	1	14	13	8	9
19	18	17	16	24	23	12	1	14	13	8	9	10

k	a	f	i	j	e	g	m	o	p	x	w
11	1	3	8	9	2	6	14	15	16	24	23
10	2	4	6	11	3	5	7	22	15	20	25
2	4	6	11	3	5	7	22	21	20	25	15

Table 4. A rearrangement of the columns of the Georges configuration, needed for the application of Steinitz's construction.